Mathematics I

Mathematics I

- Introduction

Mathematics I

- Introduction
- Limit of a sequence

Mathematics I

- Introduction
- Limit of a sequence
- Mappings

Mathematics I

- Introduction
- Limit of a sequence
- Mappings
- Functions of one real variable

Textbooks

Textbooks

- Trench: Introduction to real analysis

Textbooks

- Trench: Introduction to real analysis
- Ghorpade, Limaye: A course in calculus and real analysis

Textbooks

- Trench: Introduction to real analysis
- Ghorpade, Limaye: A course in calculus and real analysis
- Zorich: Mathematical analysis I

Textbooks

- Trench: Introduction to real analysis
- Ghorpade, Limaye: A course in calculus and real analysis
- Zorich: Mathematical analysis I
- Rudin: Principles of mathematical analysis

We take a set to be a collection of definite and distinguishable objects into a coherent whole.

We take a set to be a collection of definite and distinguishable objects into a coherent whole.

- $x \in A \ldots x$ is an element (or member) of the set A

We take a set to be a collection of definite and distinguishable objects into a coherent whole.

- $x \in A \ldots x$ is an element (or member) of the set A
- $x \notin A \ldots x$ is not a member of the set A

We take a set to be a collection of definite and distinguishable objects into a coherent whole.

- $x \in A \ldots x$ is an element (or member) of the set A
- $x \notin A \ldots x$ is not a member of the set A
- $A \subset B \ldots$ the set A is a subset of the set B (inclusion)

We take a set to be a collection of definite and distinguishable objects into a coherent whole.

- $x \in A \dots x$ is an element (or member) of the set A
- $x \notin A \dots x$ is not a member of the set A
- $A \subset B \dots$ the set A is a subset of the set B (inclusion)
- $A = B \dots$ the sets A and B have the same elements; the following holds: $A \subset B$ and $B \subset A$

We take a set to be a collection of definite and distinguishable objects into a coherent whole.

- $x \in A \ldots x$ is an element (or member) of the set A
- $x \notin A \ldots x$ is not a member of the set A
- $A \subset B \ldots$ the set A is a subset of the set B (inclusion)
- $A = B \ldots$ the sets A and B have the same elements; the following holds: $A \subset B$ and $B \subset A$
- $\emptyset \ldots$ an empty set

We take a set to be a collection of definite and distinguishable objects into a coherent whole.

- $x \in A \dots x$ is an element (or member) of the set A
- $x \notin A \dots x$ is not a member of the set A
- $A \subset B \dots$ the set A is a subset of the set B (inclusion)
- $A = B \dots$ the sets A and B have the same elements; the following holds: $A \subset B$ and $B \subset A$
- $\emptyset \dots$ an empty set
- $A \cup B \dots$ a union of the sets A and B

We take a set to be a collection of definite and distinguishable objects into a coherent whole.

- $x \in A \dots x$ is an element (or member) of the set A
- $x \notin A \dots x$ is not a member of the set A
- $A \subset B \dots$ the set A is a subset of the set B (inclusion)
- $A = B \dots$ the sets A and B have the same elements; the following holds: $A \subset B$ and $B \subset A$
- $\emptyset \dots$ an empty set
- $A \cup B \dots$ a union of the sets A and B
- $A \cap B \dots$ an intersection of the sets A and B

We take a set to be a collection of definite and distinguishable objects into a coherent whole.

- $x \in A \ldots x$ is an element (or member) of the set A
- $x \notin A \ldots x$ is not a member of the set A
- $A \subset B \ldots$ the set A is a subset of the set B (inclusion)
- $A = B \ldots$ the sets A and B have the same elements; the following holds: $A \subset B$ and $B \subset A$
- $\emptyset \ldots$ an empty set
- $A \cup B \ldots$ a union of the sets A and B
- $A \cap B \ldots$ an intersection of the sets A and B
- disjoint sets $\ldots A$ and B are disjoint if $A \cap B = \emptyset$

We take a set to be a collection of definite and distinguishable objects into a coherent whole.

- $x \in A \dots x$ is an element (or member) of the set A
- $x \notin A \dots x$ is not a member of the set A
- $A \subset B \dots$ the set A is a subset of the set B (inclusion)
- $A = B \dots$ the sets A and B have the same elements; the following holds: $A \subset B$ and $B \subset A$
- $\emptyset \dots$ an empty set
- $A \cup B \dots$ a union of the sets A and B
- $A \cap B \dots$ an intersection of the sets A and B
- disjoint sets $\dots A$ and B are disjoint if $A \cap B = \emptyset$
- $A \setminus B = \{x \in A;\ x \notin B\} \dots$ a difference of the sets A and B

We take a set to be a collection of definite and distinguishable objects into a coherent whole.

- $x \in A \ldots x$ is an element (or member) of the set A
- $x \notin A \ldots x$ is not a member of the set A
- $A \subset B \ldots$ the set A is a subset of the set B (inclusion)
- $A = B \ldots$ the sets A and B have the same elements; the following holds: $A \subset B$ and $B \subset A$
- $\emptyset \ldots$ an empty set
- $A \cup B \ldots$ a union of the sets A and B
- $A \cap B \ldots$ an intersection of the sets A and B
- disjoint sets $\ldots A$ and B are disjoint if $A \cap B = \emptyset$
- $A \setminus B = \{x \in A;\ x \notin B\} \ldots$ a difference of the sets A and B
- $A_1 \times \cdots \times A_m = \{[a_1, \ldots, a_m];\ a_1 \in A_1, \ldots, a_m \in A_m\}$ $\ldots$ a Cartesian product

Let I be a non-empty set of indices and suppose we have a system of sets A_α, where the indices α run over I.

Let I be a non-empty set of indices and suppose we have a system of sets A_α, where the indices α run over I.

- $\bigcup_{\alpha \in I} A_\alpha$ ... the set of all elements belonging to at least one of the sets A_α

Let I be a non-empty set of indices and suppose we have a system of sets A_α, where the indices α run over I.

- $\bigcup_{\alpha \in I} A_\alpha$ ... the set of all elements belonging to at least one of the sets A_α
- $\bigcap_{\alpha \in I} A_\alpha$ ... the set of all elements belonging to every A_α

Logic

Logic

A statement (or proposition) is a sentence which can be declared to be either true or false.

Logic

A statement (or proposition) is a sentence which can be declared to be either true or false.

- $\neg$, also non . . . negation

Logic

A statement (or proposition) is a sentence which can be declared to be either true or false.

- $\neg$, also non . . . negation
- & (also $\wedge$) . . . conjunction, logical “and”

Logic

A statement (or proposition) is a sentence which can be declared to be either true or false.

- $\neg$, also non . . . negation
- $\&$ (also $\wedge$) . . . conjunction, logical “and”
- $\vee$ . . . disjuction (alternative), logical “or”

Logic

A statement (or proposition) is a sentence which can be declared to be either true or false.

- $\neg$, also non . . . negation
- $\&$ (also $\wedge$) . . . conjunction, logical “and”
- $\vee$ . . . disjuction (alternative), logical “or”
- $\Rightarrow$ . . . implication

Logic

A statement (or proposition) is a sentence which can be declared to be either true or false.

- $\neg$, also non ... negation
- & (also $\wedge$) ... conjunction, logical “and”
- $\vee$ ... disjuction (alternative), logical “or”
- $\Rightarrow$ ... implication
- $\Leftrightarrow$ ... equivalence; “if and only if”

Tautology is a compound statement, which is true independently of the truthness of its elementary statements.

Tautology is a compound statement, which is true independently of the truthness of its elementary statements.
Examples of tautologies:

- $A \vee \neg A$

Tautology is a compound statement, which is true independently of the truthness of its elementary statements.
Examples of tautologies:

- $A \vee \neg A$
- $\neg(A \ \& \ \neg A)$

Tautology is a compound statement, which is true independently of the truthness of its elementary statements.
Examples of tautologies:

- $A \vee \neg A$
- $\neg(A \ \& \ \neg A)$
- $\big((A \ \& \ B) \ \& \ C\big) \Leftrightarrow \big(A \ \& \ (B \ \& \ C)\big)$

Tautology is a compound statement, which is true independently of the truthness of its elementary statements.
Examples of tautologies:

- $A \vee \neg A$
- $\neg(A \;\&\; \neg A)$
- $\big((A \;\&\; B) \;\&\; C\big) \Leftrightarrow \big(A \;\&\; (B \;\&\; C)\big)$
- $\neg(A \;\&\; B) \Leftrightarrow (\neg A \vee \neg B)$

Tautology is a compound statement, which is true independently of the truthness of its elementary statements.
Examples of tautologies:

- $A \vee \neg A$
- $\neg(A \ \& \ \neg A)$
- $\big((A \ \& \ B) \ \& \ C\big) \Leftrightarrow \big(A \ \& \ (B \ \& \ C)\big)$
- $\neg(A \ \& \ B) \Leftrightarrow (\neg A \vee \neg B)$
- $\neg(A \vee B) \Leftrightarrow (\neg A \ \& \ \neg B)$

Tautology is a compound statement, which is true independently of the truthness of its elementary statements.
Examples of tautologies:

- $A \vee \neg A$
- $\neg(A \ \& \ \neg A)$
- $\big((A \ \& \ B) \ \& \ C\big) \Leftrightarrow \big(A \ \& \ (B \ \& \ C)\big)$
- $\neg(A \ \& \ B) \Leftrightarrow (\neg A \vee \neg B)$
- $\neg(A \vee B) \Leftrightarrow (\neg A \ \& \ \neg B)$
- $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$

Tautology is a compound statement, which is true independently of the truthness of its elementary statements.
Examples of tautologies:

- $A \vee \neg A$
- $\neg(A \;\&\; \neg A)$
- $\big((A \;\&\; B) \;\&\; C\big) \Leftrightarrow \big(A \;\&\; (B \;\&\; C)\big)$
- $\neg(A \;\&\; B) \Leftrightarrow (\neg A \vee \neg B)$
- $\neg(A \vee B) \Leftrightarrow (\neg A \;\&\; \neg B)$
- $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$
- $\neg(A \Rightarrow B) \Leftrightarrow (A \;\&\; \neg B)$

Tautology is a compound statement, which is true independently of the truthness of its elementary statements.
Examples of tautologies:

- $A \vee \neg A$
- $\neg(A \;\&\; \neg A)$
- $\big((A \;\&\; B) \;\&\; C\big) \Leftrightarrow \big(A \;\&\; (B \;\&\; C)\big)$
- $\neg(A \;\&\; B) \Leftrightarrow (\neg A \vee \neg B)$
- $\neg(A \vee B) \Leftrightarrow (\neg A \;\&\; \neg B)$
- $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$
- $\neg(A \Rightarrow B) \Leftrightarrow (A \;\&\; \neg B)$
- $(A \Leftrightarrow B) \Leftrightarrow \big((A \Rightarrow B) \;\&\; (B \Rightarrow A)\big)$

Tautology is a compound statement, which is true independently of the truthness of its elementary statements.
Examples of tautologies:

- $A \vee \neg A$
- $\neg(A \;\&\; \neg A)$
- $\big((A \;\&\; B) \;\&\; C\big) \Leftrightarrow \big(A \;\&\; (B \;\&\; C)\big)$
- $\neg(A \;\&\; B) \Leftrightarrow (\neg A \vee \neg B)$
- $\neg(A \vee B) \Leftrightarrow (\neg A \;\&\; \neg B)$
- $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$
- $\neg(A \Rightarrow B) \Leftrightarrow (A \;\&\; \neg B)$
- $(A \Leftrightarrow B) \Leftrightarrow \big((A \Rightarrow B) \;\&\; (B \Rightarrow A)\big)$
- $(A \Rightarrow B) \Leftrightarrow (\neg A \vee B)$

A predicate (or propositional function) is an expression or sentence involving variables which becomes a statement once we substitute certain elements of a given set for the variables.

A predicate (or propositional function) is an expression or sentence involving variables which becomes a statement once we substitute certain elements of a given set for the variables.
General form:

$$V(x), x \in M$$

A predicate (or propositional function) is an expression or sentence involving variables which becomes a statement once we substitute certain elements of a given set for the variables.
General form:

$$V(x), x \in M$$

$$V(x_1, \ldots, x_n), x_1 \in M_1, \ldots, x_n \in M_n$$

If $A(x)$, $x \in M$ is a predicate, then the statement "$A(x)$ holds for every x from M." is shortened to

$$\forall x \in M \colon A(x).$$

If $A(x)$, $x \in M$ is a predicate, then the statement "$A(x)$ holds for every x from M." is shortened to

$$\forall x \in M\colon\ A(x).$$

The statement "There exists x in M such that $A(x)$ holds." is shortened to

$$\exists x \in M\colon\ A(x).$$

If $A(x)$, $x \in M$ is a predicate, then the statement "$A(x)$ holds for every x from M." is shortened to

$$\forall x \in M\colon\ A(x).$$

The statement "There exists x in M such that $A(x)$ holds." is shortened to

$$\exists x \in M\colon\ A(x).$$

The statement "There is only one x in M such that $A(x)$ holds." is shortened to

$$\exists! x \in M\colon\ A(x).$$

If $A(x)$, $x \in M$ is a predicate, then the statement "$A(x)$ holds for every x from M." is shortened to

$$\forall x \in M \colon A(x).$$

The statement "There exists x in M such that $A(x)$ holds." is shortened to

$$\exists x \in M \colon A(x).$$

The statement "There is only one x in M such that $A(x)$ holds." is shortened to

$$\exists ! x \in M \colon A(x).$$

If $A(x)$, $x \in M$ and $B(x)$, $x \in M$ are predicates, then

$$\forall x \in M, B(x) \colon A(x) \quad \text{means} \quad \forall x \in M \colon (B(x) \Rightarrow A(x)),$$

If $A(x)$, $x \in M$ is a predicate, then the statement "$A(x)$ holds for every x from M." is shortened to

$$\forall x \in M\colon\ A(x).$$

The statement "There exists x in M such that $A(x)$ holds." is shortened to

$$\exists x \in M\colon\ A(x).$$

The statement "There is only one x in M such that $A(x)$ holds." is shortened to

$$\exists! x \in M\colon\ A(x).$$

If $A(x)$, $x \in M$ and $B(x)$, $x \in M$ are predicates, then

$$\forall x \in M, B(x)\colon\ A(x) \quad \text{means} \quad \forall x \in M\colon\ (B(x) \Rightarrow A(x)),$$

$$\exists x \in M, B(x)\colon\ A(x) \quad \text{means} \quad \exists x \in M\colon\ (A(x)\ \ \&\ \ B(x)).$$

Negations of the statements with quantifiers:

$$\neg(\forall x \in M\colon A(x)) \quad \text{is the same as} \quad \exists x \in M\colon \neg A(x),$$

Negations of the statements with quantifiers:

$$\neg(\forall x \in M\colon A(x)) \quad \text{is the same as} \quad \exists x \in M\colon \neg A(x),$$

$$\neg(\exists x \in M\colon A(x)) \quad \text{is the same as} \quad \forall x \in M\colon \neg A(x).$$

Methods of proofs

Methods of proofs

- direct proof

Methods of proofs

- direct proof
- indirect proof

Methods of proofs

- direct proof
- indirect proof
- proof by contradiction

Methods of proofs

- direct proof
- indirect proof
- proof by contradiction
- mathematical induction

Theorem 1 (de Morgan rules)

Let S, A_α, $\alpha \in I$, where $I \neq \emptyset$, be sets. Then

$$S \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (S \setminus A_\alpha) \qquad \textit{and} \qquad S \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (S \setminus A_\alpha).$$

Theorem 2 (Cauchy inequality)

Let $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ be real numbers. Then

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right) \left(\sum_{i=1}^{n} b_i^2\right).$$

Example (irrationality of $\sqrt{2}$)

If a real number x solves the equation $x^2 = 2$, then x is not rational.

Rational numbers

- A set of natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

Rational numbers

- A set of natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

- A set of integers

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n;\ n \in \mathbb{N}\} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

Rational numbers

- A set of natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

- A set of integers

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n;\ n \in \mathbb{N}\} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

- A set of rational numbers

$$\mathbb{Q} = \left\{\frac{p}{q};\ p \in \mathbb{Z}, q \in \mathbb{N}\right\},$$

where $\frac{p_1}{q_1} = \frac{p_2}{q_2}$ if and only if $p_1 \cdot q_2 = p_2 \cdot q_1$.

Real numbers

By a set of real numbers $\mathbb{R}$ we will understand a set on which there are operations of addition and multiplication (denoted by $+$ and $\cdot$), and a relation of ordering (denoted by $\leq$), such that it has the following three groups of properties.

I. The properties of addition and multiplication and their relationships.
II. The relationships of the ordering and the operations of addition and multiplication.
III. The infimum axiom.

The properties of addition and multiplication and their relationships:

- $\forall x, y \in \mathbb{R}\colon\ x + y = y + x$ (commutativity of addition),

The properties of addition and multiplication and their relationships:

- $\forall x, y \in \mathbb{R}\colon\ x + y = y + x$ (commutativity of addition),
- $\forall x, y, z \in \mathbb{R}\colon\ x + (y + z) = (x + y) + z$ (associativity),

The properties of addition and multiplication and their relationships:

- $\forall x, y \in \mathbb{R}\colon x + y = y + x$ (commutativity of addition),
- $\forall x, y, z \in \mathbb{R}\colon x + (y + z) = (x + y) + z$ (associativity),
- There is an element in $\mathbb{R}$ (denoted by 0 and called a zero element), such that $x + 0 = x$ for every $x \in \mathbb{R}$,

The properties of addition and multiplication and their relationships:

- $\forall x, y \in \mathbb{R}\colon x + y = y + x$ (commutativity of addition),
- $\forall x, y, z \in \mathbb{R}\colon x + (y + z) = (x + y) + z$ (associativity),
- There is an element in $\mathbb{R}$ (denoted by 0 and called a zero element), such that $x + 0 = x$ for every $x \in \mathbb{R}$,
- $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\colon x + y = 0$ (y is called the negative of x, such y is only one, denoted by $-x$),

The properties of addition and multiplication and their relationships:

- $\forall x, y \in \mathbb{R}\colon x + y = y + x$ (commutativity of addition),
- $\forall x, y, z \in \mathbb{R}\colon x + (y + z) = (x + y) + z$ (associativity),
- There is an element in $\mathbb{R}$ (denoted by 0 and called a zero element), such that $x + 0 = x$ for every $x \in \mathbb{R}$,
- $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\colon x + y = 0$ (y is called the negative of x, such y is only one, denoted by $-x$),
- $\forall x, y \in \mathbb{R}\colon x \cdot y = y \cdot x$ (commutativity),

The properties of addition and multiplication and their relationships:

- $\forall x, y \in \mathbb{R}\colon x + y = y + x$ (commutativity of addition),
- $\forall x, y, z \in \mathbb{R}\colon x + (y + z) = (x + y) + z$ (associativity),
- There is an element in $\mathbb{R}$ (denoted by 0 and called a zero element), such that $x + 0 = x$ for every $x \in \mathbb{R}$,
- $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\colon x + y = 0$ (y is called the negative of x, such y is only one, denoted by $-x$),
- $\forall x, y \in \mathbb{R}\colon x \cdot y = y \cdot x$ (commutativity),
- $\forall x, y, z \in \mathbb{R}\colon x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associativity),

The properties of addition and multiplication and their relationships:

- $\forall x, y \in \mathbb{R}\colon x + y = y + x$ (commutativity of addition),
- $\forall x, y, z \in \mathbb{R}\colon x + (y + z) = (x + y) + z$ (associativity),
- There is an element in $\mathbb{R}$ (denoted by 0 and called a zero element), such that $x + 0 = x$ for every $x \in \mathbb{R}$,
- $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\colon x + y = 0$ (y is called the negative of x, such y is only one, denoted by $-x$),
- $\forall x, y \in \mathbb{R}\colon x \cdot y = y \cdot x$ (commutativity),
- $\forall x, y, z \in \mathbb{R}\colon x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associativity),
- There is a non-zero element in $\mathbb{R}$ (called identity, denoted by 1), such that $1 \cdot x = x$ for every $x \in \mathbb{R}$,

The properties of addition and multiplication and their relationships:

- $\forall x, y \in \mathbb{R}\colon x + y = y + x$ (commutativity of addition),
- $\forall x, y, z \in \mathbb{R}\colon x + (y + z) = (x + y) + z$ (associativity),
- There is an element in $\mathbb{R}$ (denoted by 0 and called a zero element), such that $x + 0 = x$ for every $x \in \mathbb{R}$,
- $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\colon x + y = 0$ (y is called the negative of x, such y is only one, denoted by $-x$),
- $\forall x, y \in \mathbb{R}\colon x \cdot y = y \cdot x$ (commutativity),
- $\forall x, y, z \in \mathbb{R}\colon x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associativity),
- There is a non-zero element in $\mathbb{R}$ (called identity, denoted by 1), such that $1 \cdot x = x$ for every $x \in \mathbb{R}$,
- $\forall x \in \mathbb{R} \setminus \{0\}\ \exists y \in \mathbb{R}\colon x \cdot y = 1$ (such y is only one, denoted by x^{-1} or $\frac{1}{x}$),

The properties of addition and multiplication and their relationships:

- $\forall x, y \in \mathbb{R}\colon x + y = y + x$ (commutativity of addition),
- $\forall x, y, z \in \mathbb{R}\colon x + (y + z) = (x + y) + z$ (associativity),
- There is an element in $\mathbb{R}$ (denoted by 0 and called a zero element), such that $x + 0 = x$ for every $x \in \mathbb{R}$,
- $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\colon x + y = 0$ (y is called the negative of x, such y is only one, denoted by $-x$),
- $\forall x, y \in \mathbb{R}\colon x \cdot y = y \cdot x$ (commutativity),
- $\forall x, y, z \in \mathbb{R}\colon x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associativity),
- There is a non-zero element in $\mathbb{R}$ (called identity, denoted by 1), such that $1 \cdot x = x$ for every $x \in \mathbb{R}$,
- $\forall x \in \mathbb{R} \setminus \{0\}\ \exists y \in \mathbb{R}\colon x \cdot y = 1$ (such y is only one, denoted by x^{-1} or $\frac{1}{x}$),
- $\forall x, y, z \in \mathbb{R}\colon (x + y) \cdot z = x \cdot z + y \cdot z$ (distributivity).

