A NEW MEMBER OF NICHOLSON'S MORPHIC FOLKS

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Abstract. The main aim of the paper is to describe the structure of modules and corresponding rings satisfying the property (P), which says that $M/\operatorname{im}(\alpha)$ is embeddable into $\ker(\alpha)$ for each endomorphism α and which generalizes the morphic property. In particular, it is proved that the class of rings with the property (P) is closed under taking products and summands and contains unit regular rings. We also explain connections between the virtually internal cancellation property and the property (P) and characterize the structure of particular classes of rings satisfying the property (P).

1. Introduction and preliminaries

Among other algebraic dualities, the concept of homomorphisms satisfying the dual condition to that from the first isomorphism theorem has appeared fruitful during the last two decades. While an arbitrary endomorphism α of a module M satisfies the condition $\operatorname{im}(\alpha) \cong M/\ker(\alpha)$, it is called *morphic* if it holds $\ker(\alpha) \cong M/\operatorname{im}(\alpha)$. The research of morphic modules (and rings), i.e. of modules over which every endomorphism is morphic, was started by Nicholson and Sánchez Campos in papers [20, 21].

The concept of morphic rings, which naturally generalizes widely studied unit regular rings [11, Theorem 1], have motivated other generalizations. Given a ring R, if $\mathcal{P}$ denotes the set of all its right principal ideals (i.e. images of endomorphisms of R_R) and $\mathcal{A}$ the set of all right annihilators (i.e. kernels of endomorphisms of R_R), then we say that R is right *quasi-morphic* (*pseudo-morphic*, *generalized morphic*, respectively) if $\mathcal{P} = \mathcal{A}$ ($\mathcal{P} \subseteq \mathcal{A}$, $\mathcal{P} \supseteq \mathcal{A}$, respectively). The structure of quasi-morphic rings and modules satisfying the corresponding condition on endomorphism rings is partially described in papers [2, 5, 6, 9], while pseudo-morphic rings are studied in [7, 27] and generalized morphic rings in [28].

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This paper is focused to another intermediate class of pseudo-morphic modules containing the class of all morphic modules (see Example 2.1). For a module M, we say that $\alpha \in \mathrm{End}(M)$ satisfies the property (P) if $M/\operatorname{im}(\alpha)$ embeds into $\ker(\alpha)$, the module M satisfies the property (P) if every its endomorphism satisfies (P), and the ring R satisfies the property (right) (P) if R_R satisfies (P). The main aim of the paper is to place the class of modules and rings satisfying the property (P) in the context of other generalizations of the idea of morphicity as well as in the general context of ring theory. In particular, we prove that the class of all rings satisfying the condition (P) is closed under taking products and summands (Proposition 2.15), regular endomorphism rings of modules satisfying the property (P) are exactly unit regular (Theorem 2.11) and reversible rings satisfying the property (P) are already morphic (Theorem 2.7). Theorem 2.9 shows that if α in any Jacobson pair $\{\alpha, \beta\}$ satisfies the property (P), then so is β. The third section of the paper is devoted to relations between properties virtually (internal) cancellation and (P). Among the other results, we obtain that a module M satisfies the virtually-IC property iff every regular element in $\mathrm{End}(M)$ satisfies the property (P) (Theorem 3.4) and hence $\mathrm{End}(M)$ is unit-regular iff M satisfies the virtually-IC property and $\mathrm{End}(M)$ is regular (Corollary 3.6) and $\mathrm{End}(M)$ is unit-regular iff M satisfies the virtually-IC property and $\mathrm{End}(M)$ is regular iff M satisfies the virtually-C property and $\mathrm{End}(M)$ is regular (Corollary 3.11). Notice that the property $(P) \Rightarrow$ the virtually-IC property and the property (P) $\Rightarrow$ the virtually-C property if a module M is either injective or satisfies the finite exchange property. The fourth section of the paper is devoted to the structure of particular classes of rings satisfying the property (P), in particular, it is proved that local semiartinian rings with the property (P) are precisely right artinian right chain rings (Theorem 4.4). The final section describes group rings that are ring satisfy the property (P). We proved that if RG_{RG} satisfies the property (P), then R_R satisfies the property (P) and G is a locally finite group (Theorem 5.1), and if $G = H \rtimes K$ with $|H| < \infty$ (i.e. G is a semidirect product of H by K) and RG_{RG} satisfies the property (P), then RK_{RK} satisfies the property (P) (Theorem 5.3).

Throughout this paper, R denotes an associative ring with identity and modules are unitary right R-modules. For a right R-module M, we use $N \subseteq M_R$, $N \leq M_R$, $N \leq_e M_R$, $N \ll M$ and $N \leq^{\oplus} M_R$, to mean that N is a subset, a submodule, an essential submodule, a superfluous submodule and a direct summand of M_R, respectively. $Rad(M)$ denotes the intersection of all maximal submodules of M, $\mathrm{Soc}(M)$ denotes the socle of M and $(S_i(M) \mid i \leq \sigma)$ denotes the socle sequence of M. In case $R = M$, we use $J(R)$ instead of $Rad(R)$ and it is called the

Jacobson radical of R. For any $x \in R$, the left and right annihilator of $x \in R$ denoted by $l(x)$ (or $l_R(x)$) and $r(x)$ (or $r_R(x)$), respectively. We write module morphisms opposite the scalars. If there exists an R-monomorphism (respectively, R-epimorphism) from X to Y then we write $X \hookrightarrow Y$ (respectively, $X \twoheadrightarrow Y$). The notations $\operatorname{End}(M)$ and $E(M)$ denote the ring of R-endomorphisms of M and the injective hull of M_R, respectively. For a set $\wedge$, let $M^{(\wedge)}$ and $M^{\wedge}$ denote direct sum and direct product of copies of M_R indexed over $\wedge$, respectively. We use $M_n(R)$ to stand for the ring of all $n \times n$ matrices over a ring R. In what follows $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Z}_n$ denote the natural numbers, integers, rational numbers and the ring of integers modulo n, respectively.

For unexplained notions and results, we refer the reader to [3, 18, 24, 25, 26].

2. THE PROPERTY (P)

Example 2.1. Morphic modules (elements) satisfy the property (P). On the other hand, as a $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ satisfies the property (P) but it is not a morphic $\mathbb{Z}$-module.

