Tasks for the CFF course

1. Tests

1.1. Algebras over a field.

1. If $A \leq B$ are subspaces of a vector space V, what is the relations of sets V^*, A^o a B^o?

1.2. Algebraic function fields.

2. If R is a domain with the field of fractions K, what is the relations of linear independence over R and over K?

3. Define an algebraic function field.

4. Give two non-isomorphic examples of an algebraic function field over $\mathbb{F}_5$.

5. Define a field of constant $\tilde{K}$ of an AFF L over K. What is the relation of $\tilde{K}$ and K?

6. Characterize transcendental elements of algebraic function fields by degree of extensions.

1.3. Valuation rings.

7. Define a valuation ring and describe a non-trivial example of it.

8. Describe relations between maximality of subring of a field and valuation rings.

9. Describe elements of noetherian local domain with a principal maximal ideal.

1.4. Discrete valuation rings.

10. Describe a discrete valuation ring by property of ideals.

11. Write an example of a normalized discrete valuation on a fraction field $K(x)$.

12. Define a discrete valuation ring and describe a non-trivial example of it.

13. Describe all discrete valuation on an AFF $K(x)$ over K.

14. Define a notion of a place of an AFF and the degree of a place.

15. Describe a notion of discrete valuation.

16. What is the relation between places and a corresponding discrete valuation rings?

17. What is the relation between places and a corresponding discrete valuations?

1.5. Weierstrass equations.

18. What does it mean that two WEPs are K-equivalent?

19. What is a Weierstrass equation polynomial?

20. Let $w(x, y) \in \mathbb{C}[x, y]$ be a WEP. Are $w(x, y)$ and $w(x, 1 - y)$ $\mathbb{C}$-equivalent? Explain your answer.

21. Are $\mathbb{F}_5$-equivalent polynomials $y^2 - x^3$ and $y^2 - (x^3 + 1) \in \mathbb{F}_5[x, y]$? Explain your answer.

22. Write one equivalent condition of K-equivalence of WEPs.

1.6. Singularities.

23. Define singular and smooth points and explain how they can be transformed using affine homomorphisms.

24. Define a notion of a tangent and singular and smooth points.

25. Find all singular points of a Weierstrass equation polynomial $y^2 - (x - 2)^3$ over $\mathbb{R}$.

1.7. **Coordinate rings.**

26. Find prime ideals $0 \neq P \subsetneq Q$ of the ring $\mathbb{C}[x,y]$ such that $P \subseteq (x^2 - y^2) \subseteq Q$.

27. Describe all prime ideals of a domain $K[x,y]$ for a field K.

28. What does it mean that an AFF L is over K given by $w(\alpha,\beta) = 0$?

1.8. **Absolutely irreducible polynomials.**

29. What is an absolutely irreducible polynomial? Which WEPs are absolutely irreducible?

30. Describe all algebraic elements of an AFF L over K pro a WEP.

1.9. **Places determined by a pair.**

31. What is m-weighted multiplicity μ of a polynomial $a(x,y)$?

32. For $w = yg(x,y) + h(x) + y \in K[x,y]$ where $h \in K[x]$, $g \in K[x,y]$, $m := \text{mult}(h) \geq 2$, $\text{mult}(g) \geq 1$, and L is an AFF over K given by $w(\alpha,\beta) = 0$ formulate the assertion describing places containing α a β.

33. Let f be smooth at $\gamma = (\gamma_1, \gamma_2) \in V_f(K)$, L be an AFF given by $f(\alpha,\beta) = 0$ and $l = l_0 + l_1 x + l_2 y \in K[x,y]$. Formulate the existence theorem about places P for which $\nu_P(\alpha - \gamma_1) > 0$ $\nu_P(\beta - \gamma_2) > 0$. How to compute $\nu_P(l(\alpha,\beta))$?

