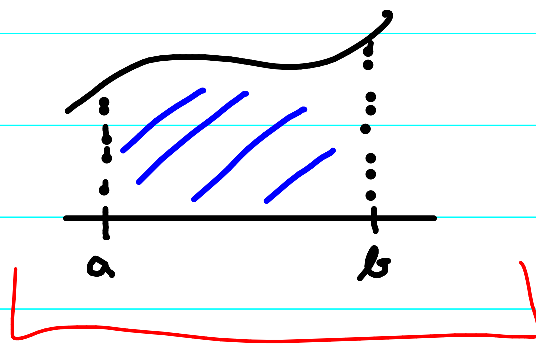
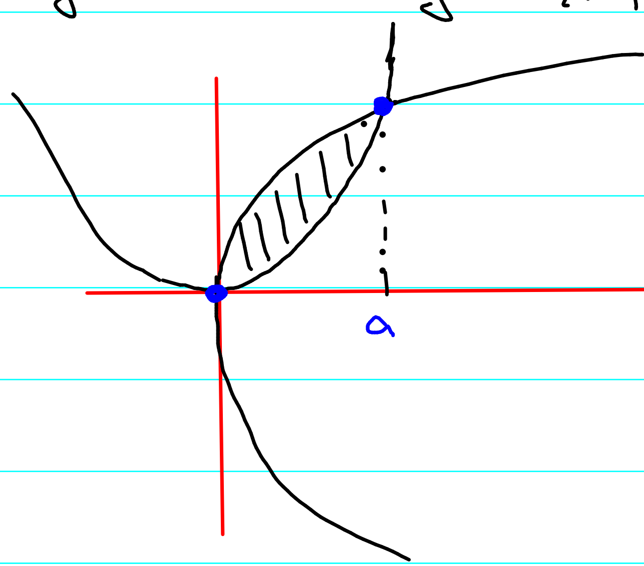
CVIČENÍ 15.5.2020

Obsah plochy:



$\boxed{1}$ Spočtěte obsah plochy vymezené křivkami $ax = y^2$, $ay = x^2$, $a > 0$; přesněji obsah množiny $M = \{[x, y] \in \mathbb{R}^2 : \sqrt{ax} \geq y \geq \frac{1}{a}x^2\}$.



$y = \sqrt{ax}$

$$ax = y^2 = \left(\frac{1}{a}x^2\right)^2$$

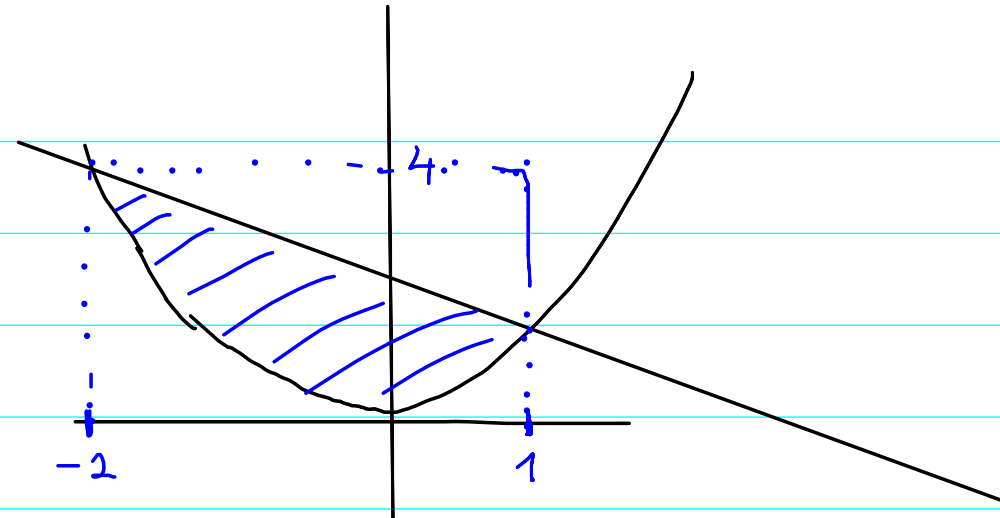
$$ax = \frac{1}{a^2}x^4 \Rightarrow \begin{cases} x = 0 \\ a^3 = x^3 \Rightarrow x = a \end{cases}$$

$$\int_0^a \left(\sqrt{ax} - \frac{1}{a}x^2\right)dx = \left[\sqrt{a}\,\frac{2}{3}\,x^{\frac{3}{2}} - \frac{1}{a}\,\frac{1}{3}\,x^3\right]_0^a = \sqrt{a}\,\frac{2}{3}a^{\frac{3}{2}} - \frac{1}{3}a^2$$

$$= \underline{\frac{1}{3}a^2}$$

$\boxed{2}$ $y=x^2,\ x+y=2$

$y=2-x$



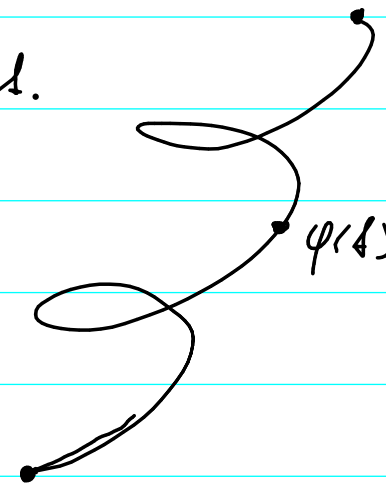
$$\int_{-2}^{1}(2-x-x^2)\,dx=\left[2x-\frac{1}{2}x^2-\frac{1}{3}x^3\right]_{-2}^{1}$$

