CVIČENÍ 20.5.2020

$\boxed{1}$ Wallisova formule: $\lim\limits_{n\to\infty} \prod\limits_{k=1}^{n} \frac{2k}{2k-1}\cdot\frac{2k}{2k+1} = \frac{\pi}{2}$

$$\prod_{k=1}^{\infty} \frac{2k}{2k-1}\cdot\frac{2k}{2k+1} = \frac{\pi}{2}$$

$$I_n = \int_0^{\pi/2} \sin^n x \, dx$$

$n \geq 2$

$$\int \sin^n x \, dx = \int \sin^{n-1} x \cdot \sin x \, dx = -\sin^{n-1} x \cdot \cos x + \int (n-1)\sin^{n-2} x \cos^2 x \, dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \cdot (1-\sin^2 x) dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cdot \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

$$I_n = \int_0^{\pi/2} \sin^n x \, dx = \underbrace{\left[-\frac{1}{n} \sin^{n-1} x \cos x\right]_0^{\pi/2}}_{=0} + \frac{n-1}{n} \underbrace{\int_0^{\pi/2} \sin^{n-2} x \, dx}$$

$$I_n = \frac{n-1}{n} I_{n-2}, \quad I_0 = \int_0^{\pi/2} 1 \, dx = \frac{\pi}{2}, \quad I_1 = \int_0^{\pi/2} \sin x \, dx = \left[-\cos x\right]_0^{\pi/2} = 1$$

$$I_{2n} = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} = \frac{\pi}{2} \cdot \prod_{k=1}^{n} \frac{2k-1}{2k}$$

$$I_{2n+1} = \underbrace{1 \cdot \frac{2}{3} \cdot \frac{4}{5} \cdots}_{I_{2n-1}} \frac{2n}{2n+1} = \prod_{k=1}^{n} \frac{2k}{2k+1}$$

$$\frac{\pi}{2} = I_{2n} \cdot \prod_{k=1}^{n} \frac{2k}{2k-1} = \frac{I_{2n}}{I_{2n+1}} \underbrace{\prod_{k=1}^{n} \frac{2k}{2k-1} \cdot \frac{2k}{2k+1}}_{a_n}$$

Platí: $I_{2n+2} \leq I_{2n+1} \leq I_{2n}$ $\qquad \sin^n x \leq \sin^{n-1} x, \; x \in [0, \frac{\pi}{2}]$

$$I_n \leq I_{n-1}$$

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{I_{2n}}{I_{2n+2}} = \frac{I_{2n}}{\frac{2n+1}{2n+2} I_{2n}} = \frac{2n+2}{2n+1} \xrightarrow{n \to \infty} 1$$

$$\Big\downarrow n \to \infty$$

$$1$$

$$a_n = \frac{\pi}{2} \frac{I_{2n+1}}{I_{2n}} \longrightarrow \frac{\pi}{2}, \; n \to \infty$$

$\boxed{2}$ (zavedení logaritmu) $\qquad F(x) = \int_1^x \frac{1}{t}\, dt, \; x \in (0, \infty).$

- $\mathcal{D}(F) = (0, \infty)$ ✓
- F je rostoucí ✓
- $x, y \in (0, \infty)$: $\qquad F(xy) = F(x) + F(y)$ ✓

$$\int_1^{xy} \frac{1}{t}\, dt = \underbrace{\int_1^x \frac{1}{t}\, dt}_{F(x)} + \underbrace{\int_x^{xy} \frac{1}{t}\, dt} = \int_1^y \frac{1}{sx} x\, ds = \int_1^y \frac{1}{s}\, ds = F(y)$$

$$s = \frac{t}{x} \qquad ds = \frac{1}{x}\, dt$$

- $\lim\limits_{x \to 1} \frac{F(x)}{x-1} = \lim\limits_{x \to 1} \frac{F(x) - F(1)}{x-1} = 1$ ✓ $\qquad (F(1) = \int_1^1 \frac{1}{t}\, dt = 0)$