1.10. **Localization in a coordinate ring.**

34. Describe the sets P_γ or/and $\mathcal{O}_\gamma$ for points γ.

35. For which points γ is a ring $\mathcal{O}_\gamma$ valuation?

36. For which points γ and places P are $\mathcal{O}_\gamma$ and $\mathcal{O}_P$ valuation?

37. Describe all places corresponding to smooth rational points.

1.11. **Weak Approximation Theorem.**

38. Formulate The Weak Approximation Theorem.

39. Write an example of AFF such that $\mathbb{P}_{L/K}$ is infinite and $\mathbb{P}^{(1)}_{L/K}$ is finite.

40. Describe all places of degree 1 over a smooth WEP.

1.12. **Divisors.**

41. Define the group of divisors.

42. What is the relation of the group of principal divisors over an AFF L over K the group L^*?

43. What is a principal divisor?

44. Compute a principal divisor (π) over a field $\mathbb{R}$.

45. Compute a negative part of a principal divisor $(x)_-$ of the AFF $K(x)$ over K.

46. Define a Riemann-Roch space $\mathcal{L}(A)$ of a divisor A and compute $\mathcal{L}(\underline{0})$ of an AFF over $\mathbb{C}$.

47. Formulate the assertion about the degree of a positive and negative part of a principal divisor.

48. What is the degree of a principal divisor?

49. Formulate Riemann theorem. and explain what is the genus.

50. Define genus of an AFF.

1.13. **Adèles and Weil differentials.**

51. Define the notion of adeles.

52. Define index of specialization.

53. What is a Weil differential?

54. What is a canonical divisor?

55. Characterize the dimension of a space of Weil differentials $\Omega_{L/K}$.

56. Characterize the dimension of a space of Weil differentials $\Omega_{L/K}(A)$ for a divisor A.

1.14. **Riemann-Roch Theorem.**

57. Formulate the Riemann-Roch Theorem (about relation of dimension of Riemann-Roch spaces and degrees of divisors).

58. Formulate the main consequence of the Riemann-Roch Theorem.

59. Determine $l(W)$, $\deg(W)$ a $i(W)$ for a canonical divisor W using the genus.

60. Write an example of an AFF of genus 0.

1.15. **Elliptic function fields.**

61. Define an elliptic function field.

62. Find an example of an elliptic function field.

63. Define the Picard group $P^0(L/K)$ of an AFF L over K.

64. Characterize elliptic function fields over a WEP w

65. Describe a group structure on a curve given by a WEP by divisors.

2. Types of computational tasks

1. Compute the field of constants $\tilde{\mathbb{R}}$ of the AFF $\mathbb{C}(x)$ over $\mathbb{R}$.

2. For a place P of an AFF L over $\mathbb{Q}$ and $a \in L$ satisfying $a \in P^4 \setminus P^5$ compute $\nu_P(a)$, $\nu_P(a^{-1}+3)$, $\nu_P(a^2+3a^3)$.

3. For the Weierstrass polynomial $w = y^2 - y - (x^3 + x^2 + 1) \in \mathbb{F}_5[x, y]$ find an $\mathbb{F}_5$-equivalent short WEP.

4. Find at least 3 maximal ideals in $\mathbb{R}[x, y]$ containing the WEP $w = y^2 - (x^3 + 1)$.

5. Decide whether is the WEP $y^2 - yx - (x^3 + 2) \in \mathbb{F}_5[x, y]$ smooth.

6. Find all singularities of the WEP $y^2 + y(2x + 1) - (x^3 + 2x^2 + 2x) \in \mathbb{F}_3[x, y]$.

7. If $\nu_P(a) = 3$ for a place P of an AFF, compute $\nu_P(a^3 + a)$ and $\nu_P(a^{-2} + a^{-1})$.

8. Find all places of the AFF $\mathbb{R}(x)$ over $\mathbb{R}$ containing $x^3 - 1$ and a place Q, for which $\nu_Q(x^3 - 1) < 0$.

9. Describe a principal divisor $(\alpha - \beta - 3)$ of an AFF L over $\mathbb{Q}$ given by $f(\alpha, \beta) = 0$, if you know that f is a smooth WEP and there is no $\gamma \in V_f(\mathbb{Q})$ satisfying $l(\gamma) = f(\gamma) = 0$ for $l(x, y) = x - y - 3$.

