11 Permutation groups and powers of elements

11.1. Let $\mathcal{G} = (G, \cdot, ', 1)$ be a group, $a, b \in G$ and $k, l \in \mathbb{Z}$. Prove that

(a) $a^{k+l} = a^k \cdot a^l$,

(b) $a^{kl} = (a^k)^l = (a^l)^k$,

(c) if $\mathcal{G}$ is Abelian, then $(ab)^k = a^k b^k$.

11.2. Write the following permutations as the product of independent cycles and for each permutation σ determine σ^{-1} and σ^{2020}:

(a) $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix} \in \mathbf{S}_5$,

(b) $\tau = (4\,6\,5\,1\,2) \in \mathbf{S}_6$,

(c) $\sigma = (1\,5\,6)(2\,3\,8\,4\,7) \in \mathbf{S}_8$,

(d) $\rho = (4\,3\,5) \circ (5\,1\,2) \in \mathbf{S}_5$.

Solutions: (a) $\pi = (1\,4)(2\,3\,5)$, $\pi^{-1} = (1\,4)(2\,5\,3)$, $\pi^{2020} = (2\,3\,5)$,
(b) $\tau = (4\,6\,5\,1\,2)$, $\tau^{-1} = (2\,1\,5\,6\,4)$, $\tau^{2020} = \mathrm{id}$,
(c) $\sigma = (1\,5\,6)(2\,3\,8\,4\,7)$, $\sigma^{-1} = (6\,5\,1)(7\,4\,8\,3\,2)$, $\sigma^{2020} = (1\,5\,6)$,
(d) $\rho = (1\,2\,4\,3\,5)$, $\rho^{-1} = (5\,3\,4\,2\,1)$, $\rho^{2020} = \mathrm{id}$.

11.3. Show that if $\pi(a) = b$ and $\sigma = \tau\pi\tau^{-1}$, then $\tau(b)$ is right after the element $\tau(a)$, i.e. $\sigma(\tau(a)) = \tau(b)$, and determine $\pi\tau\pi^{-1}$ and $\tau\pi\tau^{-1}$ for the permutations π and τ from example 11.2.

Solutions: $\pi\tau\pi^{-1} = (1\,6\,2\,4\,3)$, $\tau\pi\tau^{-1} = (1\,4\,3)(2\,6)$

Recall that the operation $\pi^\tau = \tau\pi\tau^{-1}$ is called *conjugation of the permutation π by the permutation τ* and the permutation π^τ is then *conjugated* with π.

11.4. Prove that the conjugation relation is an equivalence.

11.5. Find all permutations of α on the set $\{1, 2, 3, 4\}$ for which $\alpha \circ (1\,2\,3) \circ \alpha^{-1} = (1\,2\,4)$.

Solutions: $(3\,4)$, $(1\,2\,4\,3)$, $(1\,4\,3\,2)$

11.6. Determine the order of the following elements:

(a) 4 and 15 in $\mathbb{Z}_{75}$,

(b) 7 and 9 in $\mathbb{Z}_{20}^*$,

(c) 4 and 15 in $\mathbb{Z}$,

(d) $(1\,2\,3\,4)(5\,6\,7)(8\,9)$, $(1\,2)(5\,6\,8\,9)$ in $\mathbf{S}_9$ and $\mathbf{A}_{2020}$.

Solutions: (a) $\mathrm{ord}(4) = n = 75$, $\mathrm{ord}(15) = m = 5$, *(b)* $\mathrm{ord}(7) = 4$ *a že* $\mathrm{ord}(9) = 2$
(c) ∞, *(d)* $\mathrm{ord}((1\,2\,3\,4)(5\,6\,7)(8\,9)) = 12$, $\mathrm{ord}((1\,2)(5\,6\,8\,9)) = 4$.