The relationships of the ordering and the operations of addition and multiplication:

- $\forall x, y, z \in \mathbb{R}\colon (x \leq y \ \ \& \ \ y \leq z) \Rightarrow x \leq z$ (transitivity),

The relationships of the ordering and the operations of addition and multiplication:

- $\forall x, y, z \in \mathbb{R}\colon (x \le y \ \ \& \ \ y \le z) \Rightarrow x \le z$ (transitivity),
- $\forall x, y \in \mathbb{R}\colon (x \le y \ \ \& \ \ y \le x) \Rightarrow x = y$ (weak antisymmetry),

The relationships of the ordering and the operations of addition and multiplication:

- $\forall x, y, z \in \mathbb{R}\colon (x \leq y \ \ \& \ \ y \leq z) \Rightarrow x \leq z$ (transitivity),
- $\forall x, y \in \mathbb{R}\colon (x \leq y \ \ \& \ \ y \leq x) \Rightarrow x = y$ (weak antisymmetry),
- $\forall x, y \in \mathbb{R}\colon x \leq y \vee y \leq x$,

The relationships of the ordering and the operations of addition and multiplication:

- $\forall x, y, z \in \mathbb{R}\colon (x \le y \ \ \& \ \ y \le z) \Rightarrow x \le z$ (transitivity),
- $\forall x, y \in \mathbb{R}\colon (x \le y \ \ \& \ \ y \le x) \Rightarrow x = y$ (weak antisymmetry),
- $\forall x, y \in \mathbb{R}\colon x \le y \vee y \le x$,
- $\forall x, y, z \in \mathbb{R}\colon x \le y \Rightarrow x + z \le y + z$,

The relationships of the ordering and the operations of addition and multiplication:

- $\forall x, y, z \in \mathbb{R}$: $(x \leq y \ \ \& \ \ y \leq z) \Rightarrow x \leq z$ (transitivity),
- $\forall x, y \in \mathbb{R}$: $(x \leq y \ \ \& \ \ y \leq x) \Rightarrow x = y$ (weak antisymmetry),
- $\forall x, y \in \mathbb{R}$: $x \leq y \vee y \leq x$,
- $\forall x, y, z \in \mathbb{R}$: $x \leq y \Rightarrow x + z \leq y + z$,
- $\forall x, y \in \mathbb{R}$: $(0 \leq x \ \ \& \ \ 0 \leq y) \Rightarrow 0 \leq x \cdot y$.

Definition

We say that the set $M \subset \mathbb{R}$ is bounded from below if there exists a number $a \in \mathbb{R}$ such that for each $x \in M$ we have $x \geq a$.

Definition

We say that the set $M \subset \mathbb{R}$ is bounded from below if there exists a number $a \in \mathbb{R}$ such that for each $x \in M$ we have $x \geq a$. Such a number a is called a lower bound of the set M.

Definition

We say that the set $M \subset \mathbb{R}$ is bounded from below if there exists a number $a \in \mathbb{R}$ such that for each $x \in M$ we have $x \geq a$. Such a number a is called a lower bound of the set M. Analogously we define the notions of a set bounded from above and an upper bound.

Definition

We say that the set $M \subset \mathbb{R}$ is bounded from below if there exists a number $a \in \mathbb{R}$ such that for each $x \in M$ we have $x \geq a$. Such a number a is called a lower bound of the set M. Analogously we define the notions of a set bounded from above and an upper bound. We say that a set $M \subset \mathbb{R}$ is bounded if it is bounded from above and below.

The infimum axiom:
Let M be a non-empty set bounded from below. Then there exists a unique number $g \in \mathbb{R}$ such that

(i) $\forall x \in M\colon x \geq g$,

The infimum axiom:
Let M be a non-empty set bounded from below. Then there exists a unique number $g \in \mathbb{R}$ such that

(i) $\forall x \in M\colon x \geq g$,

(ii) $\forall g' \in \mathbb{R}, g' > g\ \exists x \in M\colon x < g'$.

The infimum axiom:
Let M be a non-empty set bounded from below. Then there exists a unique number $g \in \mathbb{R}$ such that

(i) $\forall x \in M \colon x \geq g$,

(ii) $\forall g' \in \mathbb{R}, g' > g \; \exists x \in M \colon x < g'$.

The number g is denoted by $\inf M$ and is called the infimum of the set M.

Remark

- The infimum axiom says that every non-empty set bounded from below has an infimum.

Remark

- The infimum axiom says that every non-empty set bounded from below has an infimum.
- The infimum of the set M is its greatest lower bound.

Remark

- The infimum axiom says that every non-empty set bounded from below has an infimum.
- The infimum of the set M is its greatest lower bound.
- The real numbers exist and are uniquely determined by the properties I–III.

The following hold:

(i) $\forall x \in \mathbb{R}\colon x \cdot 0 = 0 \cdot x = 0$,

The following hold:

(i) $\forall x \in \mathbb{R}\colon x \cdot 0 = 0 \cdot x = 0$,

(ii) $\forall x \in \mathbb{R}\colon -x = (-1) \cdot x$,

The following hold:

(i) $\forall x \in \mathbb{R}\colon\ x \cdot 0 = 0 \cdot x = 0$,

(ii) $\forall x \in \mathbb{R}\colon\ -x = (-1) \cdot x$,

(iii) $\forall x, y \in \mathbb{R}\colon\ xy = 0 \Rightarrow (x = 0 \vee y = 0)$,

The following hold:

(i) $\forall x \in \mathbb{R}\colon x \cdot 0 = 0 \cdot x = 0$,

(ii) $\forall x \in \mathbb{R}\colon -x = (-1) \cdot x$,

(iii) $\forall x, y \in \mathbb{R}\colon xy = 0 \Rightarrow (x = 0 \vee y = 0)$,

(iv) $\forall x \in \mathbb{R}\ \forall n \in \mathbb{N}\colon x^{-n} = (x^{-1})^n$,

The following hold:

(i) $\forall x \in \mathbb{R}\colon x \cdot 0 = 0 \cdot x = 0$,

(ii) $\forall x \in \mathbb{R}\colon -x = (-1) \cdot x$,

(iii) $\forall x, y \in \mathbb{R}\colon xy = 0 \Rightarrow (x = 0 \vee y = 0)$,

(iv) $\forall x \in \mathbb{R}\ \forall n \in \mathbb{N}\colon x^{-n} = (x^{-1})^n$,

(v) $\forall x, y \in \mathbb{R}\colon (x > 0 \wedge y > 0) \Rightarrow xy > 0$,

The following hold:

(i) $\forall x \in \mathbb{R}\colon x \cdot 0 = 0 \cdot x = 0$,

(ii) $\forall x \in \mathbb{R}\colon\ -x = (-1) \cdot x$,

(iii) $\forall x, y \in \mathbb{R}\colon xy = 0 \Rightarrow (x = 0 \vee y = 0)$,

(iv) $\forall x \in \mathbb{R}\ \forall n \in \mathbb{N}\colon x^{-n} = (x^{-1})^n$,

(v) $\forall x, y \in \mathbb{R}\colon (x > 0 \wedge y > 0) \Rightarrow xy > 0$,

(vi) $\forall x \in \mathbb{R}, x \geq 0\ \forall y \in \mathbb{R}, y \geq 0\ \forall n \in \mathbb{N}\colon x < y \Leftrightarrow x^n < y^n$.

Let $a, b \in \mathbb{R}$, $a \leq b$. We denote:

- An open interval $(a, b) = \{x \in \mathbb{R};\ a < x < b\}$,
- A closed interval $[a, b] = \{x \in \mathbb{R};\ a \leq x \leq b\}$,
- A half-open interval $[a, b) = \{x \in \mathbb{R};\ a \leq x < b\}$,
- A half-open interval $(a, b] = \{x \in \mathbb{R};\ a < x \leq b\}$.

Let $a, b \in \mathbb{R}$, $a \leq b$. We denote:

- An open interval $(a, b) = \{x \in \mathbb{R};\ a < x < b\}$,
- A closed interval $[a, b] = \{x \in \mathbb{R};\ a \leq x \leq b\}$,
- A half-open interval $[a, b) = \{x \in \mathbb{R};\ a \leq x < b\}$,
- A half-open interval $(a, b] = \{x \in \mathbb{R};\ a < x \leq b\}$.

The point a is called the left endpoint of the interval, The point b is called the right endpoint of the interval. A point in the interval which is not an endpoint is called an inner point of the interval.

Let $a, b \in \mathbb{R}$, $a \leq b$. We denote:

- An open interval $(a, b) = \{x \in \mathbb{R};\ a < x < b\}$,
- A closed interval $[a, b] = \{x \in \mathbb{R};\ a \leq x \leq b\}$,
- A half-open interval $[a, b) = \{x \in \mathbb{R};\ a \leq x < b\}$,
- A half-open interval $(a, b] = \{x \in \mathbb{R};\ a < x \leq b\}$.

The point a is called the left endpoint of the interval, The point b is called the right endpoint of the interval. A point in the interval which is not an endpoint is called an inner point of the interval.
Unbounded intervals:

$$(a, +\infty) = \{x \in \mathbb{R};\ a < x\}, \quad (-\infty, a) = \{x \in \mathbb{R};\ x < a\},$$

analogically $(-\infty, a]$, $[a, +\infty)$ and $(-\infty, +\infty)$.

We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. If we transfer the addition and multiplication from $\mathbb{R}$ to the above sets, we obtain the usual operations on these sets.

We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. If we transfer the addition and multiplication from $\mathbb{R}$ to the above sets, we obtain the usual operations on these sets.
A real number that is not rational is called irrational. The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of irrational numbers.

Complex numbers

By the set of complex numbers we mean the set of all expressions of the form $a + bi$, where $a, b \in \mathbb{R}$. The set of all complex numbers is denoted by $\mathbb{C}$. On $\mathbb{C}$ there are operations of addition and multiplication satisfying the group of properties I and moreover $i \cdot i = -1$.

Complex numbers

By the set of complex numbers we mean the set of all expressions of the form $a + bi$, where $a, b \in \mathbb{R}$. The set of all complex numbers is denoted by $\mathbb{C}$. On $\mathbb{C}$ there are operations of addition and multiplication satisfying the group of properties I and moreover $i \cdot i = -1$.

Theorem ("fundamental theorem of algebra")

Let $n \in \mathbb{N}$, $a_0, \ldots, a_n \in \mathbb{C}$, $a_n \neq 0$. Then the equation

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_1 z + a_0 = 0$$

has at least one solution $z \in \mathbb{C}$.

Consequences of the infimum axiom

Definition

Let $M \subset \mathbb{R}$. A number $G \in \mathbb{R}$ satisfying

(i) $\forall x \in M\colon\ x \leq G$,

(ii) $\forall G' \in \mathbb{R}, G' < G\ \exists x \in M\colon\ x > G'$,

is called a supremum of the set M.

Consequences of the infimum axiom

Definition

Let $M \subset \mathbb{R}$. A number $G \in \mathbb{R}$ satisfying

(i) $\forall x \in M \colon x \leq G$,

(ii) $\forall G' \in \mathbb{R}, G' < G \; \exists x \in M \colon x > G'$,

is called a supremum of the set M.

Theorem 3 (Supremum theorem)

Let $M \subset \mathbb{R}$ be a non-empty set bounded from above. Then there exists a unique supremum of the set M.

Consequences of the infimum axiom

Definition

Let $M \subset \mathbb{R}$. A number $G \in \mathbb{R}$ satisfying

(i) $\forall x \in M\colon\ x \leq G$,

(ii) $\forall G' \in \mathbb{R}, G' < G\ \exists x \in M\colon\ x > G'$,

is called a supremum of the set M.

Theorem 3 (Supremum theorem)

Let $M \subset \mathbb{R}$ be a non-empty set bounded from above. Then there exists a unique supremum of the set M.

The supremum of the set M is denoted by $\sup M$.

Consequences of the infimum axiom

Definition

Let $M \subset \mathbb{R}$. A number $G \in \mathbb{R}$ satisfying

(i) $\forall x \in M\colon\ x \leq G$,

(ii) $\forall G' \in \mathbb{R}, G' < G\ \exists x \in M\colon\ x > G'$,

is called a supremum of the set M.

Theorem 3 (Supremum theorem)

Let $M \subset \mathbb{R}$ be a non-empty set bounded from above. Then there exists a unique supremum of the set M.

The supremum of the set M is denoted by $\sup M$.
The following holds: $\sup M = -\inf(-M)$.

Definition

Let $M \subset \mathbb{R}$. We say that a is a maximum of the set M (denoted by $\max M$) if a is an upper bound of M and $a \in M$. Analogously we define a minimum of M, denoted by $\min M$.

Lemma 4 ("no holes")

Let $M \subset \mathbb{R}$ and

$$\forall x, y \in M \; \forall z \in \mathbb{R}, x < z < y \colon z \in M.$$

Then M is an interval.

Theorem 5 (Archimedean property)

For every $x \in \mathbb{R}$ *there exists* $n \in \mathbb{N}$ *satisfying* $n > x$.

Theorem 5 (Archimedean property)

For every $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ satisfying $n > x$.

Theorem 6 (existence of an integer part)

For every $r \in \mathbb{R}$ there exists an integer part of r, i.e. a number $k \in \mathbb{Z}$ satisfying $k \leq r < k + 1$. The integer part of r is determined uniquely and it is denoted by $[r]$.

Theorem 7 (nth root)

For every $x \in [0, +\infty)$ *and every* $n \in \mathbb{N}$ *there exists a unique* $y \in [0, +\infty)$ *satisfying* $y^n = x$.

Theorem 7 (nth root)

For every $x \in [0, +\infty)$ and every $n \in \mathbb{N}$ there exists a unique $y \in [0, +\infty)$ satisfying $y^n = x$.

Theorem 8 (density of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$)

Let $a, b \in \mathbb{R}$, $a < b$. Then there exist $r \in \mathbb{Q}$ satisfying $a < r < b$ and $s \in \mathbb{R} \setminus \mathbb{Q}$ satisfying $a < s < b$.

II. Limit of a sequence

II. Limit of a sequence

Definition

Suppose that to each natural number $n \in \mathbb{N}$ we assign a real number a_n. Then we say that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers.

II. Limit of a sequence

Definition

Suppose that to each natural number $n \in \mathbb{N}$ we assign a real number a_n. Then we say that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers. The number a_n is called the nth member of this sequence.

II. Limit of a sequence

Definition

Suppose that to each natural number $n \in \mathbb{N}$ we assign a real number a_n. Then we say that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers. The number a_n is called the nth member of this sequence.

A sequence $\{a_n\}_{n=1}^{\infty}$ is equal to a sequence $\{b_n\}_{n=1}^{\infty}$ if $a_n = b_n$ holds for every $n \in \mathbb{N}$.

II. Limit of a sequence

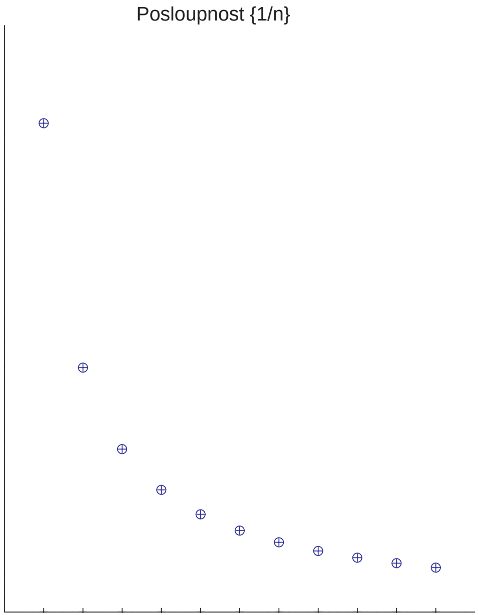
Definition

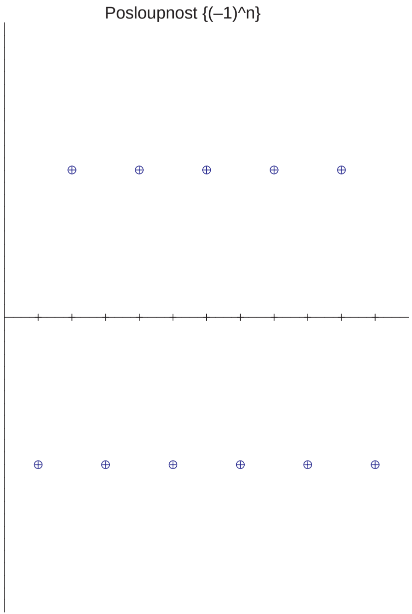
Suppose that to each natural number $n \in \mathbb{N}$ we assign a real number a_n. Then we say that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers. The number a_n is called the nth member of this sequence.

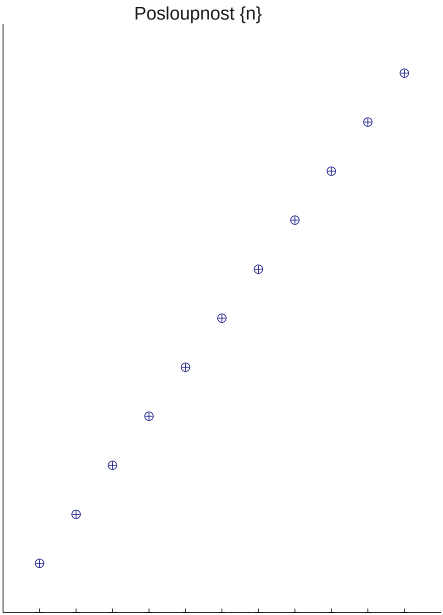
A sequence $\{a_n\}_{n=1}^{\infty}$ is equal to a sequence $\{b_n\}_{n=1}^{\infty}$ if $a_n = b_n$ holds for every $n \in \mathbb{N}$.

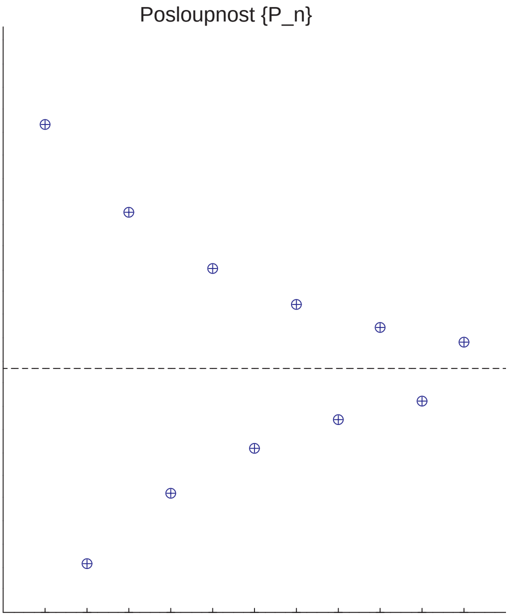
By the set of all members of the sequence $\{a_n\}_{n=1}^{\infty}$ we understand a set

$$\{x \in \mathbb{R};\ \exists n \in \mathbb{N} \colon a_n = x\}.$$


Posloupnost {1/n}


Posloupnost {(–1)^n}


Posloupnost {n}


Posloupnost {P_n}

Definition

We say that a sequence $\{a_n\}$ is

- bounded from above if the set of all members of this sequence is bounded from above,

Definition

We say that a sequence $\{a_n\}$ is

- bounded from above if the set of all members of this sequence is bounded from above,
- bounded from below if the set of all members of this sequence is bounded from below,

Definition

We say that a sequence $\{a_n\}$ is

- bounded from above if the set of all members of this sequence is bounded from above,
- bounded from below if the set of all members of this sequence is bounded from below,
- bounded if the set of all members of this sequence is bounded.