Lemma 2.2. *The following conditions are equivalent for $\alpha \in \operatorname{End}(M)$:*

1. *α satisfies the property (P).*
2. *There exists $\beta \in \operatorname{End}(M)$ such that $\alpha \circ \beta = 0$ and $\alpha(M) = \ker(\beta)$.*
3. *There exists $\beta \in \operatorname{End}(M)$ such that $\operatorname{im}(\beta) \leq \ker(\alpha)$ and $\alpha(M) = \ker(\beta)$.*
4. *There exists $\beta \in \operatorname{End}(M)$ such that $\operatorname{im}(\beta) \hookrightarrow \ker(\alpha)$ and $\alpha(M) = \ker(\beta)$.*

Proof. (1) $\Rightarrow$ (2) Consider the natural projection $\pi : M \to M/\operatorname{im}(\alpha)$, the inclusion map $\iota : \ker(\alpha) \to M$ and call $\phi : M/\operatorname{im}(\alpha) \hookrightarrow \ker(\alpha)$. Take

$$\beta = \iota \circ \phi \circ \pi : M \to M.$$

Clearly, $\alpha \circ \beta = 0$ and $\alpha(M) = \ker(\beta)$ as desired.
(2) $\Rightarrow$ (3) $\Rightarrow$ (4) The implications are obvious.
(4) $\Rightarrow$ (1) Assume that there exists $\beta \in \operatorname{End}(M)$ such that $\operatorname{im}(\beta) \hookrightarrow \ker(\alpha)$ and $\alpha(M) = \ker(\beta)$. It is easy to see that

$$M/\alpha(M) = M/\ker(\beta) \cong \beta(M) \hookrightarrow \ker(\alpha),$$

as desired. $\square$

Corollary 2.3. *The following conditions are equivalent for an element $a \in R$.*

1. *a satisfies the property (P).*
2. *There exists $b \in R$ such that $aR = r(b)$ and $bR \leq r(a)$.*

(3) *$aR = r(b)$ and $bR \hookrightarrow r(a)$ for some $b \in R$.*

Lemma 2.4. *The following conditions are equivalent for a ring R:*

(1) *R_R satisfies the property (P).*
(2) *For each $a \in R$, there exists $b \in r(a)$ such that $aR = r(b)$.*
(3) *For each $a \in R$, there exists $s \in R$ and $b \in r(a)$ such that $aR = r(b)$ and $bR = r(as)$.*

Proof. The equivalence (1)⇔(2) follows from Corollary 2.3 and (3)⇒(2) is clear.

(2)⇒(3) Since there exists $c \in r(b) = aR$ such that $bR = r(c)$, it is enough to take $s \in R$ satisfying $c = as$. □

A ring R is called *left principally injective* (*left P-injective* for short) if every R-homomorphism $Ra \to R$, $a \in R$, extends to R. It is well-known that a ring is left P-injective if and only if all principal right ideals are annihilators by [19, Lemma 1.1], cf. [5, Lemma 3]. We can formulate the following consequence identifying a class of rings that elements satisfy the property (P) of Corollary 2.3 which follows from Lemma 2.4:

Corollary 2.5. *If R_R satisfies the property (P), then the ring R is left P-injective.*

Recall that, a ring R is said to be *Dedekind-finite* if $ab = 1$ implies $ba = 1$ for any two $a \in R$ and $b \in R$. In other words, all one-sided inverses in the ring are two-sided. The following result is also a direct consequence of Lemma 2.2.

Corollary 2.6. *An endomorphism satisfying the property (P) is a monomorphism if and only if it is an isomorphism. In particular, every ring satisfying the property (P) is Dedekind-finite.*

Recall that a ring is *reversible* if $ba = 0$ whenever $ab = 0$, for details see [8, 12]. The last consequence of Lemma 2.4 generalizes [5, Corollary 4] and [27, Theorem 11(2)].

Corollary 2.7. *Let R be a reversible ring. Then R_R satisfies the property (P) if and only if R is morphic.*

Proof. Let $a \in R$. From Lemma 2.4, it infers that there exist $s \in R$ and $b \in r(a)$ such that $aR = r(b)$ and $bR = r(as)$. Since R is reversible and $ba = 0 = ax$ for each $x \in r(a)$, we have $ab = 0$ and $0 = xa = xas = asx$, hence $bR \subseteq r(a) \subseteq r(as) = bR$ and $b = r(a)$ (cf. [12, Lemma 1.8]). □

Recall that a module M is said to be *pseudo-morphic* if for any $\alpha \in \mathrm{End}(M)$ there exists $\beta \in \mathrm{End}(M)$ for which $\mathrm{im}(\alpha) = \ker(\beta)$. From Lemma 2.2, it is clear that all modules satisfying the property (P) are pseudo-morphic and the following example shows that the converse does not hold.

Example 2.8. The abelian group $\mathbb{Z}_2 \times \mathbb{Z}_4$ is a pseudo-morphic $\mathbb{Z}$-module but it does not satisfy the property (P).

Two elements $\alpha, \beta \in R$ are said to form a *Jacobson pair* if there exist elements $a, b \in R$ such that $\alpha = 1 - ab$ and $\beta = 1 - ba$.

For such a pair, "*Jacobson's Lemma*" is the statement that α is a unit if and only if β is a unit (this result is known as Jacobson's Lemma for units). There are several analogous results in the literature. In [13], the authors observed that, for any Jacobson pair $\{\alpha, \beta\}$ in any ring R, if α is left morphic (quasi-morphic), then so is β.

Theorem 2.9. *For any Jacobson pair $\{\alpha, \beta\}$ in any ring R, if α satisfies the property (P), then so is β.*

Proof. By the basic commutation rules, we have $\alpha a = a\beta$ and $b\alpha = \beta b$ for the Jacobson pair $\{\alpha, \beta\}$.

Assume that α satisfies the property (P). Then, there is $x \in R$ such that $\alpha R = r(x)$ and $xR \leq r(\alpha)$.

Claim: $\beta R = r(xa)$: The inclusion $\beta R \subseteq r(xa)$ follows from the fact that $x(a\beta) = x(\alpha a) = (x\alpha)a = 0$ which implies $\beta \in r(xa)$.

For the reverse inclusion, let $t \in r(xa)$. Then $x(at) = (xa)t = 0$, i.e. $at \in r(x) = \alpha R$. Hence, $at = \alpha s$ for some $s \in R$. We have

$$(ba)t = b(at) = b(\alpha s) = (b\alpha)s = (\beta b)s = \beta(bs)$$

which implies $(1 - \beta)t = (ba)t = \beta(bs)$, and so $t = \beta(bs + t) \in \beta R$.

Call $y := bxa$.

Claim: $yR \leq r(\beta)$ and $r(y) = \beta R$: The first follows from the fact that

$$\beta y = \beta(bxa) = (\beta b)xa = (b\alpha)xa = b(\alpha x)a = 0,$$

i.e. $y \in r(\beta)$.