$F(1) = 0$

Diferenciální rovnice se separovanými proměnnými

$y = g(y)h(x)$ $\qquad g, h \ldots$ dané funkce

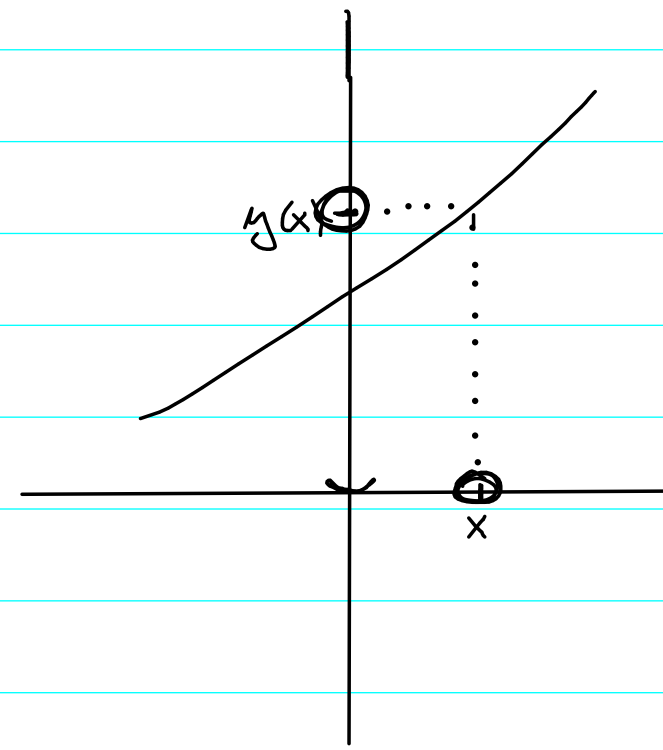
$y'(x) = g(y(x))\,h(x)$

$\boxed{3}$ $y' = y$ $\qquad g(y) = y$, $h(x) = 1$

$\mathcal{D}(h) = \mathbb{R}$

$\mathcal{D}(g) = \mathbb{R}$ $\qquad g(c) = 0 \Leftrightarrow c = 0$

- singulární řešení: $y(x) = 0$, $x \in \mathbb{R}$
- $y \in (0, \infty)$

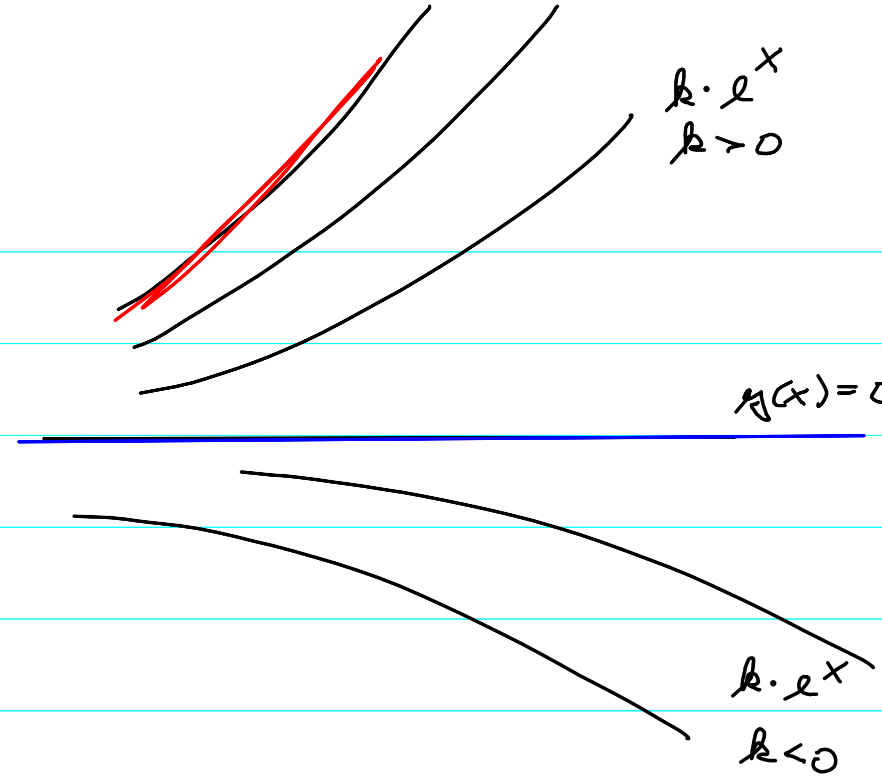


$$y'(x) = y(x)$$

$$\frac{y'(x)}{y(x)} = 1$$

$$\log y(x) = x + C$$

$$y(x) = \exp(x + C),\ x \in \mathbb{R}$$
$$= e^C \cdot e^x = k \cdot e^x$$



• $y \in (-\infty, 0)$

$$\log|y(x)| = x + c$$

$$\log(-y(x)) = x + c$$

$$y(x) = -\exp(x+c)$$

$$x \in \mathbb{R}$$

$$y(x) = k\cdot e^x, \; x \in \mathbb{R}, \; k \in \mathbb{R}$$