Definition

We say that a sequence $\{a_n\}$ is

- increasing if $a_n < a_{n+1}$ for every $n \in \mathbb{N}$,

Definition

We say that a sequence $\{a_n\}$ is

- increasing if $a_n < a_{n+1}$ for every $n \in \mathbb{N}$,
- decreasing if $a_n > a_{n+1}$ for every $n \in \mathbb{N}$,

Definition

We say that a sequence $\{a_n\}$ is

- increasing if $a_n < a_{n+1}$ for every $n \in \mathbb{N}$,
- decreasing if $a_n > a_{n+1}$ for every $n \in \mathbb{N}$,
- non-decreasing if $a_n \leq a_{n+1}$ for every $n \in \mathbb{N}$,

Definition

We say that a sequence $\{a_n\}$ is

- increasing if $a_n < a_{n+1}$ for every $n \in \mathbb{N}$,
- decreasing if $a_n > a_{n+1}$ for every $n \in \mathbb{N}$,
- non-decreasing if $a_n \leq a_{n+1}$ for every $n \in \mathbb{N}$,
- non-increasing if $a_n \geq a_{n+1}$ for every $n \in \mathbb{N}$.

Definition

We say that a sequence $\{a_n\}$ is

- increasing if $a_n < a_{n+1}$ for every $n \in \mathbb{N}$,
- decreasing if $a_n > a_{n+1}$ for every $n \in \mathbb{N}$,
- non-decreasing if $a_n \leq a_{n+1}$ for every $n \in \mathbb{N}$,
- non-increasing if $a_n \geq a_{n+1}$ for every $n \in \mathbb{N}$.

A sequence $\{a_n\}$ is monotone if it satisfies one of the conditions above.

Definition

We say that a sequence $\{a_n\}$ is

- increasing if $a_n < a_{n+1}$ for every $n \in \mathbb{N}$,
- decreasing if $a_n > a_{n+1}$ for every $n \in \mathbb{N}$,
- non-decreasing if $a_n \leq a_{n+1}$ for every $n \in \mathbb{N}$,
- non-increasing if $a_n \geq a_{n+1}$ for every $n \in \mathbb{N}$.

A sequence $\{a_n\}$ is monotone if it satisfies one of the conditions above. A sequence $\{a_n\}$ is strictly monotone if it is increasing or decreasing.

Definition

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.

- By the sum of sequences $\{a_n\}$ and $\{b_n\}$ we understand a sequence $\{a_n + b_n\}$.

Definition

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.

- By the sum of sequences $\{a_n\}$ and $\{b_n\}$ we understand a sequence $\{a_n + b_n\}$.
- Analogously we define a difference and a product of sequences.

Definition

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.

- By the sum of sequences $\{a_n\}$ and $\{b_n\}$ we understand a sequence $\{a_n + b_n\}$.
- Analogously we define a difference and a product of sequences.
- Suppose all the members of the sequence $\{b_n\}$ are non-zero. Then by the quotient of sequences $\{a_n\}$ and $\{b_n\}$ we understand a sequence $\{\frac{a_n}{b_n}\}$.

Definition

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.

- By the sum of sequences $\{a_n\}$ and $\{b_n\}$ we understand a sequence $\{a_n + b_n\}$.
- Analogously we define a difference and a product of sequences.
- Suppose all the members of the sequence $\{b_n\}$ are non-zero. Then by the quotient of sequences $\{a_n\}$ and $\{b_n\}$ we understand a sequence $\{\frac{a_n}{b_n}\}$.
- If $\lambda \in \mathbb{R}$, then by the λ-multiple of the sequence $\{a_n\}$ we understand a sequence $\{\lambda a_n\}$.

Definition

We say that a sequence $\{a_n\}$ has a limit which equals to a number $A \in \mathbb{R}$ if to every positive real number ε there exists a natural number n_0 such that for every index $n \geq n_0$ we have $|a_n - A| < \varepsilon$, i.e.

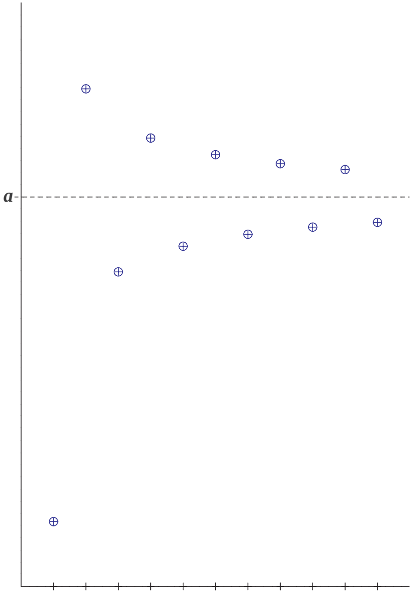
$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \; \exists n_0 \in \mathbb{N} \; \forall n \in \mathbb{N}, n \geq n_0 : \; |a_n - A| < \varepsilon.$$

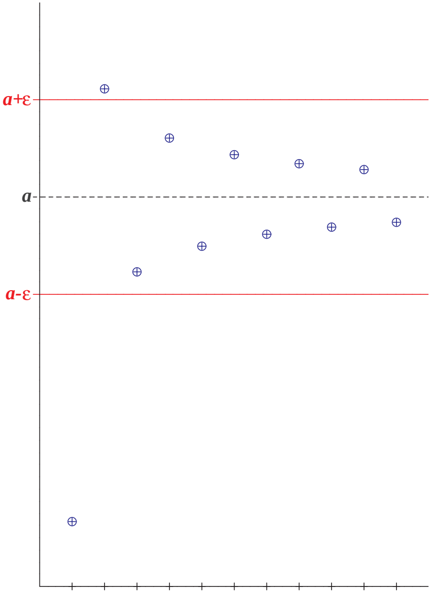
Definition

We say that a sequence $\{a_n\}$ has a limit which equals to a number $A \in \mathbb{R}$ if to every positive real number ε there exists a natural number n_0 such that for every index $n \geq n_0$ we have $|a_n - A| < \varepsilon$, i.e.

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon\ |a_n - A| < \varepsilon.$$

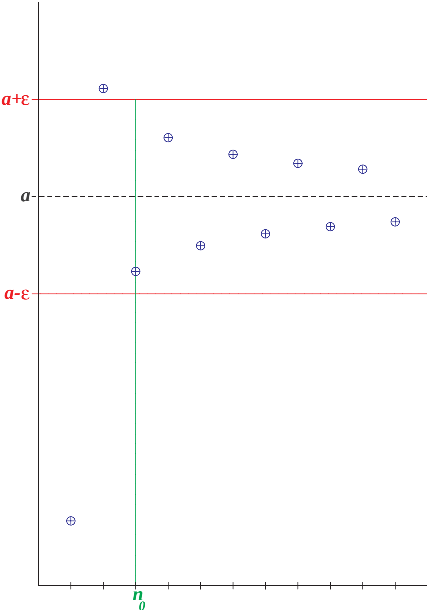
We say that a sequence $\{a_n\}$ is convergent if there exists $A \in \mathbb{R}$ which is a limit of $\{a_n\}$.

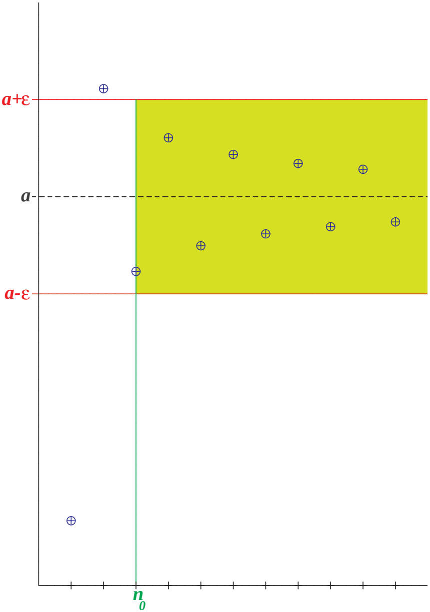

a

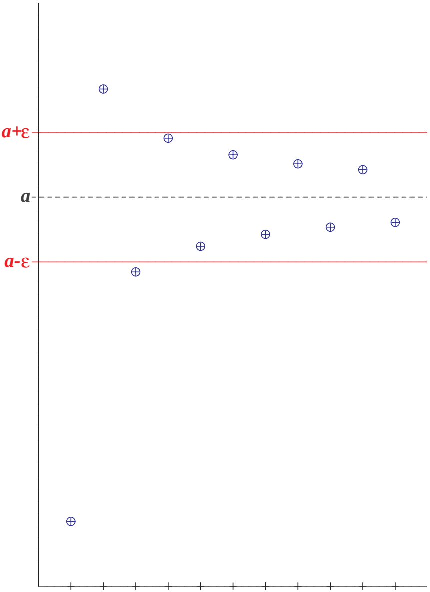


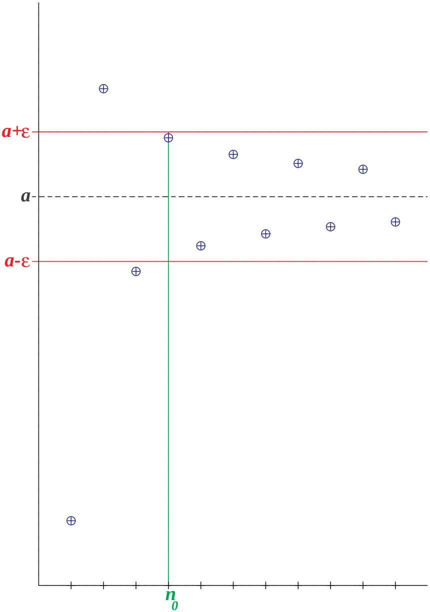
a+ε
a
a-ε

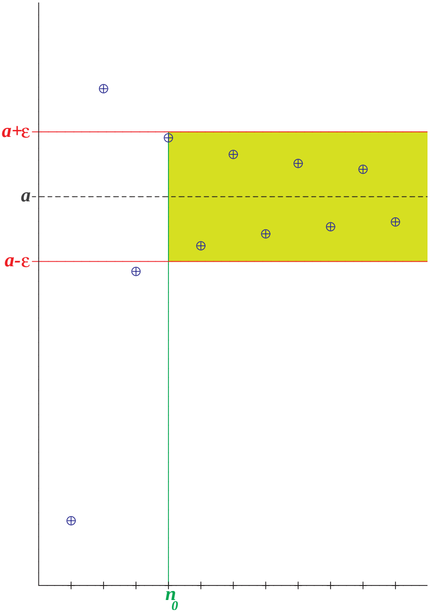
II.2. Convergence of sequences

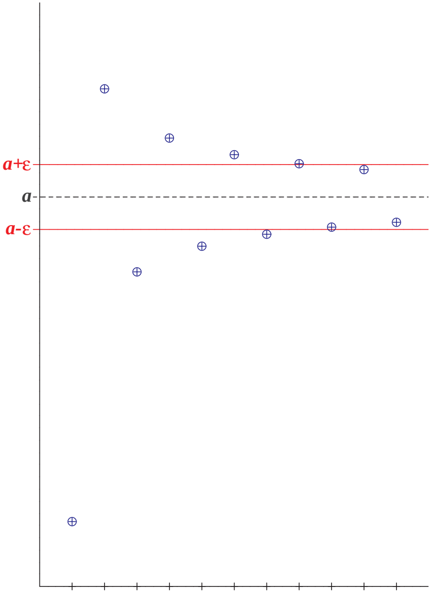


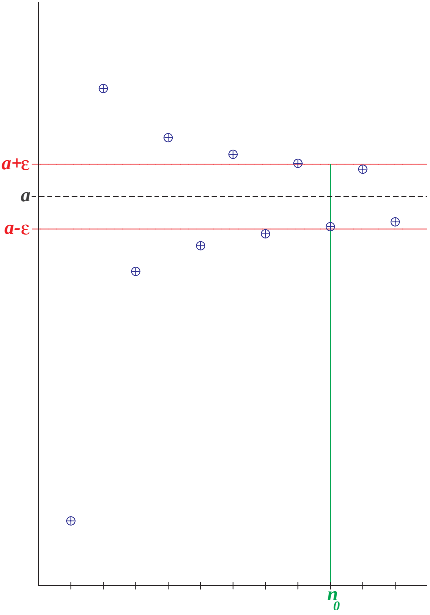

a+ε
a
a-ε
n_0


a+ε
a
a-ε

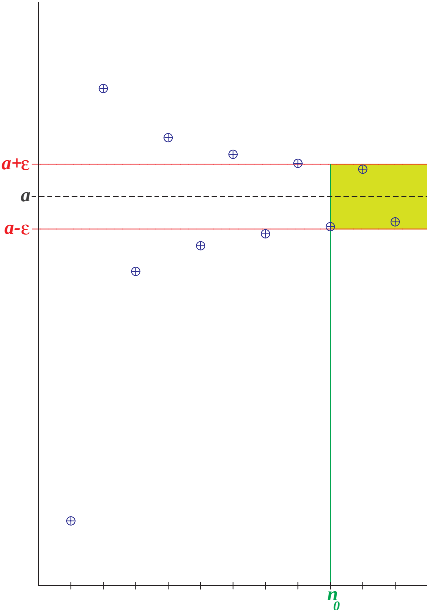

a+ε
a
a-ε
n0


a+ε
a
a-ε
n
0


a+ε
a
a-ε


a+ε
a
a-ε
n0

II.2. Convergence of sequences



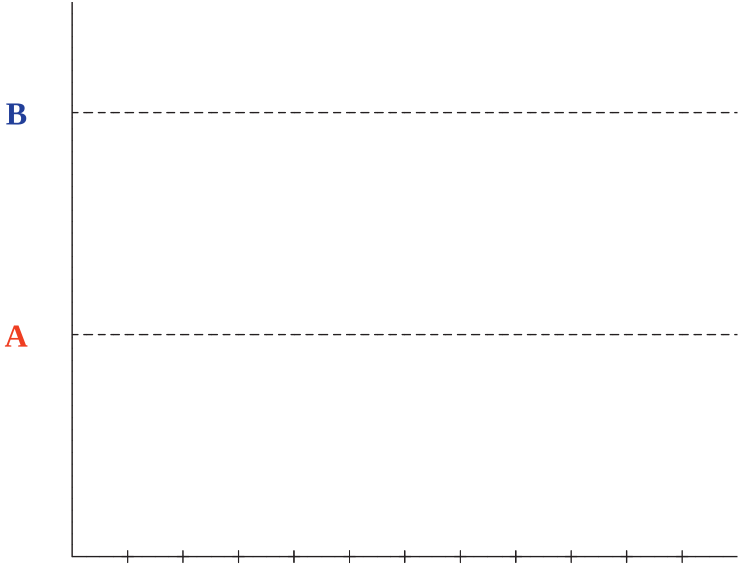
Theorem 9 (uniqueness of a limit)

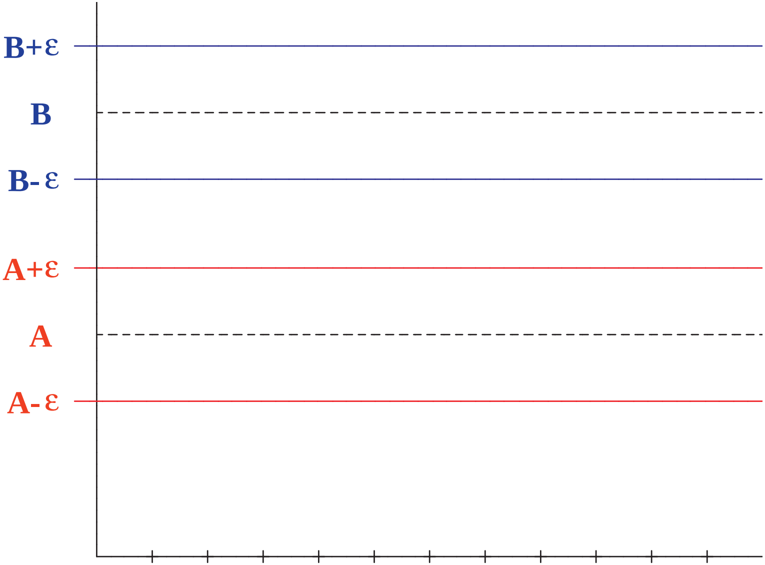
Every sequence has at most one limit.

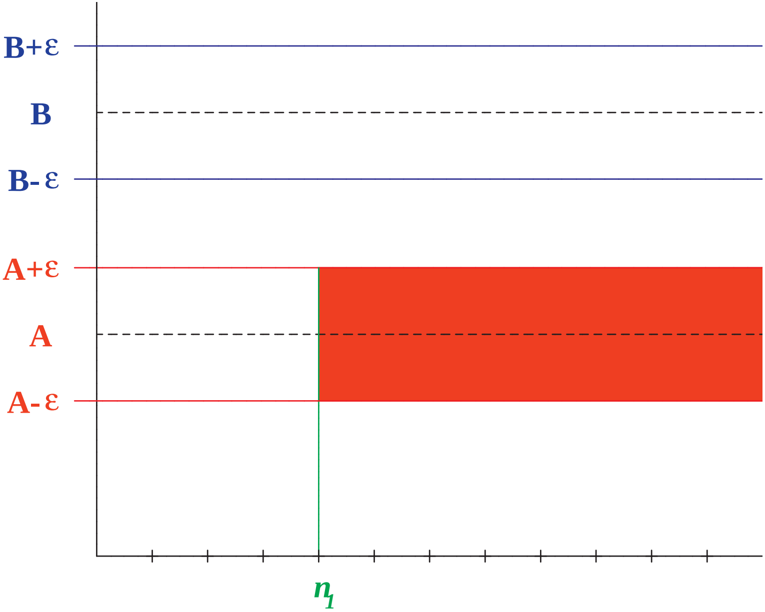
Theorem 9 (uniqueness of a limit)

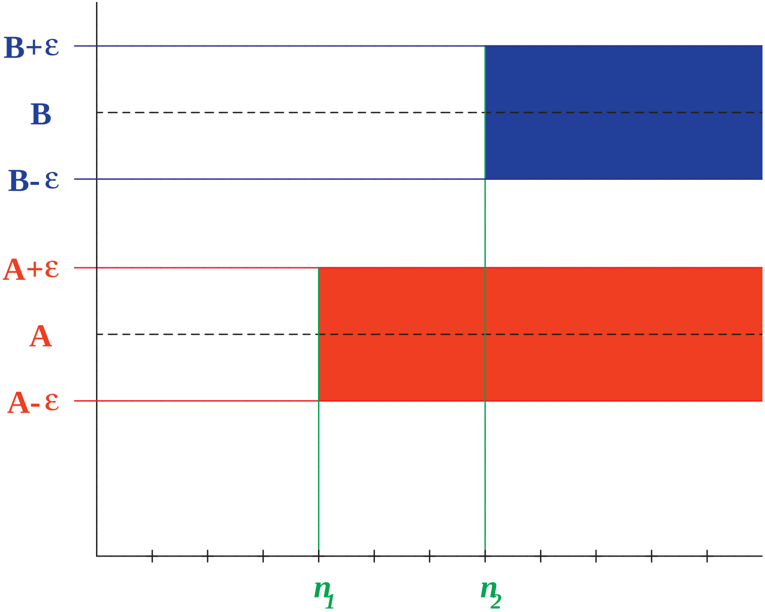
Every sequence has at most one limit.

We use the notation $\lim\limits_{n\to\infty} a_n = A$ or simply $\lim a_n = A$.