On the other hand, we have

$$y\beta = (bxa)\beta = bx(a\beta) = bx(\alpha a) = b(x\alpha)a = 0$$

It follows that $\beta \in r(y)$ or $\beta R \leq r(y)$. For the converse inclusion, if $yt = 0$ then $(bxa)t = 0$, and so $abxat = a(bxa)t = 0$. Moreover, we have

$$\alpha(xat) = (1 - ab)xat = xat - abxat = xat$$

Then, $xat = (\alpha x)at = 0$. Hence, $t \in r(xa) = \beta R$ by Claim. Now $r(y) \leq \beta R$, as desired $\square$

An element a of a ring R is said to be *unit-regular* in R if $a = aua$ for some unite $u \in R$. A ring R is called unit-regular if every element is unit-regular.

Theorem 2.10. *If $\alpha \in \mathrm{End}(M)$ satisfies the property (P) and γ is an automorphism of M, then both $\alpha \circ \gamma$ and $\gamma \circ \alpha$ satisfy the property (P).*

In particular, every unit-regular morphism satisfies the property (P).

Proof. Assume that $\alpha \in \mathrm{End}(M)$ satisfies the property (P) and γ is an automorphism of M. Then, by Lemma 2.2, there exists $\beta \in \mathrm{End}(M)$ such that $\alpha \circ \beta = 0$ and $\alpha(M) = \ker(\beta)$. Therefore, $(\alpha\gamma)(\gamma^{-1}\beta) = 0$ and $(\gamma\alpha)(\beta\gamma^{-1}) = 0$. Note that γ is an automorphism of M. It follows that

$$(\alpha\gamma)(M) = \alpha(M) = \ker(\beta) = \ker(\gamma^{-1}\beta)$$

and $(\gamma\alpha)(M) = \gamma(\ker(\beta)) = \ker(\beta\gamma^{-1})$. By Lemma 2.2, we obtain that $\alpha \circ \gamma$ and $\gamma \circ \alpha$ satisfy the property (P). $\square$

An element a of a ring R is said to be *regular* in R if $a = aba$ for some $b \in R$. A ring R is called regular if every element is regular.

Theorem 2.11. *Let M be a right R-module. Then $\mathrm{End}(M)$ is unit-regular if and only if $\mathrm{End}(M)$ is regular and M satisfies the property (P).*

Proof. Suppose that $\mathrm{End}(M)$ is a regular ring. For every $\alpha \in \mathrm{End}(M)$, $\alpha(M)$ and $\ker(\alpha)$ are direct summands of M. So, there exist submodules K and H of M such that $M = \alpha(M) \oplus K = \ker(\alpha) \oplus H$, and so $K \cong M/\alpha(M) \hookrightarrow \ker(\alpha)$ and $M = \alpha(H) \oplus K$. Call $\beta : K \hookrightarrow \ker(\alpha)$. Let $\gamma : M \to M$ be an R-homomorphism via $\gamma(\alpha(h) + k) = h + \beta(k)$. One can check that γ is a monomorphism. Now, $\alpha = \alpha \circ \gamma \circ \alpha$ and γ is an isomorphism by Corollary 2.6.

The reverse implication is clear. $\square$

Recall that a module M is called (*dual*) *Rickart* if $\ker \rho \leq^{\oplus} M$ ($\rho(M) \leq^{\oplus} M$) for every $\rho \in \mathrm{End}_R(M)$ [16, 17].

Proposition 2.12. *If M is a Rickart module and it satisfies the property (P), then M is dual-Rickart.*

Proof. Let $\alpha \in \mathrm{End}(M)$. Since M satisfies the property (P), there exists $\beta \in \mathrm{End}(M)$ such that $\operatorname{im}\alpha = \ker\beta$ and $\alpha o \beta = 0$. Then $\ker\beta \leq^{\oplus} M$ as M is Rickart. Thus $\operatorname{im}\alpha \leq^{\oplus} M$ which implies M is dual-Rickart. □

Corollary 2.13. *If M is a Rickart module and it satisfies the property (P), then* $\mathrm{End}(M)$ *is regular.*

Corollary 2.14. *Let M be a right R-module.*

(1) $\mathrm{End}(M)$ *is unit-regular if and only if M is a Rickart module and it satisfies the property (P).*
(2) *If M is quasi-projective, then* $\mathrm{End}(M)$ *is unit-regular if and only if M is a dual-Rickart module and it satisfies the property (P).*

Proof. (1) The claim is clear by Theorems 2.11, Proposition 2.12 and Corollary 2.13.
(2) Assume M is a dual-Rickart module and it satisfies the property (P). Let α be an endomorphism of M. By the assumption, $\mathrm{im}(\alpha)$ is a direct summand of M. Since $M/\ker(\alpha) \cong \mathrm{im}(\alpha)$, $M/\ker(\alpha)$ is M-projective and so $\ker(\alpha)$ is a direct summand of M. It is shown that $\mathrm{End}(M)$ is regular. Thus, $\mathrm{End}(M)$ is unit-regular by Theorem 2.11.

The reverse implication is obvious. □

Proposition 2.15. *The following statements hold:*

(1) *A direct product $\prod_I R_i$ of rings R_i satisfies the property (P) if and only if each R_i satisfies the property (P).*
(2) *A module M satisfies the property (P) if and only if whenever $\gamma : M/K \hookrightarrow M$ where K is a submodule of M, then $M/\mathrm{im}(\gamma) \hookrightarrow K$. Furthermore, A module M satisfies the property (P) if and only if whenever $M/K \cong H$ where K and H are submodules of M, then $M/H \hookrightarrow K$.*
(3) *If M and N satisfy the property (P) and $Hom(M,N) = 0 = Hom(N,M)$, then $M \oplus N$ satisfies the property (P).*
(4) *The class of modules satisfying the property (P) is closed under taking direct summands.*

Proof. (1) If $x = (x_i) \in \prod_I R_i$, then $xR = \prod_I x_i R_i$ and $r(x) = \prod_I r(x_i)$.
(2) ($\Rightarrow$) Let $\gamma : M/K \to H$ be an isomorphism for some submodule H of M ($H = \mathrm{im}(\gamma)$). Consider the natural projection $\pi : M \to M/K$, the inclusion map $\iota : H \to M$ and call $\alpha := \iota \circ \gamma \circ \pi : M \to M$. Then, we have $\ker(\alpha) = K$ and $\mathrm{im}(\alpha) = H$. By the hypothesis, there exists $\beta \in \mathrm{End}(M)$ such that $\mathrm{im}(\beta) \leq \ker(\alpha) = K$ and $\alpha(M) = \ker(\beta) = H$. It follows that $M/\mathrm{im}(\gamma) = M/\ker(\beta) \cong \mathrm{im}(\beta) \leq \ker(\alpha) = K$.