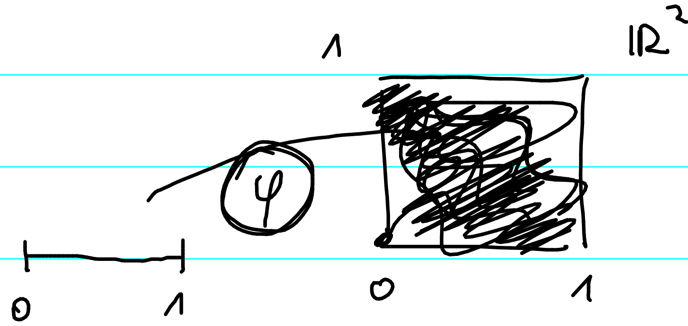
$$=\left(2-\frac{1}{2}-\frac{1}{3}\right)-\left(-4-2+\frac{8}{3}\right)=8-\frac{5}{6}-\frac{16}{6}=\frac{48-21}{6}=\frac{27}{6}=\frac{9}{2}$$

Délka křivky. $L(\varphi)=\int_a^b \|\varphi'(t)\|\,dt.$

$\varphi: [a,b] \to \mathbb{R}^m$

$t\in[a,b]$

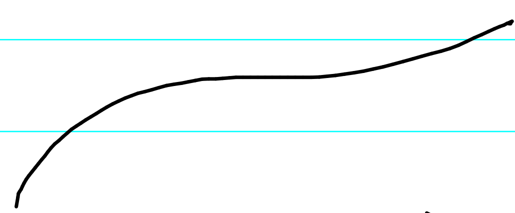




Peanova křivka: spojité $\varphi:[0,1]\to\mathbb{R}^2$ takové, že $\varphi([0,1]) = [0,1]\times[0,1]$

$\varphi([a,b])$

$\varphi:[a,b]\to\mathbb{R}^2$



$\varphi(t) = [\cos t, \sin t],\ t\in[0,2\pi]$

$\psi(t) = \text{—}\,''\,\text{—},\ t\in[0,4\pi]$

$\boxed{3}$ Vypočtěte délku křivky $y = x^{3/2}$, $x\in[0,4]$

$\varphi(t) = [t, t^{3/2}],\ t\in[0,4] \qquad \varphi'(t) = [1, \frac{3}{2}t^{\frac{1}{2}}]$

$\|\varphi'(t)\| = \left(1+\frac{9}{4}t\right)^{\frac{1}{2}} \qquad L(\varphi) = \int_0^4 \sqrt{1+\frac{9}{4}t}\,dt$

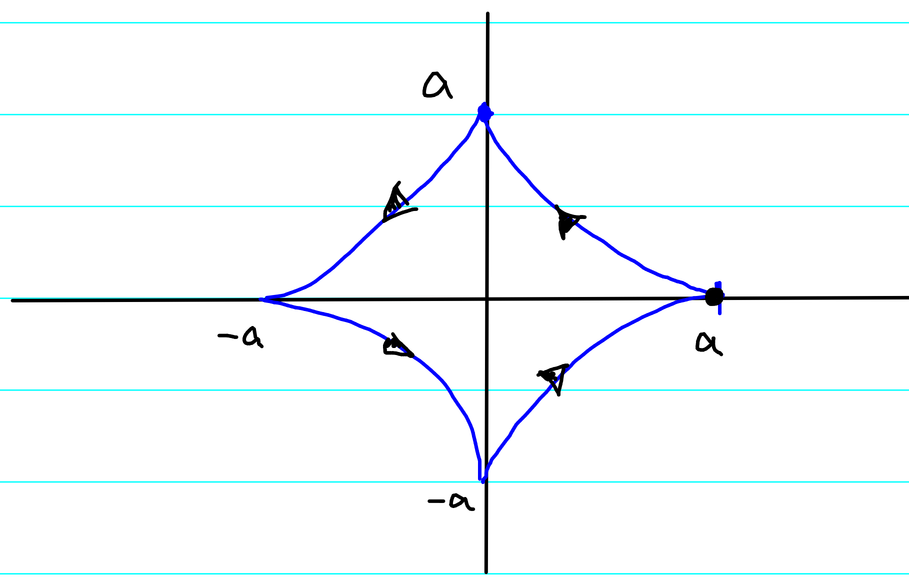
$\sqrt{1+\frac{9}{4}t} = r$

$t = (r^2-1)\frac{4}{9}$

$dt = \frac{8}{9} r\,dr$

$= \int_1^{\sqrt{10}} r\,\frac{8}{9} r\,dr = \left[\frac{8}{27} r^3\right]_1^{\sqrt{10}} = \underline{\frac{8}{27}(10\sqrt{10}-1)}$

$\boxed{4}$ Vypočtěte délku astroidy $\sqrt[3]{x^2}+\sqrt[3]{y^2}=\sqrt[3]{a^2}$, $a>0$.



$$\varphi(t)=[a\cos^3 t, a\sin^3 t];\ t\in[0,2\pi]$$

$$\varphi'(t)=[3a\cos^2 t\cdot(-\sin t),\ 3a\sin^2 t\cdot\cos t]$$

$$\|\varphi'(t)\|^2=9a^2\cos^4 t\sin^2 t+9a^2\sin^4 t\cdot\cos^2 t$$

$$=9a^2\cos^2 t\cdot\sin^2 t\,(\cos^2 t+\sin^2 t)=9a^2\cos^2 t\sin^2 t$$

$$L(\varphi)=\int_0^{2\pi}3a\,|\cos t\cdot\sin t|\,dt=4\int_0^{\pi/2}3a\cos t\cdot\sin t\,dt$$

$$=12a\left[-\frac{1}{2}\cos^2 t\right]_0^{\pi/2}=-12a\left(-\frac{1}{2}\cdot 1\right)=\underline{6a}$$