B
A


B+ε
B
B- ε
A+ε
A
A- ε


B+ε
B
B- ε
A+ε
A
A- ε
n_1


B+ε
B
B- ε
A+ε
A
A- ε
n_1
n_2

Remark

Let $\{a_n\}$ be a sequence of real numbers and $A \in \mathbb{R}$. Then

$$\lim a_n = A \Leftrightarrow \lim(a_n - A) = 0 \Leftrightarrow \lim |a_n - A| = 0.$$

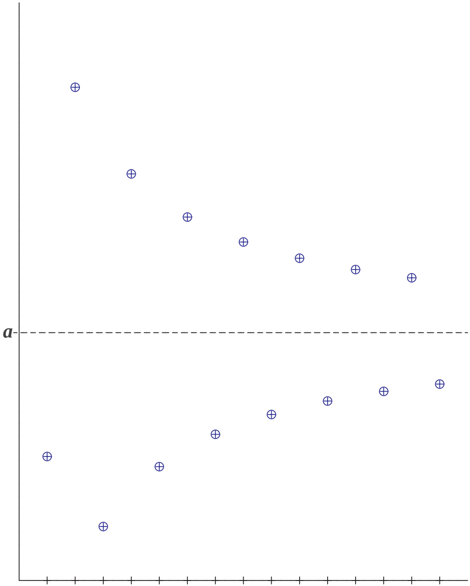
Remark

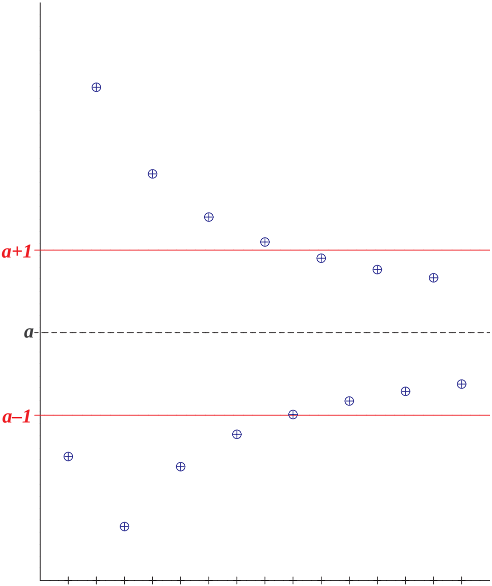
Let $\{a_n\}$ be a sequence of real numbers and $A \in \mathbb{R}$. Then

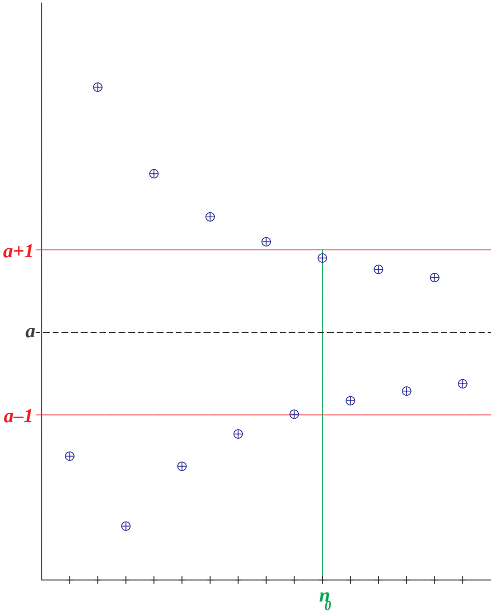
$$\lim a_n = A \Leftrightarrow \lim(a_n - A) = 0 \Leftrightarrow \lim |a_n - A| = 0.$$

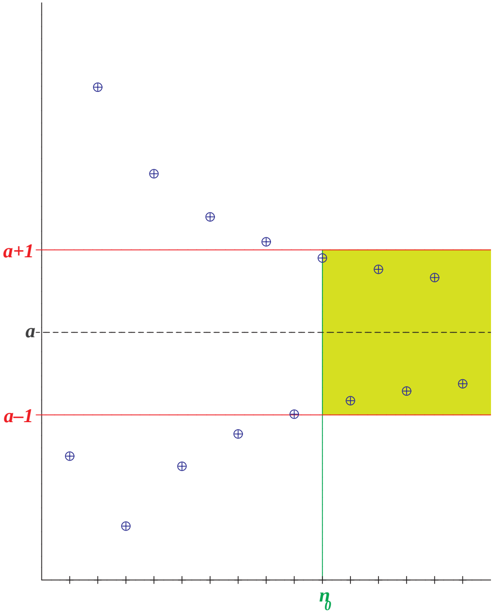
Theorem 10

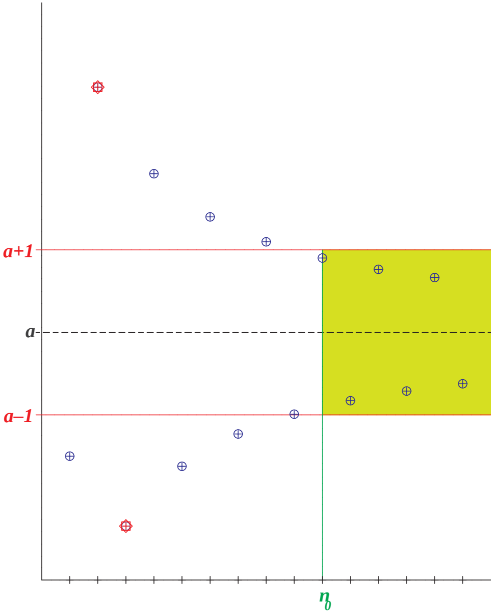
Every convergent sequence is bounded.

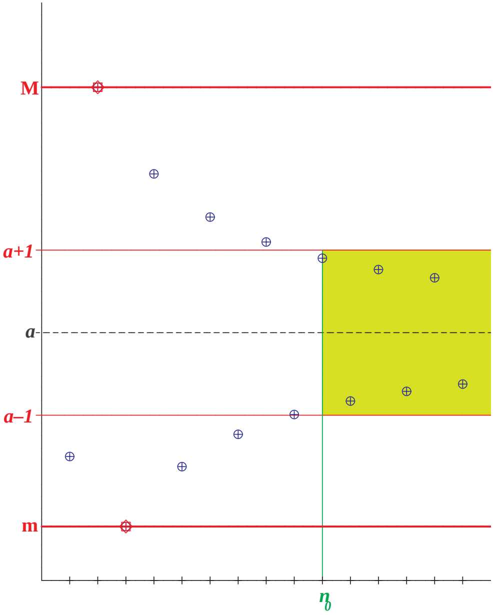

a


a+1
a
a–1


a+1
a
a–1
n0


a+1
a
a–1
n
0


a+1
a
a–1
n_0



M
a+1
a
a−1
m
n_0

Definition

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that a sequence $\{b_k\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$ if there is an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that $b_k = a_{n_k}$ for every $k \in \mathbb{N}$.

Definition

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that a sequence $\{b_k\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$ if there is an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that $b_k = a_{n_k}$ for every $k \in \mathbb{N}$.

Theorem 11 (limit of a subsequence)

Let $\{b_k\}_{k=1}^{\infty}$ *be a subsequence of* $\{a_n\}_{n=1}^{\infty}$*. If* $\lim_{n\to\infty} a_n = A \in \mathbb{R}$*, then also* $\lim_{k\to\infty} b_k = A$*.*

Remark

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, $A \in \mathbb{R}$, $K \in \mathbb{R}$, $K > 0$. If

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \; \exists n_0 \in \mathbb{N} \; \forall n \in \mathbb{N}, n \geq n_0 \colon \; |a_n - A| < K\varepsilon,$$

then $\lim a_n = A$.

Theorem 12 (arithmetics of limits)

Suppose that $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$*. Then*

(i) $\lim(a_n + b_n) = A + B$,

Theorem 12 (arithmetics of limits)

Suppose that $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$. *Then*

(i) $\lim(a_n + b_n) = A + B$,

(ii) $\lim(a_n \cdot b_n) = A \cdot B$,

Theorem 12 (arithmetics of limits)

Suppose that $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$. *Then*

(i) $\lim(a_n + b_n) = A + B$,

(ii) $\lim(a_n \cdot b_n) = A \cdot B$,

(iii) *if* $B \neq 0$ *and* $b_n \neq 0$ *for all* $n \in \mathbb{N}$, *then* $\lim(a_n/b_n) = A/B$.

Theorem 12 (arithmetics of limits)

Suppose that $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$. *Then*

(i) $\lim(a_n + b_n) = A + B$,

(ii) $\lim(a_n \cdot b_n) = A \cdot B$,

(iii) *if* $B \neq 0$ *and* $b_n \neq 0$ *for all* $n \in \mathbb{N}$, *then* $\lim(a_n/b_n) = A/B$.

Theorem 13

Suppose that $\lim a_n = 0$ *and the sequence* $\{b_n\}$ *is bounded. Then* $\lim a_n b_n = 0$.

Theorem 14 (limits and ordering)

Let $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$*.*

(i) *Suppose that there is* $n_0 \in \mathbb{N}$ *such that* $a_n \geq b_n$ *for every* $n \geq n_0$*. Then* $A \geq B$*.*

Theorem 14 (limits and ordering)

Let $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$*.*

(i) *Suppose that there is* $n_0 \in \mathbb{N}$ *such that* $a_n \geq b_n$ *for every* $n \geq n_0$*. Then* $A \geq B$*.*

(ii) *Suppose that* $A < B$*. Then there is* $n_0 \in \mathbb{N}$ *such that* $a_n < b_n$ *for every* $n \geq n_0$*.*

Theorem 14 (limits and ordering)

Let $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$.

(i) *Suppose that there is* $n_0 \in \mathbb{N}$ *such that* $a_n \geq b_n$ *for every* $n \geq n_0$. *Then* $A \geq B$.

(ii) *Suppose that* $A < B$. *Then there is* $n_0 \in \mathbb{N}$ *such that* $a_n < b_n$ *for every* $n \geq n_0$.

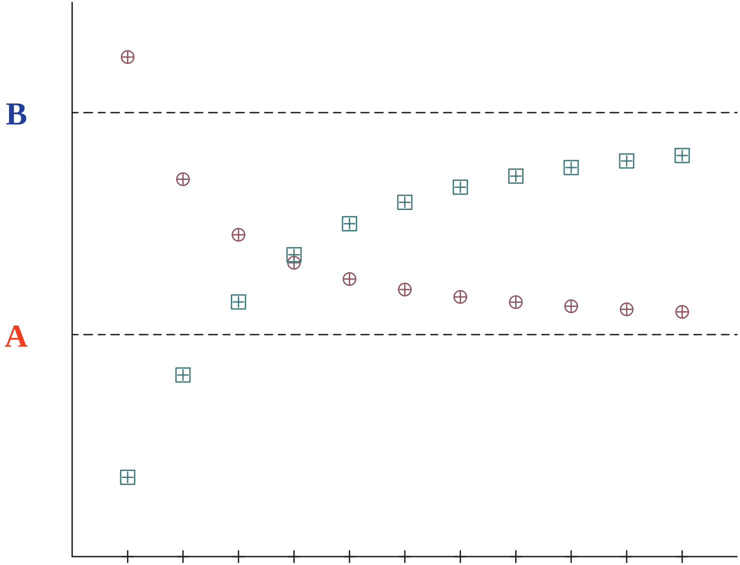
Theorem 15 (two policemen/sandwich theorem)

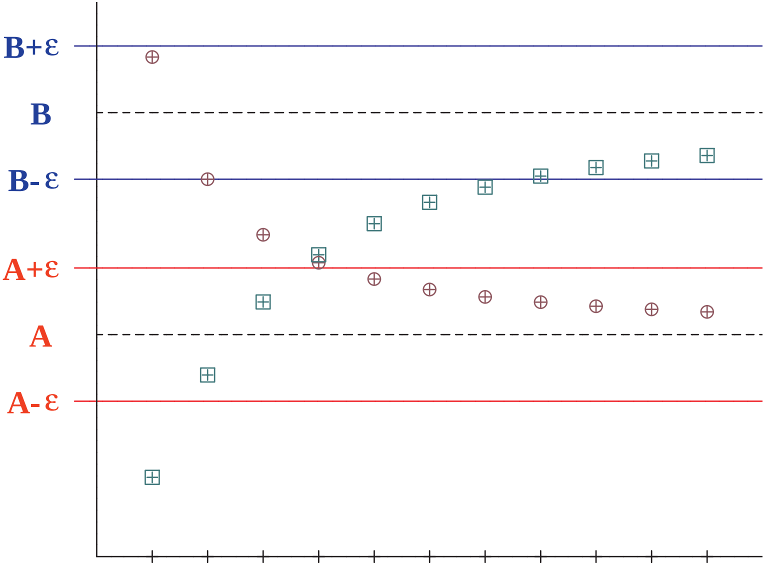
Let $\{a_n\}, \{b_n\}$ *be convergent sequences and let* $\{c_n\}$ *be a sequence such that*

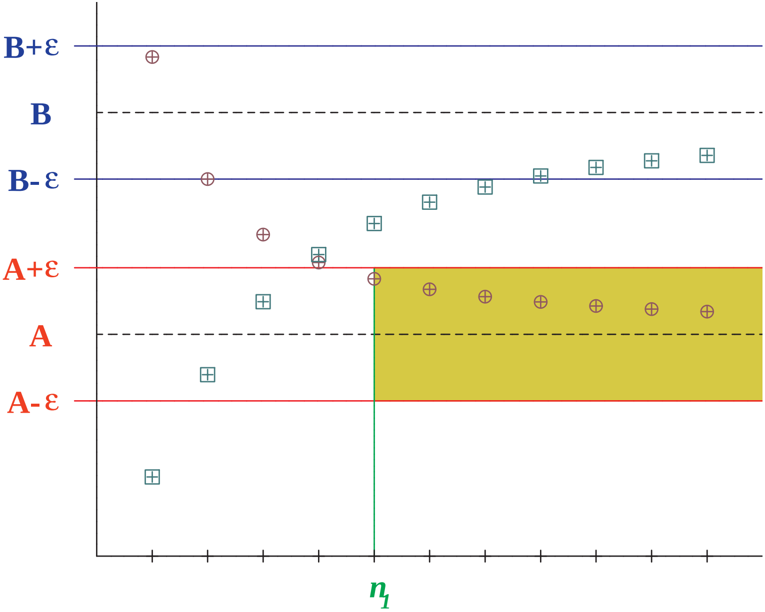
(i) $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0\colon\ a_n \leq c_n \leq b_n$,

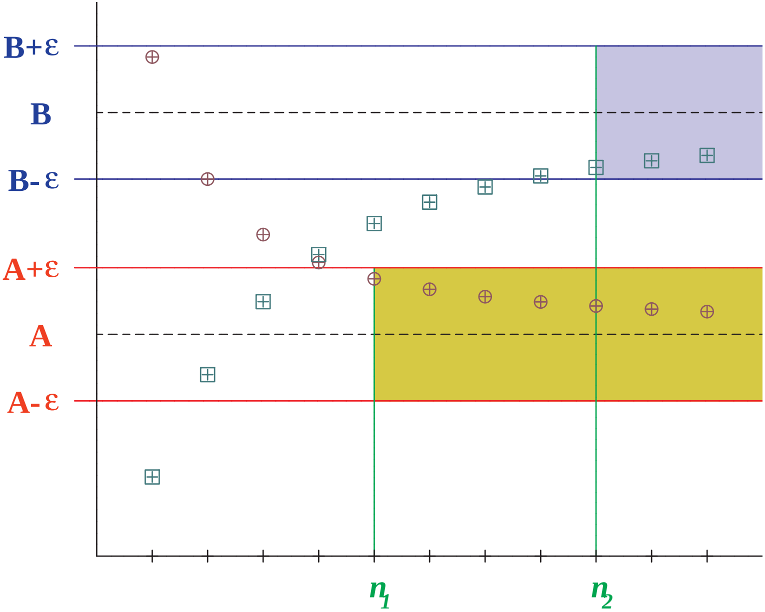
(ii) $\lim a_n = \lim b_n$.

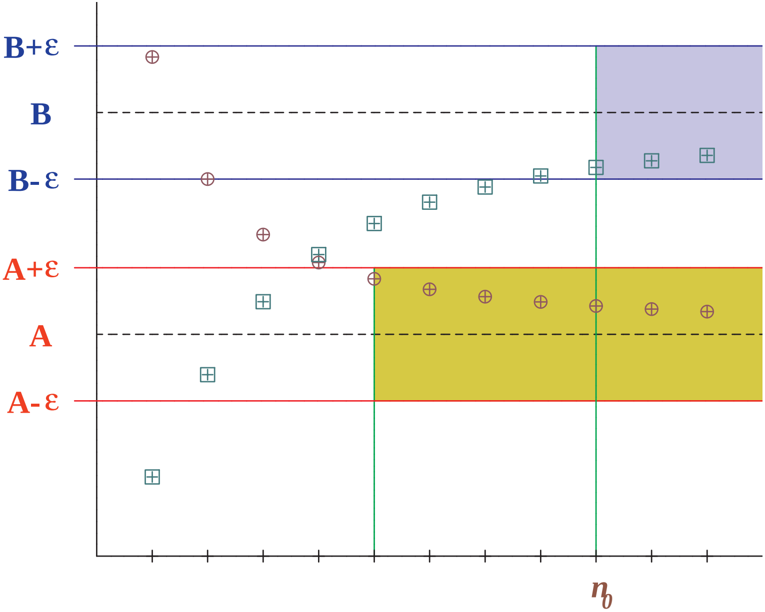
Then $\lim c_n$ *exists and* $\lim c_n = \lim a_n$.

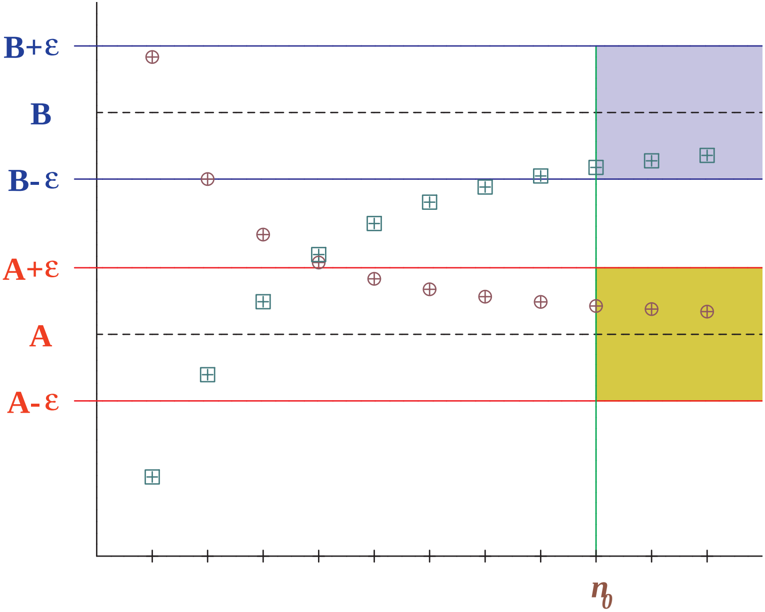

B
A


B+ε
B
B- ε
A+ε
A
A- ε


B+ε
B
B- ε
A+ε
A
A- ε
n_1


B+ε
B
B- ε
A+ε
A
A- ε
n1
n2


B+ε
B
B- ε
A+ε
A
A- ε
n_0


B+ε
B
B- ε
A+ε
A
A- ε
n_0

Definition

We say that a sequence $\{a_n\}$ has a limit $+\infty$ (plus infinity) if

$$\forall L \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon a_n > L.$$

Definition

We say that a sequence $\{a_n\}$ has a limit $+\infty$ (plus infinity) if

$$\forall L \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon a_n > L.$$

We say that a sequence $\{a_n\}$ has a limit $-\infty$ (minus infinity) if

$$\forall K \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon a_n < K.$$

Definition

We say that a sequence $\{a_n\}$ has a limit $+\infty$ (plus infinity) if

$$\forall L \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0\colon\ a_n > L.$$

We say that a sequence $\{a_n\}$ has a limit $-\infty$ (minus infinity) if

$$\forall K \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0\colon\ a_n < K.$$

Theorem 9 on the uniqueness of a limit holds also for the limits $+\infty$ and $-\infty$. If $\lim a_n = +\infty$, then we say that the sequence $\{a_n\}$ diverges to $+\infty$, similarly for $-\infty$.