($\Leftarrow$) Let $\alpha \in \mathrm{End}(M)$. Then, $M/\ker(\alpha) \cong \mathrm{im}(\alpha)$. It follows that $M/\mathrm{im}(\alpha) \hookrightarrow \ker(\alpha)$ and so α satisfies the property (P).
(3) Let $\alpha \in \mathrm{End}(M \oplus N)$. Then $\alpha = \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}$ is in the matrix form, where $\beta \in \mathrm{End}(M)$ and $\gamma \in \mathrm{End}(N)$. Now, there exist $\beta' \in \mathrm{End}(M)$ and $\gamma' \in \mathrm{End}(N)$ such that $\beta\beta' = 0$, $\mathrm{im}(\beta) = \ker(\beta')$, $\gamma\gamma' = 0$ and $\mathrm{im}(\gamma) = \ker(\gamma')$. Call $\alpha' := \begin{pmatrix} \beta' & 0 \\ 0 & \gamma' \end{pmatrix}$. One can check that $\alpha\alpha' = 0$ and $\mathrm{im}(\alpha) = \ker(\alpha')$, as desired.
(4) Assume that $M = M_1 \oplus M_2$ satisfies the property (P). Let K and H be submodules of M_1 with $M_1/K \cong H$. We show that $M_1/H \hookrightarrow K$. In fact, we have $M/K \cong M_1/K \oplus M_2 \cong H \oplus M_2$ and obtain that $M/(H \oplus M_2) \hookrightarrow K$. We deduce that $M_1/H \hookrightarrow K$. □

Remark 2.16. From Example 2.1, we infer that the class of modules satisfying the property (P) is closed under taking direct sums.

A module M is called *image-projective* if $\mathrm{im}(\beta) \subseteq \mathrm{im}(\alpha)$, where $\beta, \alpha \in \mathrm{End}(M)$, implies $\beta \in \alpha(\mathrm{End}(M))$.

Clearly, quasi-projective modules are image projective.

Proposition 2.17. *Let M be a right R-module and $S = \mathrm{End}(M)$. The following statements hold:*

1. *If S satisfies the property (P), then M is image-projective.*
2. *If M satisfies the property (P) and it is image-projective, then S satisfies the property (P).*
3. *If S satisfies the property (P) and M generates its kernels, then M satisfies the property (P).*

Proof. (1) If $f(M) \leq g(M)$ with $f, g \in S$, then $l_S(g) \leq l_S(f)$. Since S satisfies the property (P), S is left P-injective by Corollary 2.5. It follows that $fS \leq gS$. Thus, M is image-projective.
(2) Let α be an endomorphism of M. Call $\beta \in S$ with $\mathrm{im}(\beta) \leq \ker(\alpha)$ and $\mathrm{im}(\alpha) = \ker(\beta)$. One can check that $\beta S \leq r_S(\alpha)$ and $\alpha S \leq r_S(\beta)$. If $\gamma \in r_S(\beta)$ then $\gamma(M) \leq \ker(\beta) = \alpha(M)$. We have that M is image-projective and obtain that $\gamma S \leq \alpha S$ and so $\gamma \in \alpha S$. We deduce that $\alpha S = r_S(\beta)$.
(3) Let α be an endomorphism of M. Call $\beta \in S$ with $\beta S \leq r_S(\alpha)$ and $\alpha S = r_S(\beta)$. One can check that $\alpha \circ \beta = 0$ and $\alpha(M) \leq \ker(\beta)$. Since M generates its kernels, we get $\ker(\beta) = \sum_{f \in H} f(M)$ for some $H \subseteq S$. For all $f \in H$, we have

$\beta \circ f = 0$ and so $f \in r_S(\beta) = \alpha S$. It follows that $f(M) \leq \alpha(M)$. It is shown that $\ker(\beta) \leq \alpha(M)$. We deduce that $\ker(\beta) = \alpha(M)$. $\square$

We recall the following notion used in Proposition 2.17: a module M_R *generates its submodule*, say K, if

$$K = \sum \{M\lambda : \lambda \in \mathrm{End}(M), M\lambda \subseteq K\},$$

and M generates its kernels if it generates $\ker(\beta)$ for all $\beta \in \mathrm{End}(M)$.

Theorem 2.18. *The following conditions are equivalent for a right R-module M which generates its kernels.*

(1) *M is morphic and image-projective.*
(2) $\mathrm{End}(M)$ *satisfies the property (P).*

Proof. The claim follows from Propositions 2.15 and 2.17. $\square$

Theorem 2.19. *Let R be a ring and $n > 1$.*

(1) *$M_n(R)$ satisfies the property (P) if and only if R^n satisfies the property (P).*
(2) *If R satisfies the property (P) and $e^2 = e \in R$, then eRe satisfies the property (P).*

Proof. (1) The claim follows from Theorem 2.18 and because of the fact that R^n is image-projective and generates its submodules.
(2) The claim follows from Theorem 2.18 and because of the fact that eR is a projective right R-module that satisfies the property (P) by Theorem 2.15. $\square$

Recall that if $\{S_i\}$ are the homogeneous components of M, $S = \mathrm{End}(M)$ and $T_i = \mathrm{End}(S_i)$, then $S \cong \prod_I T_i$.

Theorem 2.20. *The following conditions are equivalent for a semisimple right R-module M:*

(1) *M satisfies the property (P).*
(2) $\mathrm{End}(M)$ *is unit-regular.*
(3) *Each homogeneous component of M is artinian.*

In this case $\mathrm{End}(M)$ *is a direct product of matrix rings over division rings.*

Proof. (1) $\Rightarrow$ (2). Assume that M satisfies the property (P). Then M is image-projective (being semisimple) and obtain that S satisfies the property (P) by Proposition 2.17(2). Note that S is regular. By Theorem 2.11, the ring $S =$

End(M) is unit-regular.
(2) $\Rightarrow$ (3) By Proposition 2.15, each T_i satisfies the property (P). Suppose that each S_i is not artinian. Then, $S_i = H \oplus H \oplus \cdots$ where H is simple. Hence the monomorphism $(h_1, h_2, h_3 \dots) \to (0, h_2, h_3, \dots)$ in T_i is not an epimorphism, which contradicts to Corollary 2.6. Thus, S_i is artinian.
(3) $\Rightarrow$ (1) Assume that each homogeneous component of M is artinian. Then each T_i is artinian and obtain that $S \cong \prod_I T_i$ is unit- regular, and hence it satisfies the property (P). Moreover M generates its kernels because M is semisimple, and so M satisfies the property (P) by Proposition 2.17.

The last statement follows from (3) because $S \cong \prod_I T_i$. □

3. VIRTUALLY INTERNAL CANCELLATION PROPERTY

An R-module M is said to have *internal cancellation property* (IC) if it satisfies the condition:

$$M = N \oplus K = N_1 \oplus K_1 \text{ and } N \cong N_1 \text{ implies that } K \cong K_1.$$

Notice that summand-morphic modules are precisely modules with IC property, where M is said to be a *summand-morphic* module if $M/A \cong B$ where $A, B \leq^{\oplus} M$, then $M/B \cong A$ ([23, Proposition 3.2]).

