Mathematics I, test 1
WS 2017/2018

1) Find all real solutions of the inequality

$$\frac{x+1}{x-2} \geq \frac{2x+3}{4-3x}. \tag{1}$$

Solution: Clearly, $x-2 \neq 0, 4-3x \neq 0$. So, $x \notin \left\{\frac{4}{3}, 2\right\}$. We multiply the inequality by denominators $(x-2)(4-3x)$.

$$(x-2)(4-3x) \begin{cases} \geq 0: & x \in \left[\frac{4}{3}, 2\right], \\ \leq 0: & x \in \left(-\infty, \frac{4}{3}\right] \cup [2, +\infty). \end{cases}$$

Assume $x \in \left(\frac{4}{3}, 2\right)$. Then (1) is equivalent to:

$$\begin{aligned} (x+1)(4-3x) &\geq (2x+3)(x-2), \\ -3x^2+x+4) &\geq 2x^2-x-6, \\ -5x^2+2x+10 &\geq 0. \end{aligned} \tag{2}$$

Roots of $-5x^2+2x+10$ are $\left\{\frac{1-\sqrt{51}}{5}, \frac{1+\sqrt{51}}{5}\right\}$. Since the leading coefficient -5 is negative, the solution of (2) is

$$x \in \left[\frac{1-\sqrt{51}}{5}, \frac{1+\sqrt{51}}{5}\right].$$

Since we are interested only about $x \in \left(\frac{4}{3}, 2\right)$ and $\frac{1-\sqrt{51}}{5} < \frac{4}{3} < \frac{1+\sqrt{51}}{5} < 2$ we have

$$x \in \left(\frac{4}{3}, \frac{1+\sqrt{51}}{5}\right].$$

Now, assume $x \in \left(-\infty, \frac{4}{3}\right) \cup (2, +\infty)$. Then (1) is equivalent to:

$$\begin{aligned} (x+1)(4-3x) &\leq (2x+3)(x-2), \\ -3x^2+x+4) &\leq 2x^2-x-6, \\ -5x^2+2x+10 &\leq 0. \end{aligned} \tag{3}$$

Roots of $-5x^2+2x+10$ are $\left\{\frac{1-\sqrt{51}}{5}, \frac{1+\sqrt{51}}{5}\right\}$. Since the leading coefficient -5 is negative, the solution of (3) is

$$x \in \left(-\infty, \frac{1-\sqrt{51}}{5}\right] \cup \left[\frac{1+\sqrt{51}}{5}, +\infty\right).$$

Since we are interested only about $x \left(-\infty, \frac{4}{3}\right] \cup [2, +\infty)$ and $\frac{1-\sqrt{51}}{5} < \frac{4}{3} < \frac{1+\sqrt{51}}{5} < 2$ we have

$$x \in \left(-\infty, \frac{1-\sqrt{51}}{5}\right] \cup (2, +\infty).$$

Putting those two cases together, we obtain

$$x \in \left(-\infty, \frac{1-\sqrt{51}}{5}\right] \cup \left(\frac{4}{3}, \frac{1+\sqrt{51}}{5}\right] \cup (2, +\infty).$$

□

2) Find all real solutions of the inequality

$$||2x+3|-5| > x. \tag{4}$$

Solution: Clearly

$$2x+3\begin{cases}\geq 0: & x\in\left[-\frac{3}{2},+\infty\right),\\ \leq 0: & x\in\left(-\infty,-\frac{3}{2}\right].\end{cases}$$

First, assume $x \in \left[-\infty, -\frac{3}{2}\right]$. So, (4) is equivalent to

$$\begin{aligned} |-(2x+3)-5| &> x, \\ |-2x-8| &> x, \\ |2x+8| &> x. \end{aligned} \tag{5}$$

$$2x+8\begin{cases}\geq 0: & x\in[-4,+\infty),\\ \leq 0: & x\in(-\infty,-4].\end{cases}$$

Assume $x \in \left[-4, -\frac{3}{2}\right]$. By (5), we obtain

$$\begin{aligned} 2x+8 &> x, \\ x &> -8. \end{aligned}$$

The last inequality clearly holds for every $x \in \left[-4, -\frac{3}{2}\right]$.

Assume $x \leq -4$. By (5), we obtain

$$\begin{aligned} -2x-8 &> x, \\ -8 &> 3x, \\ -\frac{8}{3} &> x. \end{aligned}$$

The last inequality clearly holds for every $x \leq -4$.

Now assume $x \geq -\frac{3}{2}$. So, (4) is equivalent to

$$\begin{aligned} |2x+3-5| &> x, \\ |2x-2| &> x. \end{aligned} \tag{6}$$

$$2x-2\begin{cases}\geq 0: & x\geq 1,\\ \leq 0: & x\leq 1.\end{cases}$$

Assume $x \in \left[-\frac{3}{2}, 1\right]$. By (6), we obtain

$$\begin{aligned} -(2x-2) &> x, \\ -2x+2 &> x, \\ 2 &> 3x, \\ \frac{2}{3} &> x. \end{aligned}$$

The last inequality clearly holds for every $x \in \left[-\frac{3}{2}, \frac{2}{3}\right)$.

Assume $x \geq 1$. By (6), we obtain

$$\begin{aligned} 2x-2 &> x, \\ x &> 2. \end{aligned}$$

The last inequality clearly holds for every $x \in (2, +\infty)$.

Putting those four cases together, we obtain $x \in \left(-\infty, \frac{2}{3}\right) \cup (2, +\infty)$.

□