Definition

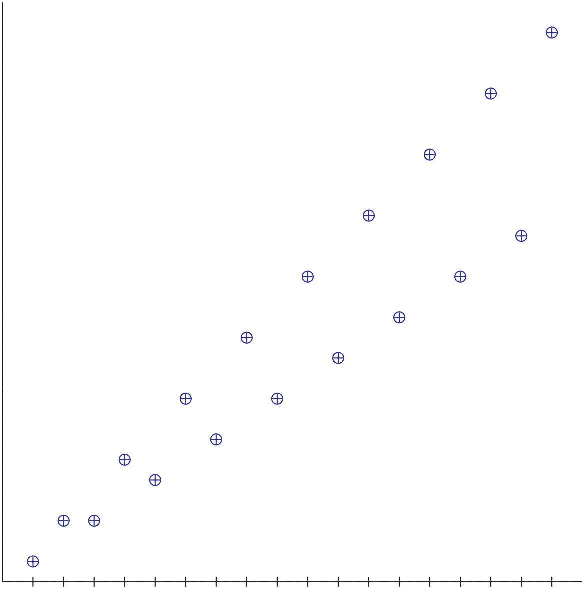
We say that a sequence $\{a_n\}$ has a limit $+\infty$ (plus infinity) if

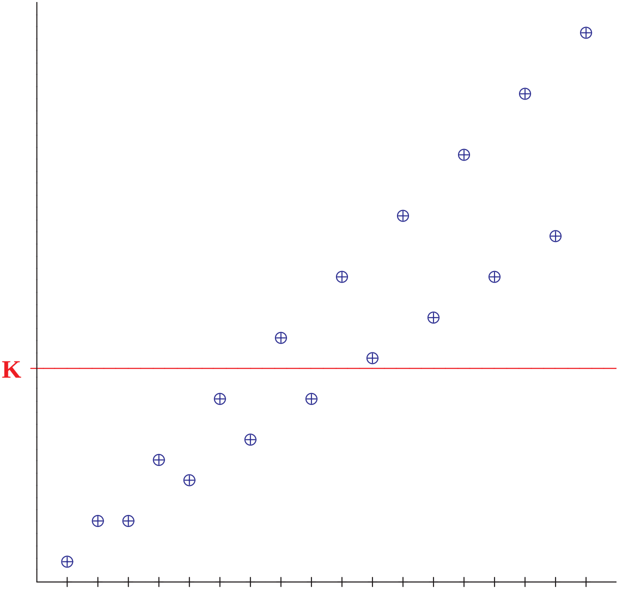
$$\forall L \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon a_n > L.$$

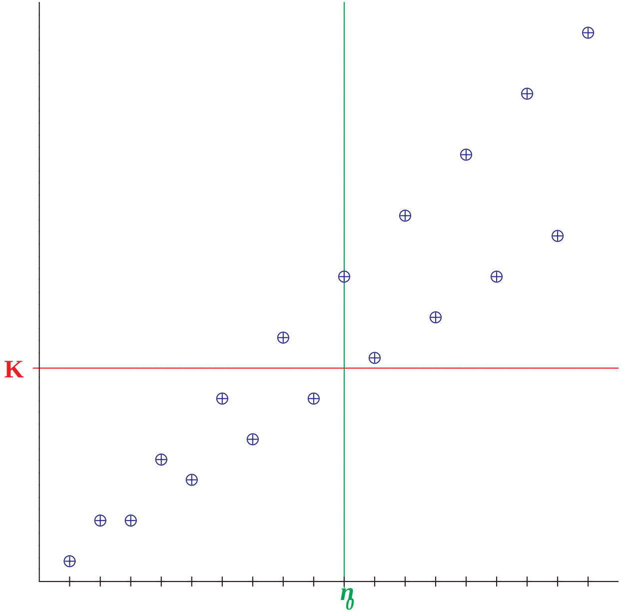
We say that a sequence $\{a_n\}$ has a limit $-\infty$ (minus infinity) if

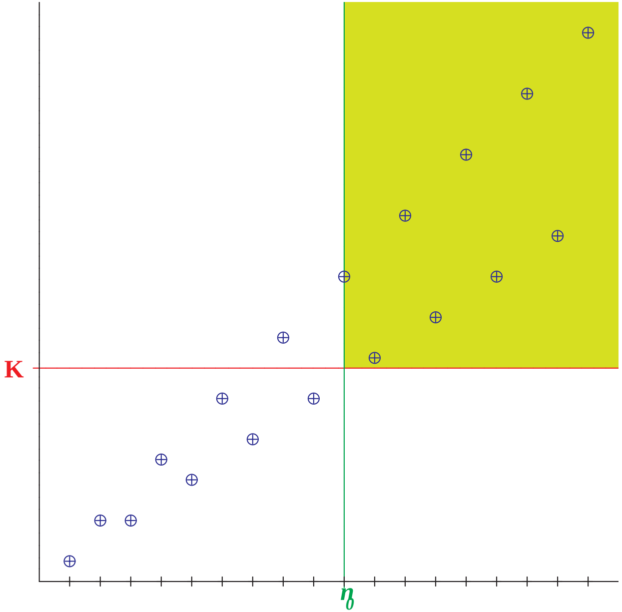
$$\forall K \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon a_n < K.$$

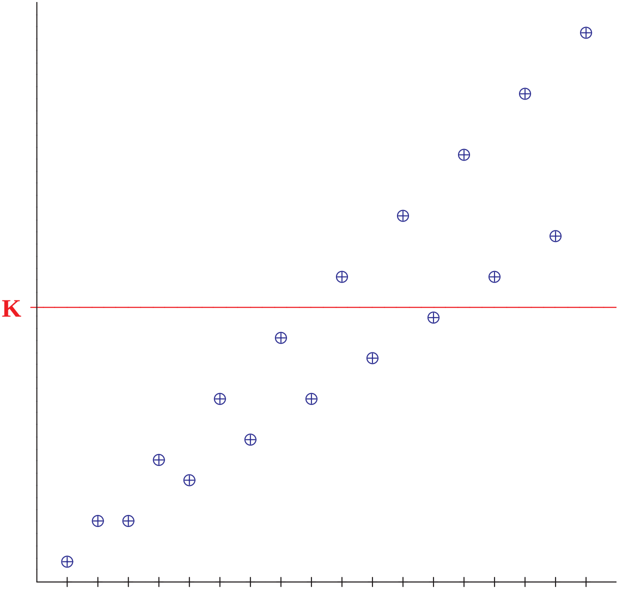
Theorem 9 on the uniqueness of a limit holds also for the limits $+\infty$ and $-\infty$. If $\lim a_n = +\infty$, then we say that the sequence $\{a_n\}$ diverges to $+\infty$, similarly for $-\infty$. If $\lim a_n \in \mathbb{R}$, then we say that the limit is finite, if $\lim a_n = +\infty$ or $\lim a_n = -\infty$, then we say that the limit is infinite.

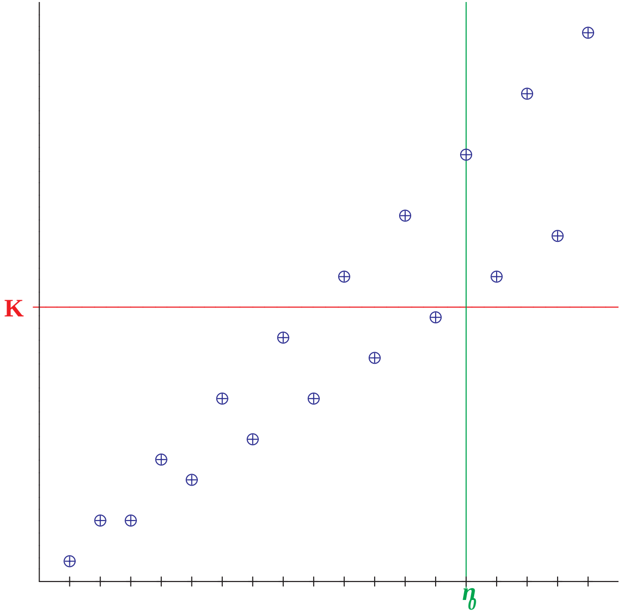


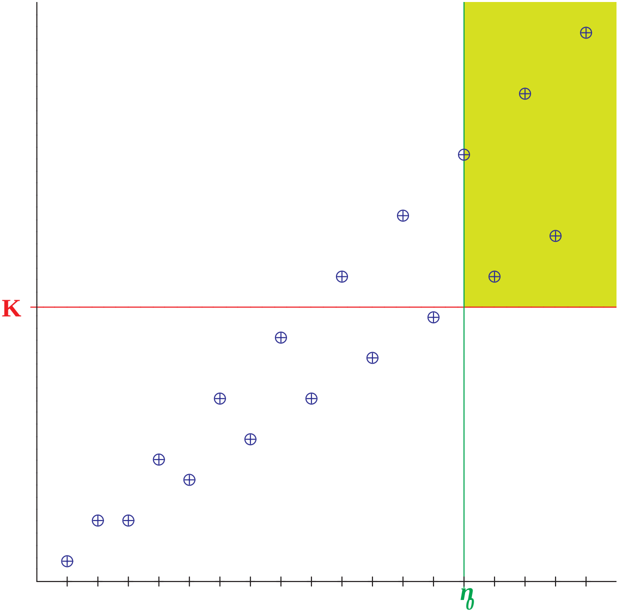

K

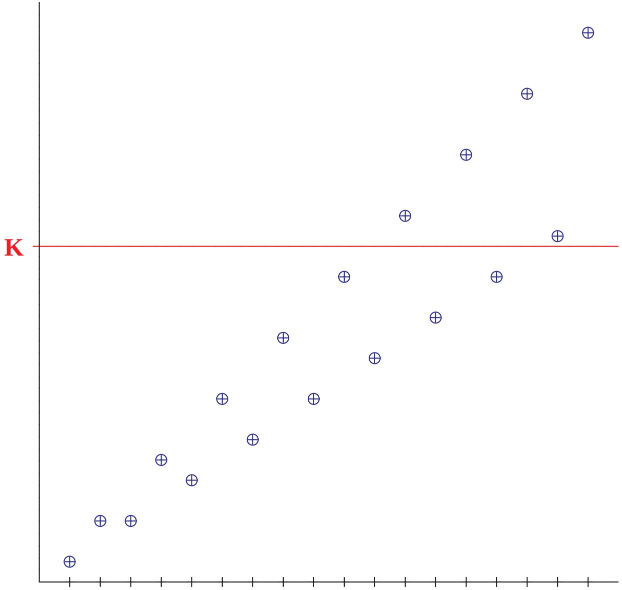

K
n_0

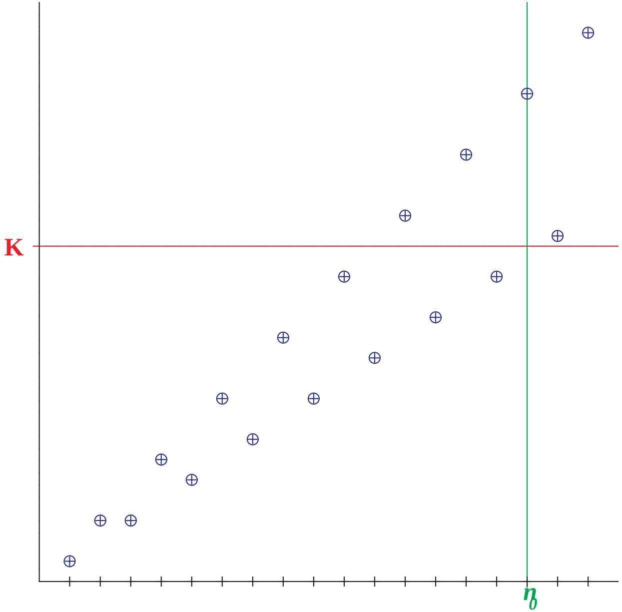

K
n_0

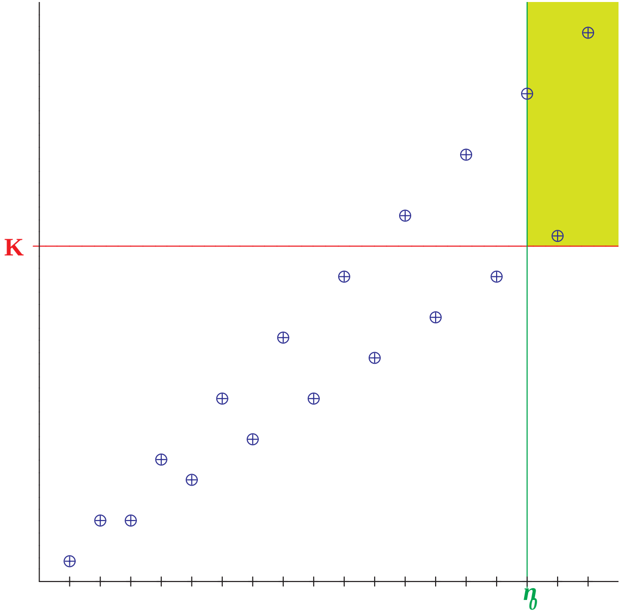

K

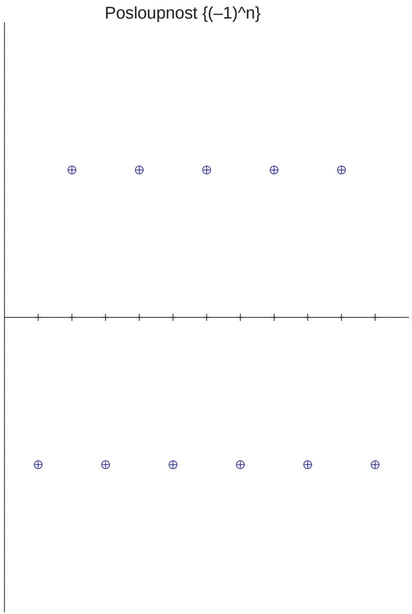

K
n_0

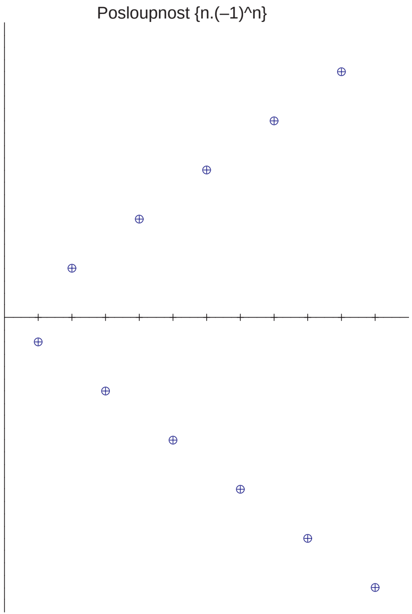

K
n_0

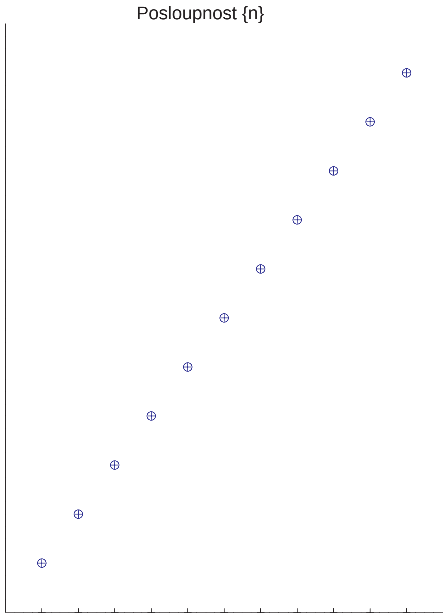

K

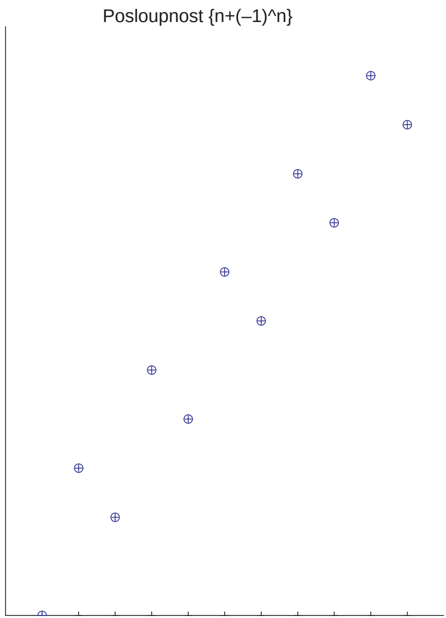

K
n_0

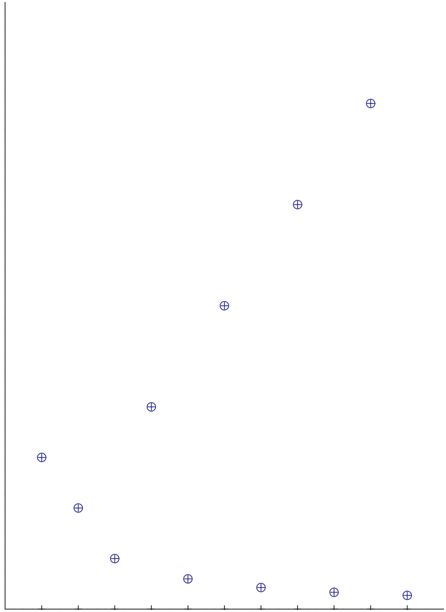

K
n_0


Posloupnost {(–1)^n}


Posloupnost {n.(–1)^n}


Posloupnost {n}


Posloupnost {n+(–1)^n}



Theorem 10 does not hold for infinite limits. But:

Theorem 10'

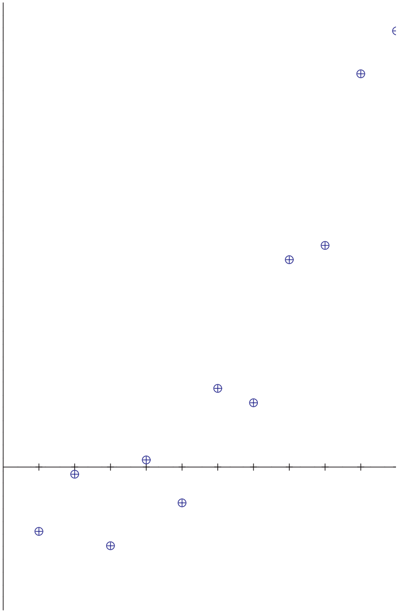
- *Suppose that* $\lim a_n = +\infty$. *Then the sequence* $\{a_n\}$ *is not bounded from above, but is bounded from below.*
- *Suppose that* $\lim a_n = -\infty$. *Then the sequence* $\{a_n\}$ *is not bounded from below, but is bounded from above.*

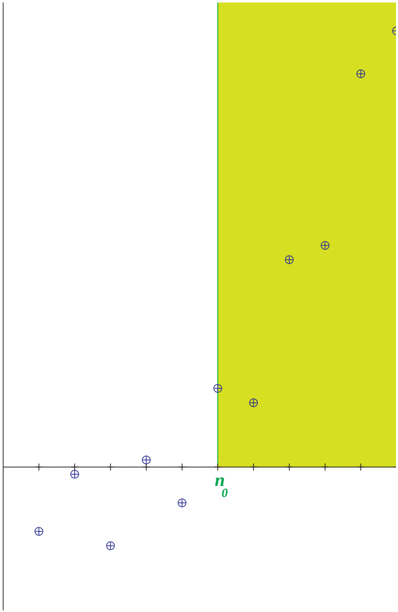
Theorem 10 does not hold for infinite limits. But:

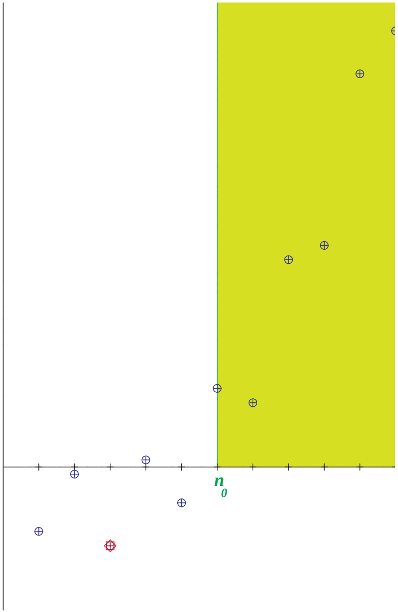
Theorem 10'

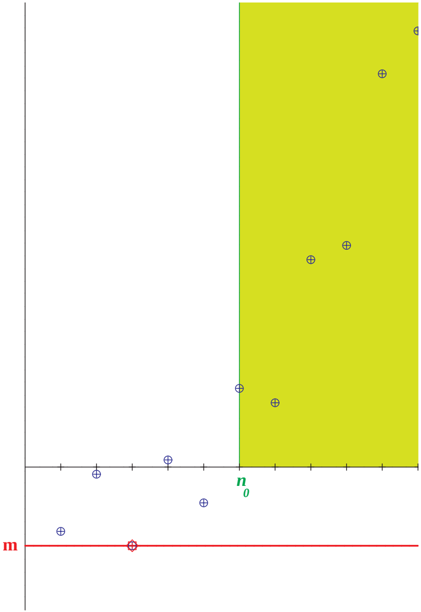
- *Suppose that* $\lim a_n = +\infty$. *Then the sequence* $\{a_n\}$ *is not bounded from above, but is bounded from below.*
- *Suppose that* $\lim a_n = -\infty$. *Then the sequence* $\{a_n\}$ *is not bounded from below, but is bounded from above.*

Theorem 11 (limit of a subsequence) holds also for infinite limits.




n
0


n_0


n
0
m

Definition

We define the extended real line by setting $\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\}$ with the following extension of operations and ordering from $\mathbb{R}$:

- $a < +\infty$ and $-\infty < a$ for $a \in \mathbb{R}$, $-\infty < +\infty$,

Definition

We define the extended real line by setting $\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\}$ with the following extension of operations and ordering from $\mathbb{R}$:

- $a < +\infty$ and $-\infty < a$ for $a \in \mathbb{R}$, $-\infty < +\infty$,
- $a + (+\infty) = (+\infty) + a = +\infty$ for $a \in \mathbb{R}^* \setminus \{-\infty\}$,

Definition

We define the extended real line by setting $\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\}$ with the following extension of operations and ordering from $\mathbb{R}$:

- $a < +\infty$ and $-\infty < a$ for $a \in \mathbb{R}$, $-\infty < +\infty$,
- $a + (+\infty) = (+\infty) + a = +\infty$ for $a \in \mathbb{R}^* \setminus \{-\infty\}$,
- $a + (-\infty) = (-\infty) + a = -\infty$ for $a \in \mathbb{R}^* \setminus \{+\infty\}$,

Definition

We define the extended real line by setting $\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\}$ with the following extension of operations and ordering from $\mathbb{R}$:

- $a < +\infty$ and $-\infty < a$ for $a \in \mathbb{R}$, $-\infty < +\infty$,
- $a + (+\infty) = (+\infty) + a = +\infty$ for $a \in \mathbb{R}^* \setminus \{-\infty\}$,
- $a + (-\infty) = (-\infty) + a = -\infty$ for $a \in \mathbb{R}^* \setminus \{+\infty\}$,
- $a \cdot (\pm\infty) = (\pm\infty) \cdot a = \pm\infty$ for $a \in \mathbb{R}^*$, $a > 0$,

Definition

We define the extended real line by setting $\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\}$ with the following extension of operations and ordering from $\mathbb{R}$:

- $a < +\infty$ and $-\infty < a$ for $a \in \mathbb{R}$, $-\infty < +\infty$,
- $a + (+\infty) = (+\infty) + a = +\infty$ for $a \in \mathbb{R}^* \setminus \{-\infty\}$,
- $a + (-\infty) = (-\infty) + a = -\infty$ for $a \in \mathbb{R}^* \setminus \{+\infty\}$,
- $a \cdot (\pm\infty) = (\pm\infty) \cdot a = \pm\infty$ for $a \in \mathbb{R}^*$, $a > 0$,
- $a \cdot (\pm\infty) = (\pm\infty) \cdot a = \mp\infty$ for $a \in \mathbb{R}^*$, $a < 0$,

Definition

We define the extended real line by setting $\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\}$ with the following extension of operations and ordering from $\mathbb{R}$:

- $a < +\infty$ and $-\infty < a$ for $a \in \mathbb{R}$, $-\infty < +\infty$,
- $a + (+\infty) = (+\infty) + a = +\infty$ for $a \in \mathbb{R}^* \setminus \{-\infty\}$,
- $a + (-\infty) = (-\infty) + a = -\infty$ for $a \in \mathbb{R}^* \setminus \{+\infty\}$,
- $a \cdot (\pm\infty) = (\pm\infty) \cdot a = \pm\infty$ for $a \in \mathbb{R}^*$, $a > 0$,
- $a \cdot (\pm\infty) = (\pm\infty) \cdot a = \mp\infty$ for $a \in \mathbb{R}^*$, $a < 0$,
- $\frac{a}{\pm\infty} = 0$ pro $a \in \mathbb{R}$.