According to [4] (see also [12]), a module X is said to be *subisomorphic* to a module Y if X is isomorphic to a submodule of Y, and denoted by $X \preceq Y$, and a module M is called a virtually semisimple module if every submodule of M is isomorphic to a direct summand of M.

Remark 3.1. A virtually semisimple module is morphic iff it is iso-summand-morphic ([23, Corollary 3.8]), where M is said to be an *iso-summand-morphic* module if $M/A \cong B$ where $A, B \preceq^{\oplus} M$, then $M/B \cong A$.

A module M is said to satisfy *virtually internal cancellation property* (for short, virtually-IC) or *iso internal cancellation property* (for short, iso-IC) if it satisfies the following condition:

$$\text{If } M = N \oplus K = N_1 \oplus K_1 \text{ and } N \cong N_1, \text{ then } K \preceq K_1 \text{ and } K_1 \preceq K$$

Example 3.2. The IC property $\Rightarrow$ the virtually-IC property. The converse implication holds when on the C2-property (a module M is C2 if whenever A is a direct summand of M and B is a submodule of M isomorphic to A, then B is also a direct summand of M).

Proof. Let M be a C2 module with the virtually-IC property. Assume that $M = A \oplus B = A_1 \oplus B_1$ and $A \cong A_1$. We show that $B \cong B_1$. Then there a monomorphism $\alpha : B \to B_1$. Hence, $B \cong \alpha(B)$ and so $\alpha(B)$ is a direct summand of M. It follows that $\alpha(B)$ is a direct summand of B_1. Write $B_1 = \alpha(B) \oplus B_2$ for some direct summand B_2 of B_1. Thus, we have

$$M = A \oplus B = (A_1 \oplus \alpha(B)) \oplus B_2$$

Note that $A \oplus B \cong A_1 \oplus \alpha(B)$, which implies $B_2 = 0$. Thus, $B_1 = \alpha(B) \cong B$, as required. □

Proposition 3.3. *The class of modules satisfying the virtually-IC property is closed under taking direct summands.*

Proof. Assume that $M = N \oplus K$ satisfies the virtually-IC property and N has a decomposition

$$N = N_1 \oplus N_2 = N' \oplus N''$$

where $N_1 \cong N'$. Then $M = (N_1 \oplus K) \oplus N_2 = (N' \oplus K) \oplus N''$ and $N_1 \oplus K \cong N' \oplus K$. By the hypothesis, we get $N_2 \preceq N''$ and $N'' \preceq N_2$, as required. □

Theorem 3.4. *The following conditions are equivalent for a right R-module M:*

(1) *M satisfies the virtually-IC property.*
(2) *Every regular element in* $\mathrm{End}(M)$ *satisfies the property* (P).

Proof. (1) $\Rightarrow$ (2) Let α be regular in $\mathrm{End}(M)$. Then, the module M has a decomposition $M = \alpha(M) \oplus K = \ker(\alpha) \oplus N$. Clearly, $\alpha(M) \cong M/\ker(\alpha) \cong N$, and so $K \hookrightarrow \ker(\alpha)$. Hence $M/\alpha(M) \cong K \hookrightarrow \ker \alpha$, i.e. α satisfies the property (P).
(2) $\Rightarrow$ (1) Assume the module M has a decomposition $M = N \oplus K = N_1 \oplus K_1$ and $\gamma : N \to N_1$ is an isomorphism. Consider $\alpha : M \to M$ by $\alpha(n + k) = \gamma(n)$ for all $n \in N$ and $k \in K$. Then $\alpha \in \mathrm{End}(M)$ and $\alpha(M) = \gamma(N) = N_1$, $\ker(\alpha) = K$ are both direct summands of M. Thus α is regular in $\mathrm{End}(M)$ and so our hypothesis gives $M/\alpha(M) \hookrightarrow \ker(\alpha)$. Hence $K_1 \cong M/N_1 = M/\alpha(M) \hookrightarrow \ker(\alpha) = K$. The above process is completely similar if we replace γ by γ^{-1}, it infers that $K \hookrightarrow K_1$, as required. □

Corollary 3.5. *The property (P) implies the virtually-IC property.*

Corollary 3.6. *The following conditions are equivalent for a right R-module M:*

(1) $\mathrm{End}(M)$ *is unit-regular.*
(2) *M satisfies the virtually-IC property and* $\mathrm{End}(M)$ *is regular.*

Proof. (2)$\Rightarrow$ (1) If M satisfies the virtually-IC property and $\mathrm{End}(M)$ is regular, then $\mathrm{End}(M)$ satisfies the property (P) by Theorem 3.4. By Theorem 2.11, $\mathrm{End}(M)$ is unit-regular.
(2)$\Rightarrow$ (1) This implication follows from Theorem 2.11 and 3.4. $\square$

Remark 3.7. By Theorem 2.20, a semisimple right R-module M satisfies the property (P) iff $\mathrm{End}(M)$ is unit-regular.

We recall that every ring satisfying the property (P) is Dedekind-finite (Corollary 2.6).

Corollary 3.8. *Every module satisfying the virtually-IC property is Dedekind-finite.*

Proof. Assume a module M satisfies the virtually-IC property and M has a proper direct summand that is isomorphic to M. It means that $M = A \oplus B$ with $A \cong M$ and $A \neq M$. Then, $M = A \oplus B = M \oplus 0$ and so B is embedded to 0. It follows that $B = 0$ and $A = M$, a contradiction. Thus, M is Dedekind-finite. $\square$

Corollary 3.9. *The following conditions are equivalent for a semisimple module M:*

(1) *M satisfies the property (P).*
(2) *M satisfies the virtually-IC property.*
(3) *There exist a cardinal κ, simple modules S_α and natural numbers n_α for each $\alpha < \kappa$ such that $M \cong \bigoplus_{\alpha<\kappa} S^{n_\alpha}$ and $S_\alpha \ncong S_\beta$ for each $\alpha \neq \beta$.*

Proof. (1)$\Rightarrow$ (2) It is clear form Theorem 3.4.
(2)$\Rightarrow$ (3) Since infinite direct powers are not Dedekind finite, the claim follows from Corollary 3.8.
(3)$\Rightarrow$ (1) As $\mathrm{End}(M) \cong \prod_{\alpha<\kappa} M_{n_\alpha}(\mathrm{End}(S_\alpha))$ is a unit regular ring, M satisfies the property (P) by Theorem 2.11. $\square$

A module M satisfies the *cancellation property* if $M \oplus A \cong M \oplus B$ implies $A \cong B$. We say that a module M satisfies the *virtually cancellation property* (virtually-C) or the *iso cancellation property* (iso-C) if $M \oplus A \cong M \oplus B$ implies $A \preceq B$ and $B \preceq A$.