10. Let w be a smooth WEP and L be an AFF over $\mathbb{F}_{32}$ given by $w(\alpha, \beta) = 0$. Compute $\deg(\alpha\beta)$, $\deg(\alpha\beta)_+$, and $\deg(\alpha^{-1})_-$.

11. If $f = y^2 - (x^3 - x + 1) \in \mathbb{C}[x, y]$ a $A = 1P_{(1,1)} + 3P_{(0,1)} + 5P_{(-1,1)} - 8P_\infty$, explain, why is A well-defined divisor and compute $\deg(A)$. Is A principal?

12. Consider the AFF given by $f(\alpha, \beta) = 0$ for $f = y^2 + y + x^3 + 1 \in \mathbb{F}_2[x, y]$. Compute degrees of positive and negative parts of principal divisors $(\alpha + 1)$ and (α).

13. For $f = y^2 + 4x^3 + x^2 + 3 \in \mathbb{F}_5[x, y]$ compute the genus of an AFF over $\mathbb{F}_5$ given by $f(\alpha, \beta) = 0$.

14. If $w = y^2 - x^3 \in \mathbb{F}_5[x, y]$, describe $\mathbb{F}_5$-isomorphism of fields $\mathbb{F}_5(z) \to \mathbb{F}_5(V_w)$.

15. Decide whether $\mathbb{F}_2(V_w)$ is an EFF if $f = y^2 - (x^3 + x) \in \mathbb{F}_3[x, y]$.

3. Sketch of proofs of key results

2. Formulate and proof characterization transcendental elements of algebraic function fields by degree of extensions and roots.

The corresponding claim: 2.8

3. Formulate and proof the assertion about existence of valuation (over)rings.

The corresponding claim: 3.6

4. If L is an AFF over K, $P \in \mathbb{P}_{L/K}$, prove that $\mathcal{O}_P$ is a uniquely defined discrete valuation ring and that $\deg P$ is finite.

The corresponding claim: 4.8

5. Describe all prime ideals of a domain $K[x,y]$ and prove your assertion.

The corresponding claim: 7.4

6. Let $w = yg(x,y) + h(x) + y \in K[x,y]$ where $h \in K[x]$, $g \in K[x,y]$, $m := \mathrm{mult}(h) \geq 2$, $\mathrm{mult}(g) \geq 1$ and L be an AFF over K given by $w(\alpha,\beta) = 0$. Formulate and prove the assertion describing places containing α and β and the corresponding discrete valuation.

The corresponding claim: 9.5

7. For an AFF given by $w(\alpha,\beta) = 0$ a bod $(\gamma_1,\gamma_2) \in V_w(K)$ formulate and prove the assertion describing the valuation of $l_1\alpha + l_2\beta + l_0$ in place containing $\alpha - \gamma_1$ and $\beta - \gamma_2$

The corresponding claim: 9.7

8. Describe places of degree one corresponding to a WEP smooth at rational points and prove your assertion.

The corresponding claim: 10.7

9. Formulate and prove the Weak Approximation Theorem and its consequence about number of places.

Corresponding claims: 11.2, 11.3

10. Formulate and prove the theorem on the degree of the positive and negative part of a principal divisor.

Corresponding claim: 12.6

11. Formulate and prove Riemann theorem and explain what is the genus.

Corresponding claim: 12.10

12. Describe the structure of vector spaces of Weil differentials $\Omega_{L/K}$ and $\Omega_{L/K}(A)$ as subspaces of the space L.

Corresponding claim: 13.5

13. Formulate and prove the Riemann-Roch Theorem and its main consequence (about relation of dimension of Riemann-Roch spaces and degrees of divisors).

Corresponding claims: 14.1, 14.3

14. Characterize elliptic function fields by a property of WEP. Prove your assertion.

Corresponding claim: 15.4