The following operations are not defined:

- $(-\infty) + (+\infty)$, $(+\infty) + (-\infty)$, $(+\infty) - (+\infty)$, $(-\infty) - (-\infty)$,

The following operations are not defined:

- $(-\infty) + (+\infty)$, $(+\infty) + (-\infty)$, $(+\infty) - (+\infty)$, $(-\infty) - (-\infty)$,
- $(+\infty) \cdot 0$, $0 \cdot (+\infty)$, $(-\infty) \cdot 0$, $0 \cdot (-\infty)$,

The following operations are not defined:

- $(-\infty) + (+\infty)$, $(+\infty) + (-\infty)$, $(+\infty) - (+\infty)$, $(-\infty) - (-\infty)$,
- $(+\infty) \cdot 0$, $0 \cdot (+\infty)$, $(-\infty) \cdot 0$, $0 \cdot (-\infty)$,
- $\frac{+\infty}{+\infty}$, $\frac{+\infty}{-\infty}$, $\frac{-\infty}{-\infty}$, $\frac{-\infty}{+\infty}$, $\frac{a}{0}$ for $a \in \mathbb{R}^*$.

Theorem 12' (arithmetics of limits)

Suppose that $\lim a_n = A \in \mathbb{R}^*$ *and* $\lim b_n = B \in \mathbb{R}^*$. *Then*

(i) $\lim(a_n \pm b_n) = A \pm B$ *if the right-hand side is defined,*

Theorem 12' (arithmetics of limits)

Suppose that $\lim a_n = A \in \mathbb{R}^*$ *and* $\lim b_n = B \in \mathbb{R}^*$. *Then*

(i) $\lim(a_n \pm b_n) = A \pm B$ *if the right-hand side is defined,*

(ii) $\lim(a_n \cdot b_n) = A \cdot B$ *if the right-hand side is defined,*

Theorem 12' (arithmetics of limits)

Suppose that $\lim a_n = A \in \mathbb{R}^*$ *and* $\lim b_n = B \in \mathbb{R}^*$. *Then*

(i) $\lim(a_n \pm b_n) = A \pm B$ *if the right-hand side is defined,*

(ii) $\lim(a_n \cdot b_n) = A \cdot B$ *if the right-hand side is defined,*

(iii) $\lim a_n/b_n = A/B$ *if the right-hand side is defined.*

Theorem 12' (arithmetics of limits)

Suppose that $\lim a_n = A \in \mathbb{R}^*$ *and* $\lim b_n = B \in \mathbb{R}^*$. *Then*

(i) $\lim(a_n \pm b_n) = A \pm B$ *if the right-hand side is defined,*

(ii) $\lim(a_n \cdot b_n) = A \cdot B$ *if the right-hand side is defined,*

(iii) $\lim a_n/b_n = A/B$ *if the right-hand side is defined.*

Theorem 16

Suppose that $\lim a_n = A \in \mathbb{R}^*$, $A > 0$, $\lim b_n = 0$ *and there is* $n_0 \in \mathbb{N}$ *such that we have* $b_n > 0$ *for every* $n \in \mathbb{N}$, $n \geq n_0$. *Then* $\lim a_n/b_n = +\infty$.

Theorem 14 (limits and ordering) and Theorem 15 (sandwich theorem) hold also for infinite limits. Even the following modification holds:

Theorem 15' (one policeman)

Let $\{a_n\}$ and $\{b_n\}$ be two sequences.

- *If $\lim a_n = +\infty$ and there is $n_0 \in \mathbb{N}$ such that $b_n \geq a_n$ for every $n \in \mathbb{N}$, $n \geq n_0$, then $\lim b_n = +\infty$.*
- *If $\lim a_n = -\infty$ and there is $n_0 \in \mathbb{N}$ such that $b_n \leq a_n$ for every $n \in \mathbb{N}$, $n \geq n_0$, then $\lim b_n = -\infty$.*

Definition

Let $A \subset \mathbb{R}$ be non-empty. If A is not bounded from above, then we define $\sup A = +\infty$. If A is not bounded from below, then we define $\inf A = -\infty$.

Definition

Let $A \subset \mathbb{R}$ be non-empty. If A is not bounded from above, then we define $\sup A = +\infty$. If A is not bounded from below, then we define $\inf A = -\infty$.

Lemma 17

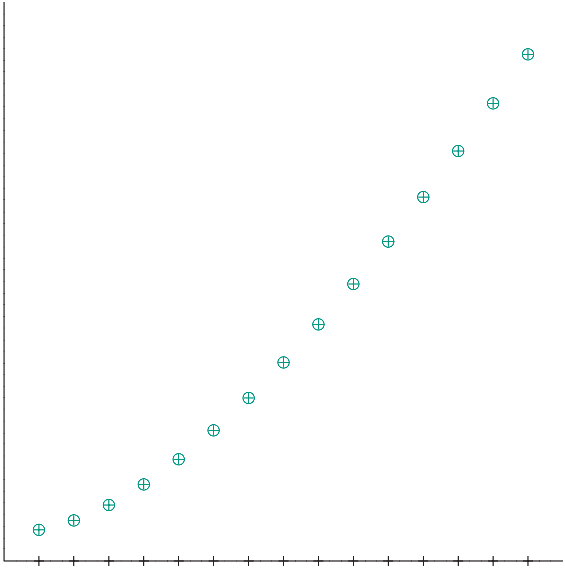
Let $M \subset \mathbb{R}$ be non-empty and $G \in \mathbb{R}^*$. Then the following statements are equivalent:

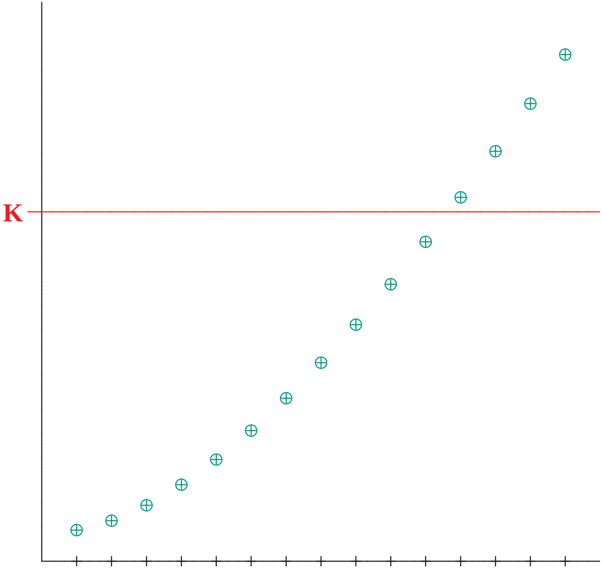
(i) $G = \sup M$.

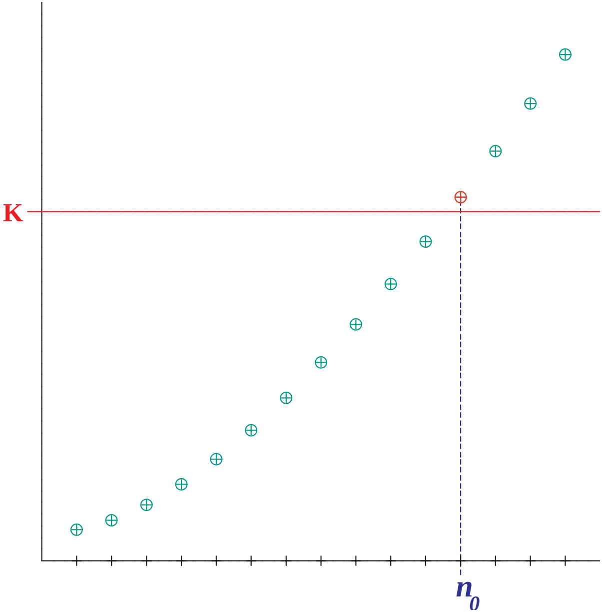
(ii) *The number G is an upper bound of M and there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of members of M such that $\lim x_n = G$.*

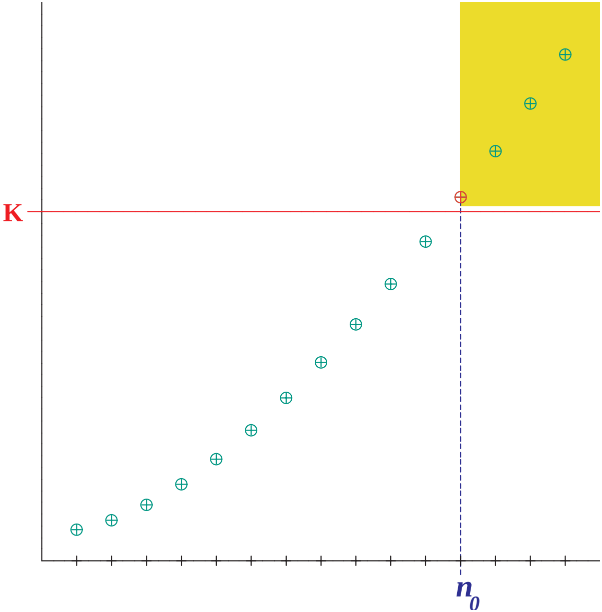
Theorem 18 (limit of a monotone sequence)

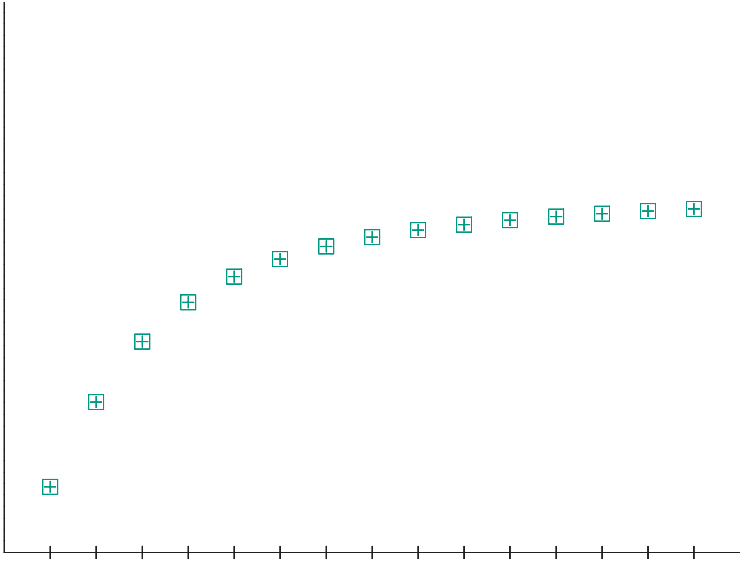
Every monotone sequence has a limit. If $\{a_n\}$ *is non-decreasing, then* $\lim a_n = \sup\{a_n;\ n \in \mathbb{N}\}$*. If* $\{a_n\}$ *is non-increasing, then* $\lim a_n = \inf\{a_n;\ n \in \mathbb{N}\}$*.*