Proposition 3.10. *The virtually-C property implies the virtually-IC property.*

Proof. Assume M has the virtually cancellation property. Let $M = A \oplus B = A_1 \oplus B_1$ with $A \cong A_1$. Then, we have

$$M \oplus B_1 = A \oplus B \oplus B_1 \cong A_1 \oplus B \oplus B_1 \cong M \oplus B$$

Since M has the weak cancellation property, $B \preceq B_1$ and $B_1 \preceq B$, as desired. □

It is well-known that if $\mathrm{End}(M)$ is unit-regular, then M has the cancellation property.

From Proposition 3.10 and Corollary 3.6, we have the following result.

Corollary 3.11. *The following conditions are equivalent for a right R-module M.*

(1) $\mathrm{End}(M)$ *is unit-regular.*
(2) M *satisfies the virtually-IC property and* $\mathrm{End}(M)$ *is regular.*
(3) M *satisfies the virtually-C property and* $\mathrm{End}(M)$ *is regular.*

Given a cardinal $\aleph$, an R-module M is said to have the $\aleph$-*exchange property* if for any module X and decompositions $X = M' \oplus Y = \bigoplus_{i\in I} N_i$, where $M' \cong M$ and $|I| \leq \aleph$, there exist submodules $N_i' \subset N_i$ such that $X = M' \oplus (\bigoplus_{i\in I} N_i')$.

A module M has *the exchange property* if it has the $\aleph$-exchange property for every cardinal $\aleph$.

A module M has *the finite exchange property* if it has the $\aleph$-exchange property for every finite cardinal $\aleph$.

Proposition 3.12. *Assuming the finite exchange property, we have*

$$\textit{the virtually-C property} \Leftrightarrow \textit{the virtually-IC property.}$$

Proof. Let M be a module with the finite exchange property.

$(:\Rightarrow)$ This implication is Proposition 3.10.

$(\Leftarrow:)$ Assume that M satisfies the virtually-IC property. Let $M \oplus A = N \oplus B$ with $M \cong N$. From the finite exchange property, we infer that $M \oplus A = M \oplus N_1 \oplus B_1$ for some submodules N_1 of N and B_1 of B. It follows that $A \cong N_1 \oplus B_1$. Write $N = N_1 \oplus N_2$ and $B = B_1 \oplus B_2$ for some submodules N_2 of N and B_2 of B. Therefore, we have

$$M \oplus N_1 \oplus B_1 = M \oplus A = N \oplus B = N_2 \oplus B_2 \oplus N_1 \oplus B_1$$

It follows that $M \cong N_2 \oplus B_2$ and so $N_2 \oplus B_2 \cong M \cong N = N_1 \oplus N_2$. Since M satisfies the virtually-IC property, we obtain $B_2 \hookrightarrow N_1$ and $N_1 \hookrightarrow B_2$. Thus, we have

$$A \cong N_1 \oplus B_1 \hookrightarrow B_2 \oplus B_1 = B$$

$$B = B_1 \oplus B_2 \hookrightarrow B_1 \oplus N_1 \cong A$$

which shows that M satisfies the virtually-C property. □

Theorem 3.13. *For a module M, we have*

$$\textit{the property } (P) \;\Rightarrow\; \textit{the virtually-C property},$$

if M is either injective or satisfies the finite exchange property.

Proof. Recall that if a module M satisfies the property (P), then M is Directly-finite (i.e. $\mathrm{End}(M)$ is Dedekind-finite) by Corollary 2.6.

If M is injective, then M has the cancellation property by [18, Theorem 1.29] and hence M has the virtually-C property.

Now suppose that M has the finite exchange property. Since M satisfies the virtually-IC property, it satisfies the virtually-C property by Proposition 3.12. □

The following theorem characterizes the (unit-)regularity of the endomorphism ring a right R-module in terms of the virtually-C property in view of Corollary 3.11.

Theorem 3.14. *The following conditions are equivalent for a right R-module M with $E := \mathrm{End}(M)$:*

(1) *E is unit-regular.*
(2) *E is regular and $(\gamma - \gamma^2)(M)$ satisfies the virtually-C property for any $\gamma \in E$.*
(3) *E is regular and $M/[\mathrm{im}(\gamma) + \ker(\gamma)]$ is embeddable into $\mathrm{im}(\gamma) \cap \ker(\gamma)$ for any $\gamma \in E$.*

Proof. Assume that E is regular. For any $\gamma \in E$,

$$\mathrm{im}(\gamma), \ker(\gamma), \mathrm{im}(\gamma) \cap \ker(\gamma) \text{ and } \mathrm{im}(\gamma) + \ker(\gamma)$$

are direct summands of M by [1, Theorem 2.7]. Call A and B submodules of M such that

$$[\mathrm{im}(\gamma) \cap \ker(\gamma)] \oplus A = \ker(\gamma)$$
$$\mathrm{im}(\gamma) \oplus A = \mathrm{im}(\gamma) + \ker(\gamma)$$
$$\mathrm{im}(\gamma) + \ker(\gamma) \oplus B = M.$$

On the other hand, we have $M = \ker(\gamma) \oplus \ker(1-\gamma) \oplus C$ for some submodule C of M. It is straightforward to verify that

$$\gamma(\ker(1-\gamma)) = \ker(1-\gamma), C \cong \gamma(1-\gamma)(C) = \gamma(1-\gamma)(M) \text{ and } C \cong \gamma(C).$$

Hence,

$$\begin{aligned}
[\operatorname{im}(\gamma)\cap\ker(\gamma)]\oplus A\oplus\ker(1-\gamma)\oplus C &= \ker(\gamma)\oplus\ker(1-\gamma)\oplus C\\
&= M=\operatorname{im}(\gamma)\oplus A\oplus B\\
&= \ker(1-\gamma)\oplus C\oplus A\oplus B\\
&= \ker(1-\gamma)\oplus C\oplus A\oplus B,
\end{aligned}$$

i.e. $[\operatorname{im}(\gamma)\cap\ker(\gamma)]\oplus C\cong\gamma(C)\oplus B\cong C\oplus M/[\operatorname{im}(\gamma)+\ker(\gamma)]$.