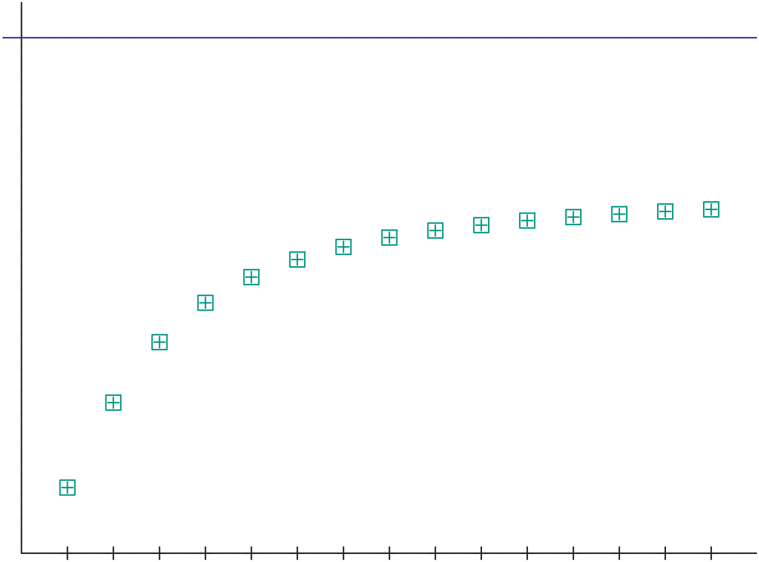


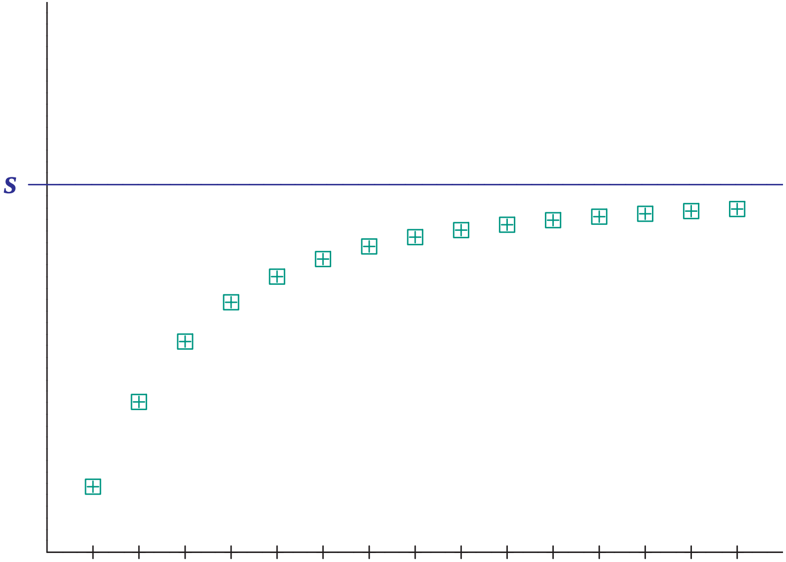

K

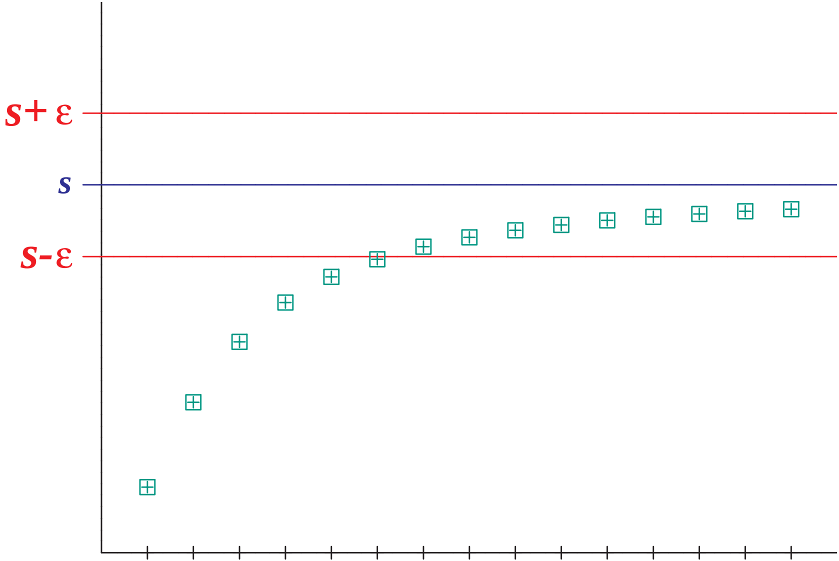

K
n_0

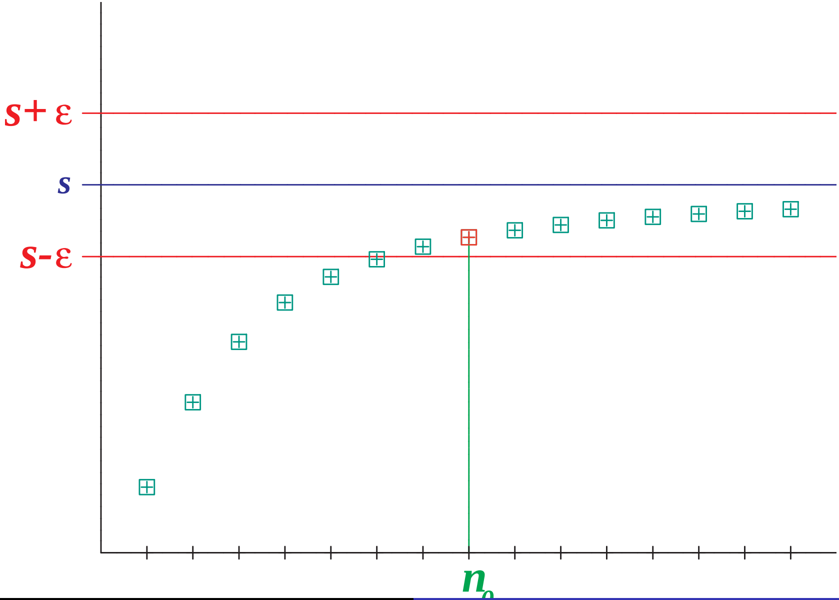

K
n0

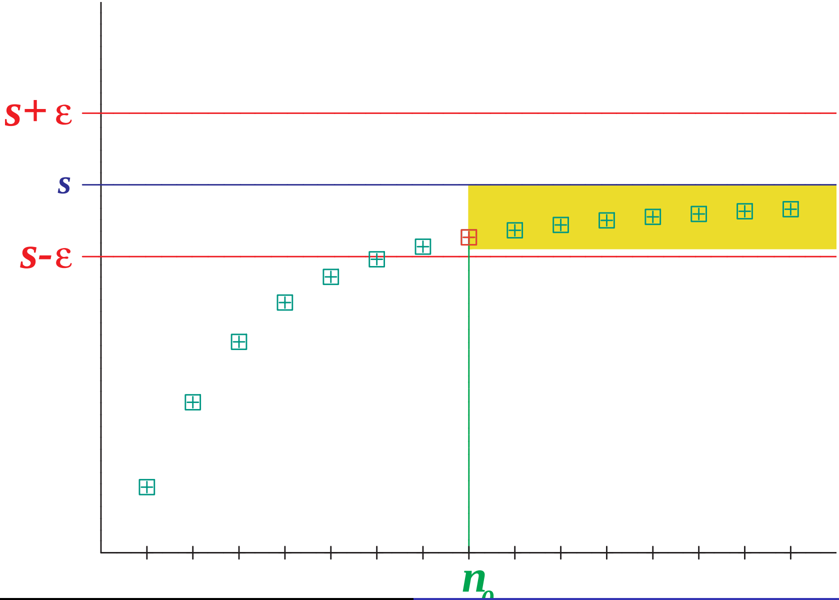





s


s+ ε
s
s- ε

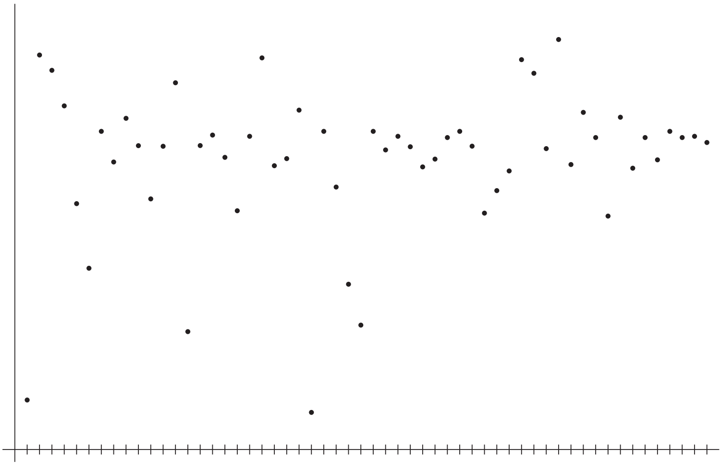

s+ ε
s
s-ε
n0

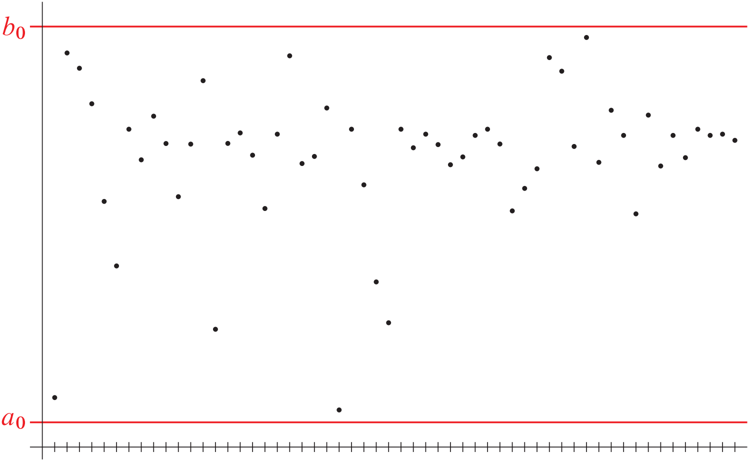


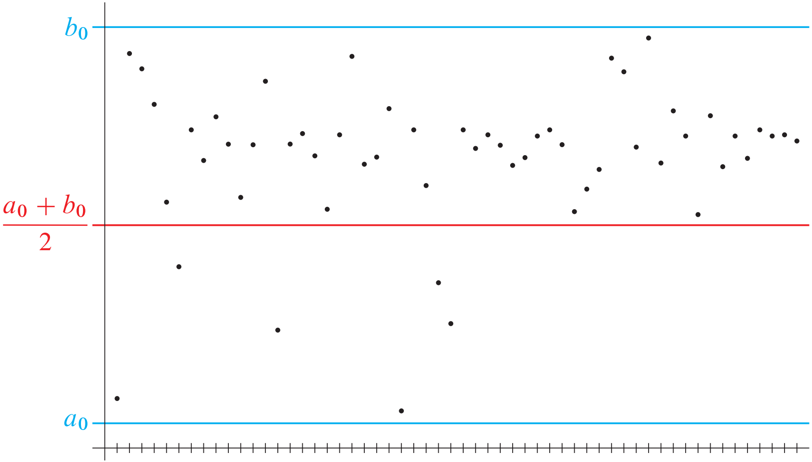
s+ ε
s
s- ε
n0

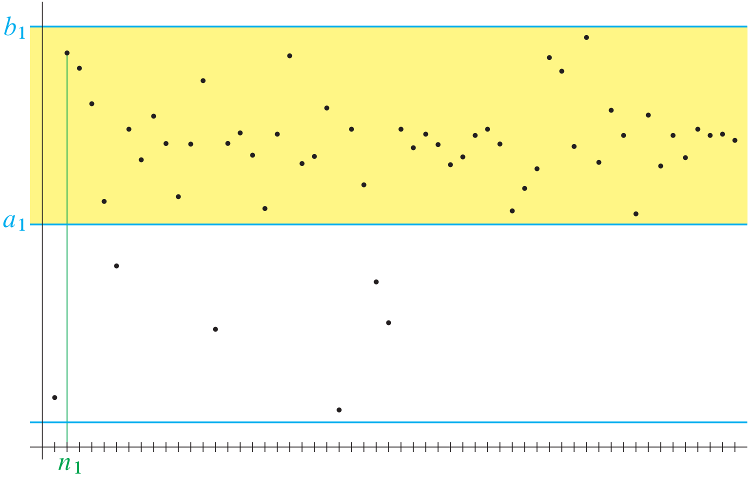
Theorem 19 (Bolzano-Weierstraß)

Every bounded sequence contains a convergent subsequence.

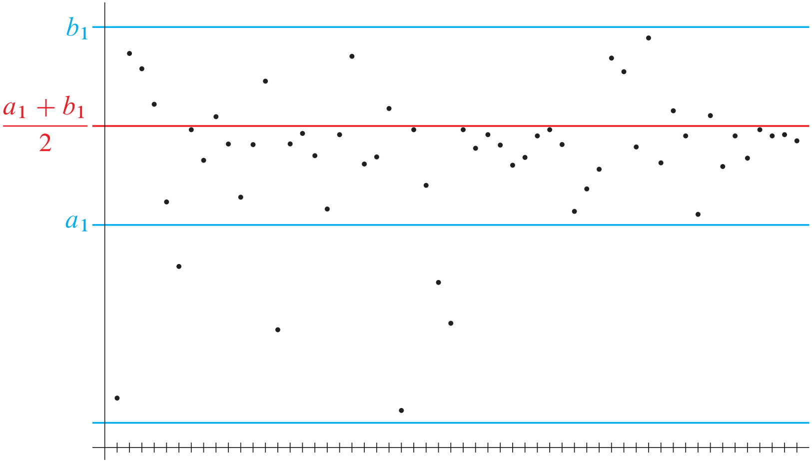


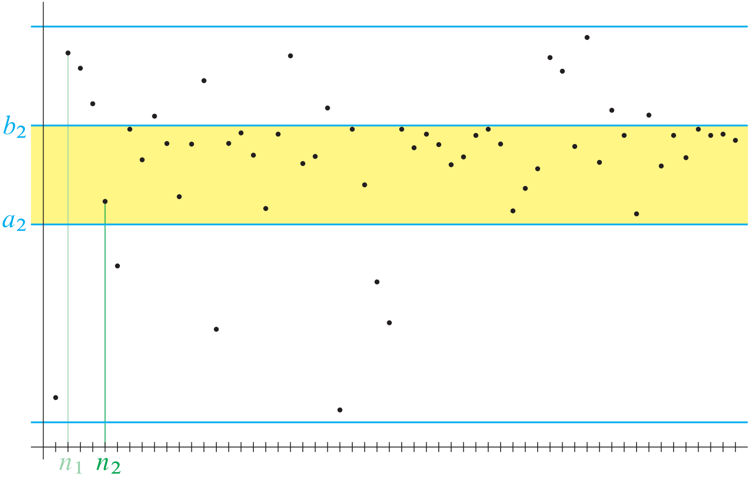

b_0
a_0


b_0
$\frac{a_0 + b_0}{2}$
a_0

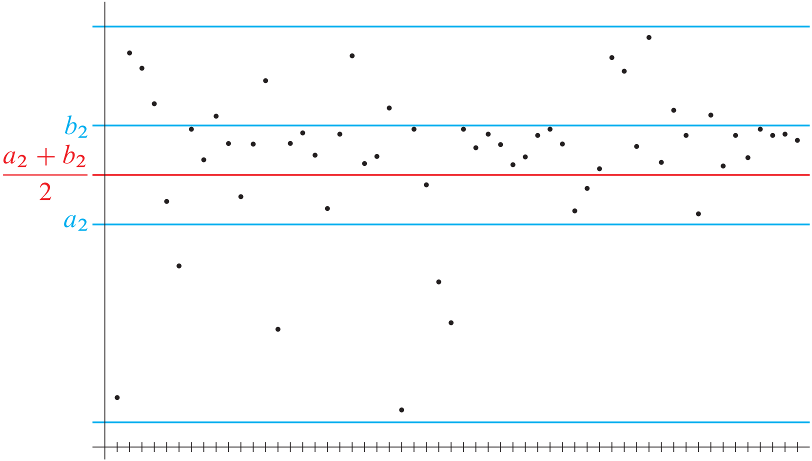


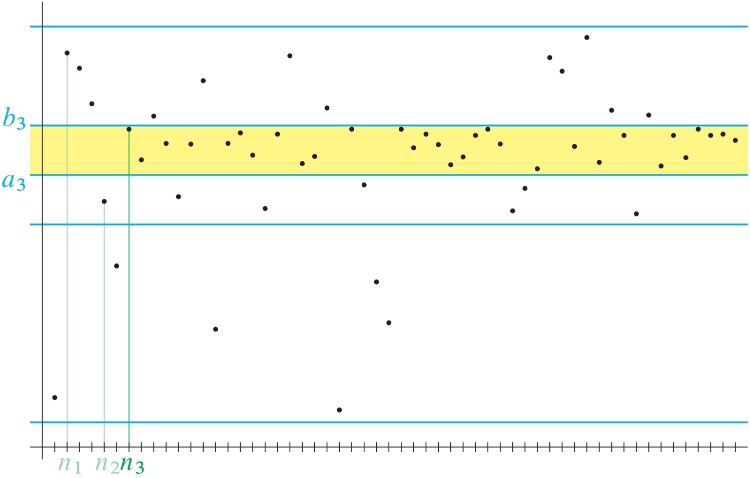
b_1
a_1
n_1

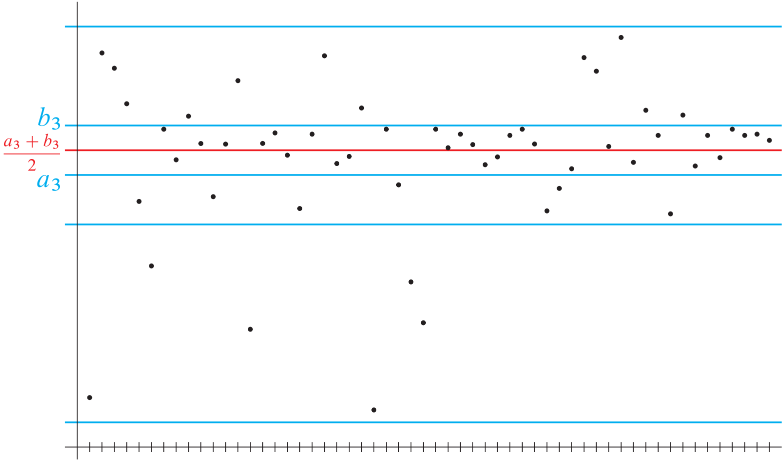

b_1
$\frac{a_1 + b_1}{2}$
a_1



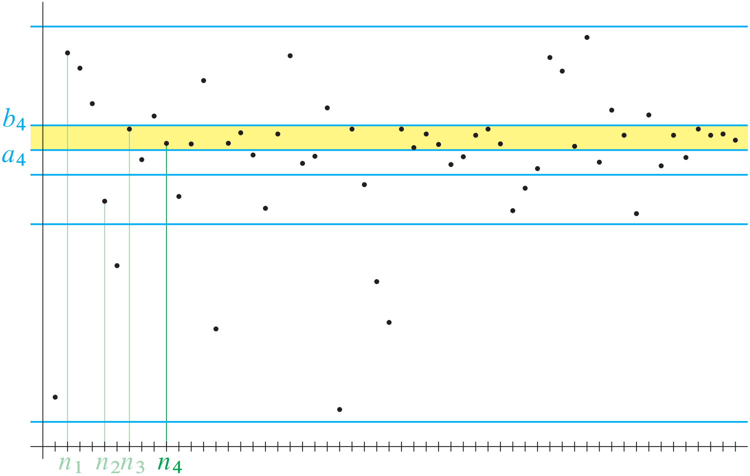
b_2
a_2
n_1
n_2


b_2
$\frac{a_2 + b_2}{2}$
a_2

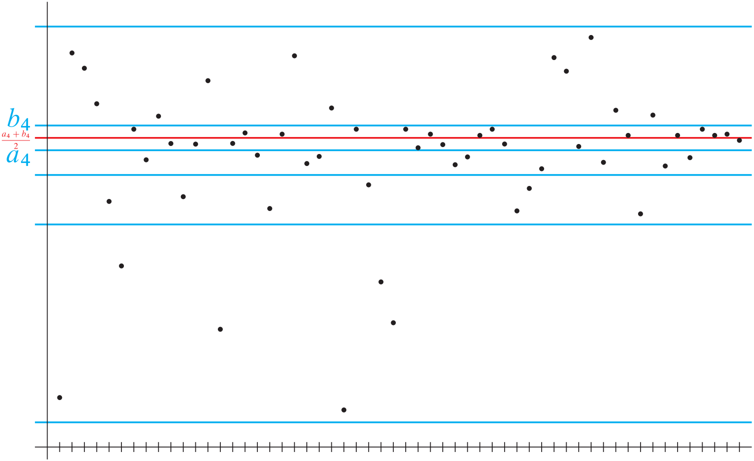

b_3
a_3
n_1
n_2
n_3



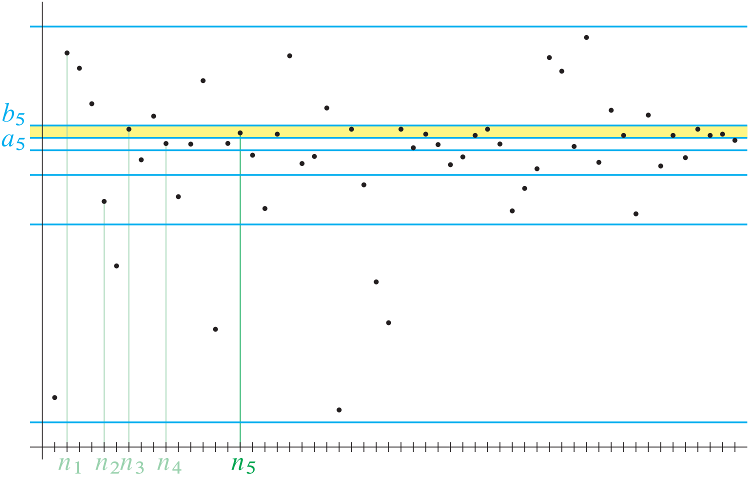
b_3
$\frac{a_3 + b_3}{2}$
a_3



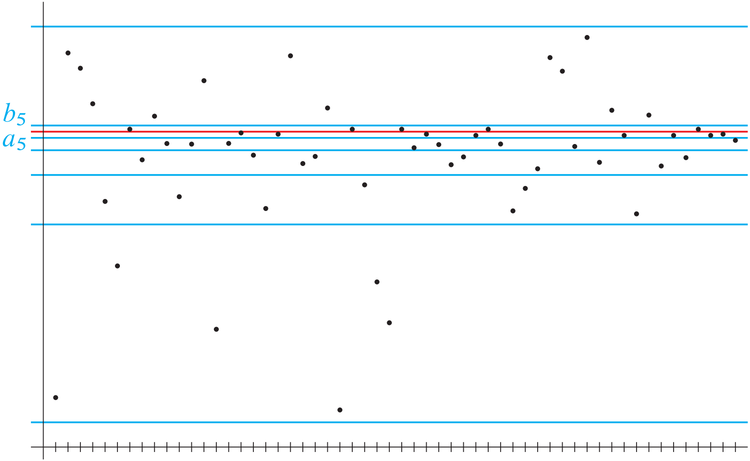
b_4
a_4
n_1
n_2
n_3
n_4

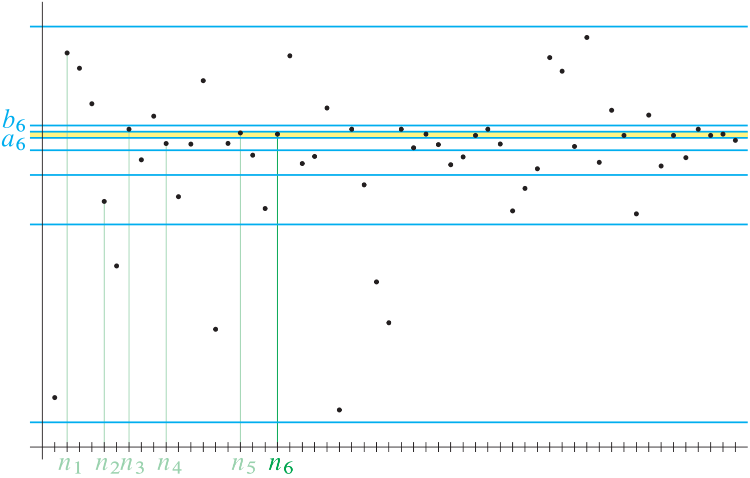


b_4
$\frac{a_4+b_4}{2}$
a_4



b_5
a_5
n_1
n_2
n_3
n_4
n_5


b_5
a_5



b_6
a_6
n_1
n_2
n_3
n_4
n_5
n_6

Definition

Let A and B be sets. A mapping f from A to B is a rule which assigns to each member x of the set A a unique member y of the set B. This element y is denoted by the symbol $f(x)$.

Definition

Let A and B be sets. A mapping f from A to B is a rule which assigns to each member x of the set A a unique member y of the set B. This element y is denoted by the symbol $f(x)$. The element y is called an image of x and the element x is called a pre-image of y.

Definition

Let A and B be sets. A mapping f from A to B is a rule which assigns to each member x of the set A a unique member y of the set B. This element y is denoted by the symbol $f(x)$. The element y is called an image of x and the element x is called a pre-image of y.

- By $f\colon A \to B$ we denote the fact that f is a mapping from A to B.

Definition

Let A and B be sets. A mapping f from A to B is a rule which assigns to each member x of the set A a unique member y of the set B. This element y is denoted by the symbol $f(x)$. The element y is called an image of x and the element x is called a pre-image of y.

- By $f\colon A \to B$ we denote the fact that f is a mapping from A to B.
- By $f\colon x \mapsto f(x)$ we denote the fact that the mapping f assigns $f(x)$ to an element x.

Definition

Let A and B be sets. A mapping f from A to B is a rule which assigns to each member x of the set A a unique member y of the set B. This element y is denoted by the symbol $f(x)$. The element y is called an image of x and the element x is called a pre-image of y.

- By $f\colon A \to B$ we denote the fact that f is a mapping from A to B.
- By $f\colon x \mapsto f(x)$ we denote the fact that the mapping f assigns $f(x)$ to an element x.
- The set A from the definition of the mapping f is called the domain of f and it is denoted by D_f.

Definition

Let $f\colon A \to B$ be a mapping.

- The subset $G_f = \{[x, y] \in A \times B;\ x \in A, y = f(x)\}$ of the Cartesian product $A \times B$ is called the graph of the mapping f.

Definition

Let $f\colon A \to B$ be a mapping.

- The subset $G_f = \{[x, y] \in A \times B;\ x \in A, y = f(x)\}$ of the Cartesian product $A \times B$ is called the graph of the mapping f.
- The image of the set $M \subset A$ under the mapping f is the set

$$f(M) = \{y \in B;\ \exists x \in M\colon f(x) = y\} \quad (= \{f(x);\ x \in M\}).$$

Definition

Let $f\colon A \to B$ be a mapping.

- The subset $G_f = \{[x, y] \in A \times B;\ x \in A, y = f(x)\}$ of the Cartesian product $A \times B$ is called the graph of the mapping f.
- The image of the set $M \subset A$ under the mapping f is the set

$$f(M) = \{y \in B;\ \exists x \in M\colon f(x) = y\} \quad (= \{f(x);\ x \in M\}).$$

- The set $f(A)$ is called the range of the mapping f, it is denoted by R_f.

Definition

Let $f\colon A \to B$ be a mapping.

- The subset $G_f = \{[x, y] \in A \times B;\ x \in A, y = f(x)\}$ of the Cartesian product $A \times B$ is called the graph of the mapping f.
- The image of the set $M \subset A$ under the mapping f is the set

$$f(M) = \{y \in B;\ \exists x \in M\colon f(x) = y\} \quad (= \{f(x);\ x \in M\}).$$

- The set $f(A)$ is called the range of the mapping f, it is denoted by R_f.
- The pre-image of the set $W \subset B$ under the mapping f is the set

$$f_{-1}(W) = \{x \in A;\ f(x) \in W\}.$$

Remark

Let $f\colon A \to B$, $X, Y \subset A$, $U, V \subset B$. Then

- $f_{-1}(U \cup V) = f_{-1}(U) \cup f_{-1}(V)$,

Remark

Let $f\colon A \to B$, $X, Y \subset A$, $U, V \subset B$. Then

- $f_{-1}(U \cup V) = f_{-1}(U) \cup f_{-1}(V)$,
- $f_{-1}(U \cap V) = f_{-1}(U) \cap f_{-1}(V)$,

Remark

Let $f\colon A \to B$, $X, Y \subset A$, $U, V \subset B$. Then

- $f_{-1}(U \cup V) = f_{-1}(U) \cup f_{-1}(V)$,
- $f_{-1}(U \cap V) = f_{-1}(U) \cap f_{-1}(V)$,
- $f(X \cup Y) = f(X) \cup f(Y)$,

Remark

Let $f\colon A \to B$, $X, Y \subset A$, $U, V \subset B$. Then

- $f_{-1}(U \cup V) = f_{-1}(U) \cup f_{-1}(V)$,
- $f_{-1}(U \cap V) = f_{-1}(U) \cap f_{-1}(V)$,
- $f(X \cup Y) = f(X) \cup f(Y)$,
- $f(X \cap Y) \subset f(X) \cap f(Y)$.

Definition

Let A, B, C be sets, $C \subset A$ and $f\colon A \to B$. The mapping $\tilde{f}\colon C \to B$ given by the formula $\tilde{f}(x) = f(x)$ for each $x \in C$ is called the restriction of the mapping f to the set C. It is denoted by $f|_C$.