(1) $\Rightarrow$ (2) Assume that E is unit-regular. Let $\gamma\in E$ (and so $(\gamma-\gamma^2)(M)$ is a direct summand of M). One can check that $\operatorname{End}((\gamma-\gamma^2)(M))$ is unit-regular. Consequently, $(\gamma-\gamma^2)(M)$ satisfies the virtually-C property by Corollary 3.11.
(2) $\Rightarrow$ (3) Note that $[\operatorname{im}(\gamma)\cap\ker(\gamma)]\oplus C\cong C\oplus M/[\operatorname{im}(\gamma)+\ker(\gamma)]$ and $C\cong(\gamma-\gamma^2)(M)$. By (2), we obtain that $M/[\operatorname{im}(\gamma)+\ker(\gamma)]$ is embedded in $\operatorname{im}(\gamma)\cap\ker(\gamma)$.
(3) $\Rightarrow$ (1) Let γ be an arbitrary endomorphism of M. Then,

$$M/\operatorname{im}(\gamma)\cong A\oplus B\cong A\oplus M/[\operatorname{im}(\gamma)+\ker(\gamma)]\hookrightarrow A\oplus[\operatorname{im}(\gamma)\cap\ker(\gamma)]=\ker(\gamma),$$

which implies M satisfies the property (P). By Theorem 2.11, E is unit-regular. $\square$

4. On more rings satisfying the property (P)

A ring R is said to be right *semi-artinian* if every non-zero right R-module has a non-zero socle.

Lemma 4.1. *If R_R satisfies the property (P) and S is a simple submodule of R_R, then there exists principal maximal right ideal mR such that $S\cong R/mR$.*

Proof. Since S is a nonzero principal right ideal, there exists $m\in R\backslash R^*$ such that $0\neq R/mR$ embeds into S by Lemma 2.4. As S is simple, we get $R/mR\cong S$, and so mR is a maximal right ideal. $\square$

An epimorphism $\pi:P\to M$ is said to be a *projective cover* of M provided P is projective and the kernel $\ker\pi\ll P$, and a projective module is *semiperfect* if every homomorphic image has a projective cover.

Lemma 4.2. *Let R be a semiperfect ring such that R_R satisfies the property (P) and $s\in R$. If sR is simple, then $r_R(s)$ is a right principal ideal.*

Proof. Since R is semiperfect, there exists a projective cover $\pi : P \to sR$. Moreover, by Lemma 4.1 there exists $m \in R$ and an epimorphism $\rho : R \to sR$ such that mR is the kernel of ρ. Since ρ factorizes through π, the kernel of π is a homomorphic image of mR, so it is a cyclic submodule. If $\tau_s : R \to sR$ is defined by $\tau_s(r) = sr$, then it factors through π again which implies that the kernel of τ_s is $mR \oplus Q$ where $R \cong P \oplus Q$, hence it is a principal right ideal. □

A ring R is said to be *right Kasch* if $l_R(I) \neq 0$ for every maximal ideal I of R.

Lemma 4.3. *Let R be a semiperfect right Kash ring satisfying the property (P). Then the following statements hold:*

(1) *All maximal right ideals of a ring R are principal.*
(2) *Every maximal submodule of a cyclic R-module is cyclic.*
(3) *For each $c_0 \in R\backslash\{0\}$, there exists $\kappa \leq \omega$ and a strictly decreasing chain of principal ideals $(c_iR \mid i < \kappa)$ such that $c_iR/c_{i+1}R$ is simple and $c_{\kappa-1} = 0$ for κ finite.*

Proof. (1) Since a right Kasch ring contains as submodules copies of all simple modules, the claim follows from Lemmas 4.1 and 4.2.
(2) The claim follows from (1) as maximal submodules of cyclic modules are factors of maximal ideals.
(3) Constructing by induction, we have in a nonzero cyclic module, say c_iR, its cyclic maximal submodule $c_{i+1}R$ by (2). □

Recall that a ring in which the right and left ideals are linearly ordered by inclusion is called a *chain ring*. An artinian ring R is called *Frobenius* if as right R-modules $Soc(R_R) \cong R/J(R)$, and as left R-modules $Soc({}_RR) \cong R/J(R)$.

Since every local ring with nilpotent (or even left T-nilpotent) Jacobson radical is semiartinian, the following assertion generalizes the results [5, Lemma 9] and [28, Proposition 2.8].

Theorem 4.4. *The following conditions are equivalent for a local right semiartinian ring R:*

(1) *R_R satisfies the property (P).*
(2) *R is right morphic.*
(3) *R is right chain right artinian.*
(4) *R is Frobenius right chain.*

Proof. (1)⇒(3) Since any local ring is semiperfect and a semiartinian ring contains simple submodules, $J(R)$ is right principal by Lemmas 4.1 and 4.2. Furthermore, a local right semiartinian ring is right Kasch and it contains no infinite strictly decreasing chain of principal right ideals. Hence R_R is of finite length by Lemma 4.3. Finally, since $J(cR) = cJ(R)$ is a maximal ideal of cR for each nonzero $c \in R$, we obtain that each ideal $J(R)^n$ are right principal and superfluous in $J(R)^{n-1}$, which implies that R is a right chain ring.
(3)⇒(2) The implication follows from [5, Theorem 19].
(2)⇒(1) The implication is obvious.
(4)⇔(3) The implication is well known (see, e.g. [15]). □

A module M is called *uniserial* if the family of its submodules is linearly ordered under inclusion. A ring R is said to be serial if R_R as well as $_RR$ are finite direct sums of uniserial modules.

Corollary 4.5. *The following conditions are equivalent for a commutative semilocal semiartinian ring R:*

1. *R satisfies the property (P).*
2. *R is morphic.*
3. *R is an artinian uniserial ring.*

Proof. Since a commutative semilocal semiartinian ring has a T-nilpotent Jacobson radical by [22, Proposition 3.2], we obtain that it is semiperfect, and hence it is a finite product of local semiartinian rings. Now the assertion follows from Theorem 4.4. □

Proposition 4.6. *If R is a left perfect right Kasch ring such that R_R satisfies the property (P), then it is right artinian such that all right ideals are principal.*

Proof. Note that a left perfect ring is semiperfect and right semiartinian. Since a right semiartinian ring contains no infinite strictly decreasing chain of principal right ideals, we get that R is a right artinian ring by Lemma 4.3(3).

Now, using the argument of the proof of Theorem 4.4, assume that R_R contains a non-principal right ideal and suppose that I is a maximal such an ideal. Since $R/I \neq 0$ is semiartinian, there exists a right ideal K such that K/I is simple. By the maximality of choice of I, the ideal K is principal and I is its maximal submodule, which contradicts Lemma 4.3. □

If M is a finite length module, denote by *length*(M) the composition length of M.

Theorem 4.7. *Let R be a left perfect right Kasch ring. Then R_R satisfies the property (P) if and only if R is right morphic and all right ideals are principal.*

Proof. Let R_R satisfy the property (P) and $x \in R$. Note that R_R is of finite length by Proposition 4.6. Since R/xR embeds into $r_R(x)$ and $R/r_R(x) \cong xR$, we get that $length(R/xR) = length(R) - l(xR) = length(r_R(x))$, hence $R/xR \cong r_R(x)$.

The reverse implication is clear. □

5. On group rings which satisfy the property (P)

Given a ring R and a group G, we denote the group ring of G over R by RG. An arbitrary element of RG, say $\alpha \in RG$, is of the form $\alpha = \sum_{g\in G} r_g g$ where $r_g \in R$ and $\{g \in G|\ r_g \neq 0\}$ is finite.

The *augmentation mapping* is of the form:

$$\varepsilon : RG \longrightarrow R$$
$$\sum_g r_g g \longmapsto \sum_g r_g.$$

Theorem 5.1. *Let R be a ring and G be an arbitrary group. If RG_{RG} satisfies the property (P), then R_R satisfies the property (P) and G is a locally finite group.*

Proof. Assume that RG_{RG} satisfies the property (P). Corollary 2.5 infers that RG is left P-injective. Then, G is locally finite by [19, Theorem 4.1]. Next, we show that R_R satisfies the property (P). Let $a \in R$. Then there exists a $u \in RG$ such that $a(RG) = r_{RG}(u)$ and $u(RG) \leq r_{RG}(a)$. Take $u = \sum_{i=1}^{n} a_i g_i$ and $H =< g_1, g_2, \dots, g_n >$. Since G is a locally finite group, we get H is a finite group. Now $ua = 0 = au$ and hence $\varepsilon(u)a = \varepsilon(ua) = 0$ and $a\varepsilon(u) = \varepsilon(au) = 0$. Call $b := \varepsilon(u)$. Then we have the following cases $aR \leq r_R(b)$ and $bR \leq r_R(a)$.
Case: $aR \leq r_R(b)$: Then a is satisfies the property (P) and hence R_R satisfies the property (P).
Case: $bR \leq r_R(a)$: Let $y \in r_R(b)$. Then $by = 0$. Put $\hat{H} = \sum_{h\in H} h$. From $u \in RH$, we infer that $u\hat{H} = \hat{H}\varepsilon(u) = \hat{H}b$ and so $u\hat{H}y = \hat{H}by = 0$. This implies that $\hat{H}y \in r_{RG}(u) = a(RG)$. Write $\hat{H}y = \sum_g aa_g g$. Comparing the coefficients of the identity on both sides, we obtain that $y = aa_e \in aR$. This gives that $r_R(b) = aR$. Therefore, a is satisfies the property (P) and so is R. □

Corollary 5.2. *If $G = H \times K$, where H and K are subgroups of G, and RG_{RG} satisfies the property (P), then RH_{RH} and RK_{RK} satisfy the property (P).*

Proof. It is well-known that $RG = R(H\times K) \cong RH(K)$. By Theorem 5.1, RH_{RH} satisfies the property (P). Similarly RK_{RK} satisfies the property (P). □

A group G is called a *semidirect product* of H by K, denoted by $G = H \rtimes K$, if G contains subgroups H and K such that $H \rhd G$, $G = HK$ and $H \cap K = \{1\}$.

Theorem 5.3. *Let $G = H \rtimes K$ with $|H| < \infty$. If RG_{RG} satisfies the property (P), then RK_{RK} satisfies the property (P).*

Proof. Let a be an arbitrary element of RK. We show that a satisfies the property (P). Since a is an element of RG and RG_{RG} satisfies the property (P), there exists $u \in RG$ such that $a(RG) = r_{RG}(u)$ and $u(RG) \leq r_{RG}(a)$. Let $a = \sum_j a_j k_j$ with $a_j \in R$, $k_j \in K$ and $u = \sum_i u_i k_i$ where $u_i \in RH$, $k_i \in K$ (since $G = HK$, the expression of u is unique). Denote $b = \sum_i \varepsilon(u_i)k_i$, and so $b \in RK$. We show that $a(RK) = r_{RK}(b)$ and $b(RK) \leq r_{RK}(a)$.

Let $\omega : G \to G/H$ be the natural group homomorphism. We extend to a ring homomorphism (still denote it by ω).

$$\begin{aligned} \omega : RG &\longrightarrow R(G/H) \\ \sum_g r_g g &\longmapsto \sum_g r_g \omega(g) \end{aligned}$$

One can check that $Ker(\omega) \cap RK = 0$ and $\omega(v) = \varepsilon(v)$ for all $v \in RH$. Since $ua = 0$, we have

$$\begin{aligned} 0 = \omega(ua) = \omega(u)\omega(a) &= \omega(\textstyle\sum_i u_i k_i)\omega(a) \\ &= (\textstyle\sum_i \omega(u_i)\omega(k_i))\omega(a) \\ &= \omega(\textstyle\sum_i \omega(u_i)k_i)\omega(a) \\ &= \omega(b)\omega(a) = \omega(ba) \end{aligned}$$

Since $ba \in RK$, we conclude that $ba = 0$. Similarly, $ab = 0$. This shows that $a(Rk) \leq r_{RK}(b)$ and $b(Rk) \leq r_{RK}(a)$. Let $y \in r_{RK}(b)$. Then $by = 0$. We have that $H \rhd G$, $\hat{H} = \sum_{h\in H} h$ is central in RG. We now have

$$u\hat{H}y = \sum_i u_i k_i \hat{H} y = (\sum_i u_i k_i \hat{H})y = \sum_i \varepsilon(u_i)k_i\hat{H}y = \hat{H}by = 0.$$

It implies that $\hat{H}y \in r_{RG}(u) = a(RG)$ and so $\hat{H}y = y\hat{H} = aw$ with $w = \sum_j h_j u_j$, $h_j \in H$ and $u_j \in RK$. We have $y\sum_j h_j = y\hat{H} = aw = \sum_j h_j(au_j)$. Since $H \cap K = \{1\}$, the expression of aw is unique. Comparing the coefficients of the identity $h_0 = e$, we obtain $y = au_0 \in a(RK)$. It is shown that $r_{RK}(b) \leq a(RK)$ and so $r_{RK}(b) = a(RK)$.

Example 5.4. (1) Let K be a field and G a torsion abelian group. Then KG is commutative P-injective by [19, Corollary 4.1], hence it satisfies the property (P) by Lemma 2.4.

(2) Let K be a field and G be a locally finite group containing no element of order 2 such that all units of KG are of the form ag for $a \in K^*$ and $g \in G$. Then KG is reduced by [14, Proposition 6.21] and left P-injective by [19, Corollary 4.1]. Then for each $a \in KG$ there exists b such that $r(b) = aR$. Since KG is reduced and $(ab)^2 = 0$ we have $b \in r(a)$, hence KG satisfies the property (P) again by Lemma 2.4.

□

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