Definition

Let $f\colon A \to B$ and $g\colon B \to C$ be two mappings. The symbol $g \circ f$ denotes a mapping from A to C defined by

$$(g \circ f)(x) = g(f(x)).$$

This mapping is called a compound mapping or a composition of the mapping f and the mapping g.

Definition

We say that a mapping $f\colon A \to B$

- maps the set A onto the set B if $f(A) = B$, i.e. if to each $y \in B$ there exist $x \in A$ such that $f(x) = y$;

Definition

We say that a mapping $f\colon A \to B$

- maps the set A onto the set B if $f(A) = B$, i.e. if to each $y \in B$ there exist $x \in A$ such that $f(x) = y$;
- is one-to-one (or injective) if images of different elements differ, i.e.

$$\forall x_1, x_2 \in A\colon x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2),$$

Definition

We say that a mapping $f\colon A \to B$

- maps the set A onto the set B if $f(A) = B$, i.e. if to each $y \in B$ there exist $x \in A$ such that $f(x) = y$;
- is one-to-one (or injective) if images of different elements differ, i.e.

$$\forall x_1, x_2 \in A\colon x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2),$$

- is a bijection of A onto B (or a bijective mapping), if it is at the same time one-to-one and maps A onto B.

Definition

Let $f\colon A \to B$ be bijective (i.e. one-to-one and onto). An inverse mapping $f^{-1}\colon B \to A$ is a mapping that to each $y \in B$ assigns a (uniquely determined) element $x \in A$ satisfying $f(x) = y$.

IV. Functions of one real variable

IV. Functions of one real variable

Definition

A function f of one real variable (or a function for short) is a mapping $f\colon M \to \mathbb{R}$, where M is a subset of real numbers.

Definition

A function $f\colon J \to \mathbb{R}$ is increasing on an interval J, if for each pair $x_1, x_2 \in J$, $x_1 < x_2$ the inequality $f(x_1) < f(x_2)$ holds. Analogously we define a function decreasing (non-decreasing, non-increasing) on an interval J.

Definition

A function $f\colon J \to \mathbb{R}$ is increasing on an interval J, if for each pair $x_1, x_2 \in J$, $x_1 < x_2$ the inequality $f(x_1) < f(x_2)$ holds. Analogously we define a function decreasing (non-decreasing, non-increasing) on an interval J.

Definition

A monotone function on an interval J is a function which is non-decreasing or non-increasing on J.

Definition

A function $f\colon J \to \mathbb{R}$ is increasing on an interval J, if for each pair $x_1, x_2 \in J$, $x_1 < x_2$ the inequality $f(x_1) < f(x_2)$ holds. Analogously we define a function decreasing (non-decreasing, non-increasing) on an interval J.

Definition

A monotone function on an interval J is a function which is non-decreasing or non-increasing on J. A strictly monotone function on an interval J is a function which is increasing or decreasing on J.

Definition

Let f be a function and $M \subset D_f$. We say that f is

- bounded from above on M if there is $K \in \mathbb{R}$ such that $f(x) \leq K$ for all $x \in M$,

Definition

Let f be a function and $M \subset D_f$. We say that f is

- bounded from above on M if there is $K \in \mathbb{R}$ such that $f(x) \leq K$ for all $x \in M$,
- bounded from below on M if there is $K \in \mathbb{R}$ such that $f(x) \geq K$ for all $x \in M$,

Definition

Let f be a function and $M \subset D_f$. We say that f is

- bounded from above on M if there is $K \in \mathbb{R}$ such that $f(x) \leq K$ for all $x \in M$,
- bounded from below on M if there is $K \in \mathbb{R}$ such that $f(x) \geq K$ for all $x \in M$,
- bounded on M if there is $K \in \mathbb{R}$ such that $|f(x)| \leq K$ for all $x \in M$,

Definition

Let f be a function and $M \subset D_f$. We say that f is

- bounded from above on M if there is $K \in \mathbb{R}$ such that $f(x) \leq K$ for all $x \in M$,
- bounded from below on M if there is $K \in \mathbb{R}$ such that $f(x) \geq K$ for all $x \in M$,
- bounded on M if there is $K \in \mathbb{R}$ such that $|f(x)| \leq K$ for all $x \in M$,
- odd if for each $x \in D_f$ we have $-x \in D_f$ and $f(-x) = -f(x)$,

Definition

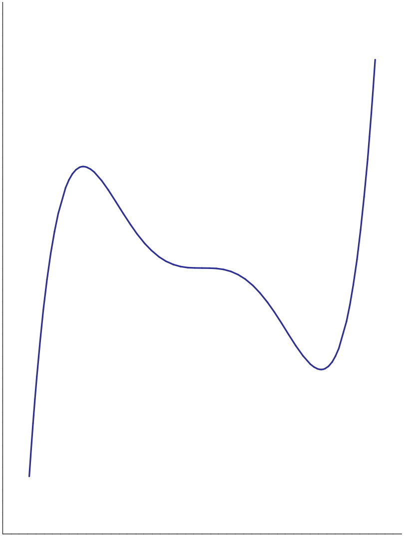
Let f be a function and $M \subset D_f$. We say that f is

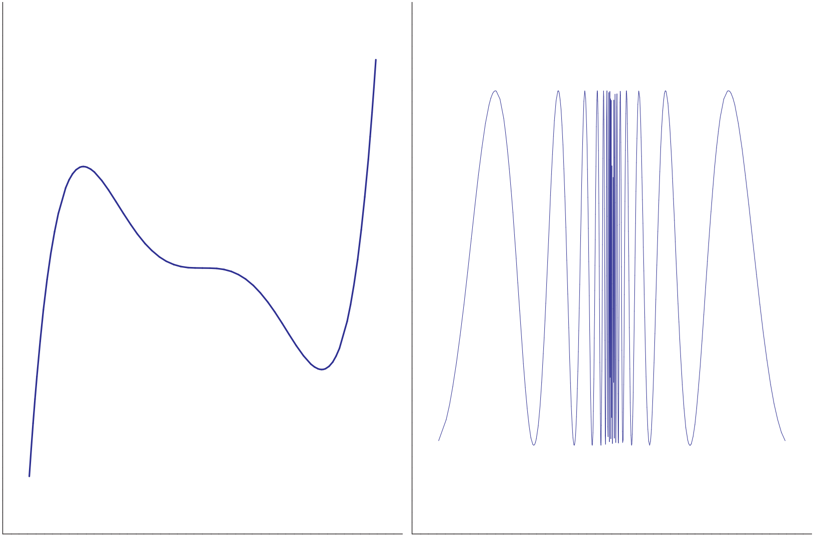
- bounded from above on M if there is $K \in \mathbb{R}$ such that $f(x) \leq K$ for all $x \in M$,
- bounded from below on M if there is $K \in \mathbb{R}$ such that $f(x) \geq K$ for all $x \in M$,
- bounded on M if there is $K \in \mathbb{R}$ such that $|f(x)| \leq K$ for all $x \in M$,
- odd if for each $x \in D_f$ we have $-x \in D_f$ and $f(-x) = -f(x)$,
- even if for each $x \in D_f$ we have $-x \in D_f$ and $f(-x) = f(x)$,

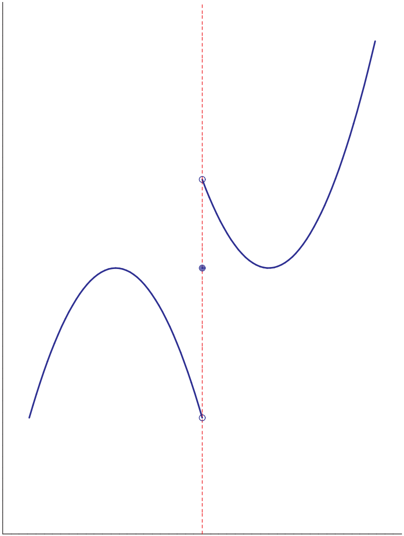
Definition

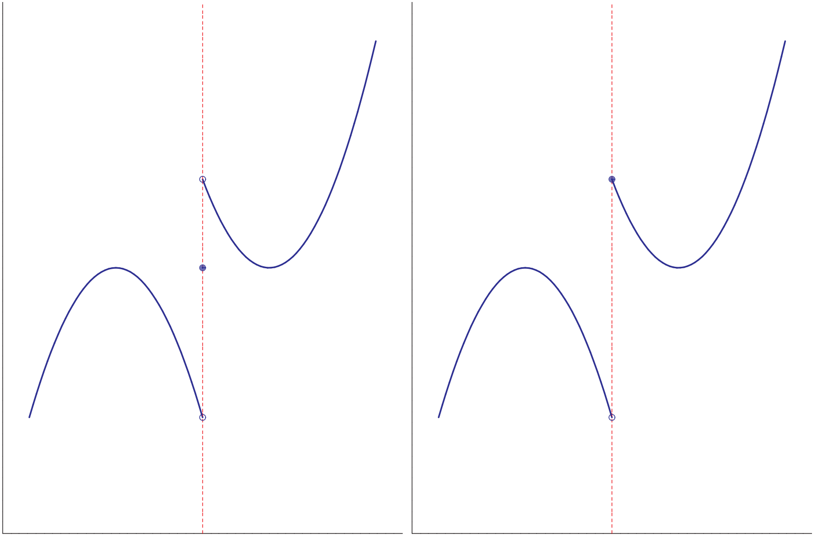
Let f be a function and $M \subset D_f$. We say that f is

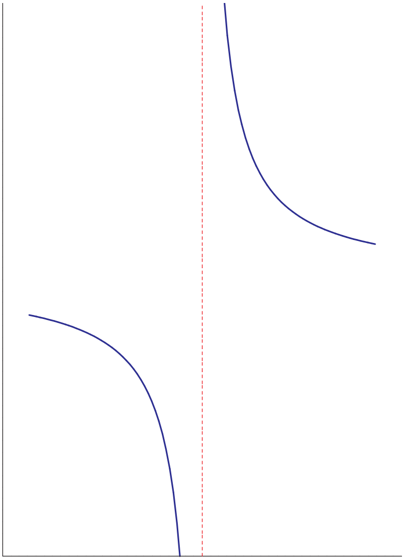
- bounded from above on M if there is $K \in \mathbb{R}$ such that $f(x) \leq K$ for all $x \in M$,
- bounded from below on M if there is $K \in \mathbb{R}$ such that $f(x) \geq K$ for all $x \in M$,
- bounded on M if there is $K \in \mathbb{R}$ such that $|f(x)| \leq K$ for all $x \in M$,
- odd if for each $x \in D_f$ we have $-x \in D_f$ and $f(-x) = -f(x)$,
- even if for each $x \in D_f$ we have $-x \in D_f$ and $f(-x) = f(x)$,
- periodic with a period a, where $a \in \mathbb{R}$, $a > 0$, if for each $x \in D_f$ we have $x + a \in D_f$, $x - a \in D_f$ and $f(x + a) = f(x - a) = f(x)$.

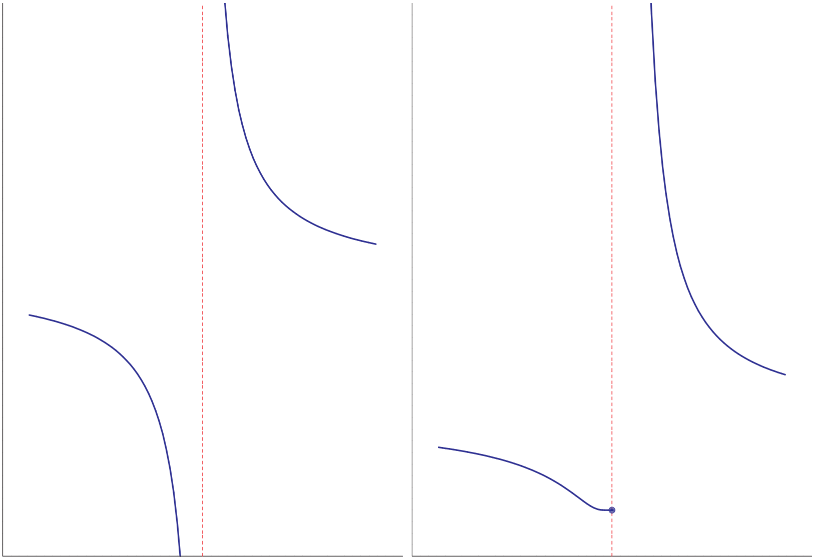












Definition

Let $c \in \mathbb{R}$ and $\varepsilon > 0$. We define

- a neighbourhood of a point c with radius ε by $B(c, \varepsilon) = (c - \varepsilon, c + \varepsilon)$,

Definition

Let $c \in \mathbb{R}$ and $\varepsilon > 0$. We define

- a neighbourhood of a point c with radius ε by $B(c, \varepsilon) = (c - \varepsilon, c + \varepsilon)$,
- a punctured neighbourhood of a point c with radius ε by $P(c, \varepsilon) = (c - \varepsilon, c + \varepsilon) \setminus \{c\}$.

Definition

We say that $A \in \mathbb{R}$ is a limit of a function f at a point $c \in \mathbb{R}$ if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \; \exists \delta \in \mathbb{R}, \delta > 0 \; \forall x \in P(c, \delta) \colon f(x) \in B(A, \varepsilon).$$

Definition

We say that $A \in \mathbb{R}$ is a limit of a function f at a point $c \in \mathbb{R}$ if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists \delta \in \mathbb{R}, \delta > 0\ \forall x \in P(c, \delta)\colon\ f(x) \in B(A, \varepsilon).$$

Theorem 20 (uniqueness of a limit)

Let f be a function and $c \in \mathbb{R}$. Then f has a most one limit $A \in \mathbb{R}$ at c.

Definition

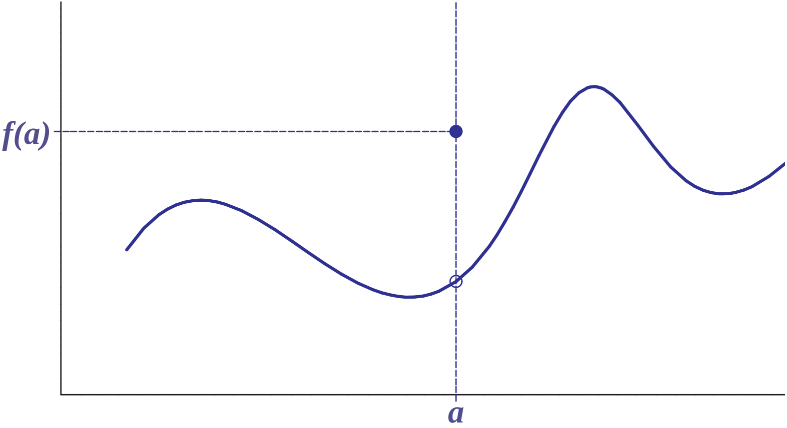
We say that $A \in \mathbb{R}$ is a limit of a function f at a point $c \in \mathbb{R}$ if

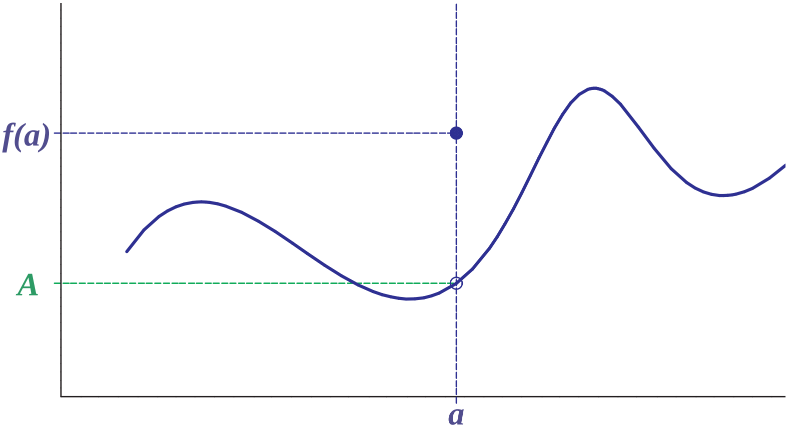
$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists \delta \in \mathbb{R}, \delta > 0\ \forall x \in P(c, \delta) \colon f(x) \in B(A, \varepsilon).$$

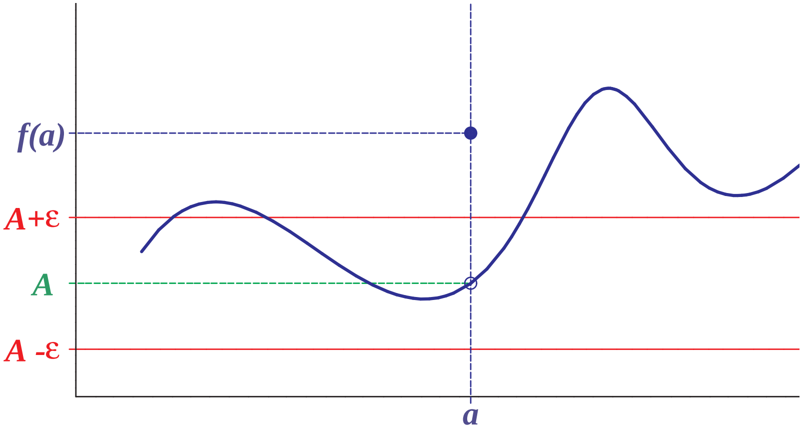
Theorem 20 (uniqueness of a limit)

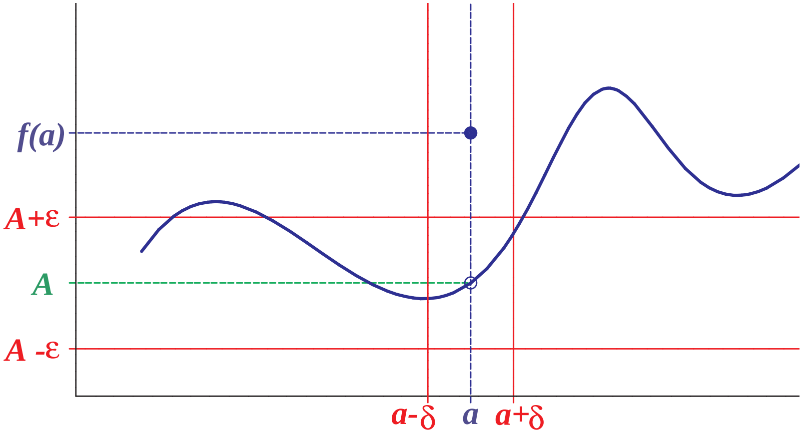
Let f be a function and $c \in \mathbb{R}$. Then f has a most one limit $A \in \mathbb{R}$ at c.

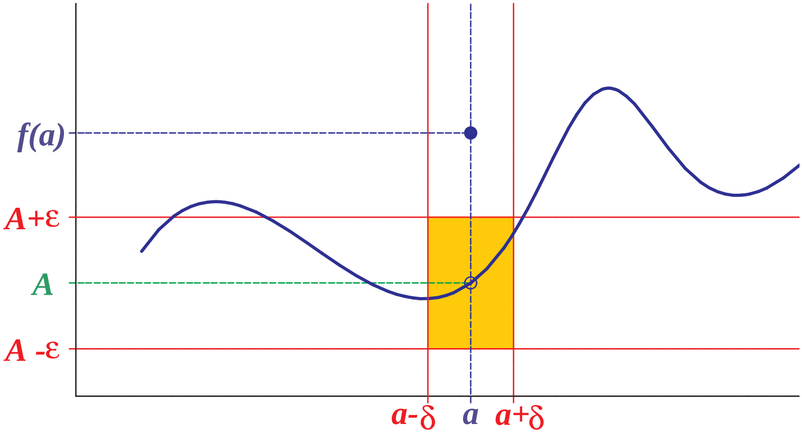
The fact that f has a limit $A \in \mathbb{R}$ at $c \in \mathbb{R}$ is denoted by $\lim\limits_{x \to c} f(x) = A$.


f(a)
a

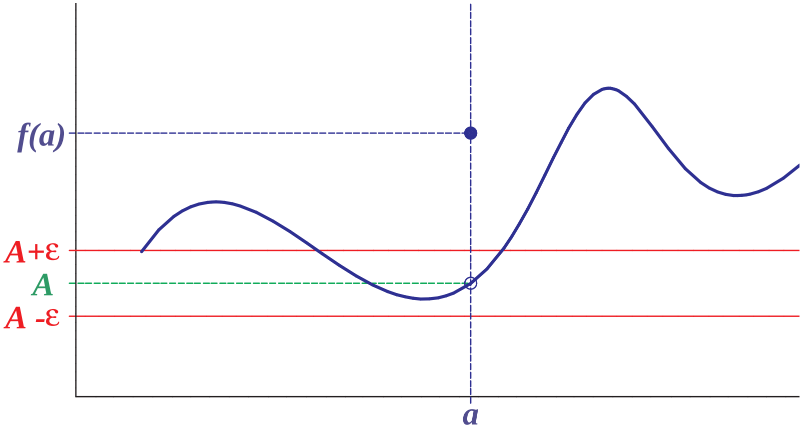

f(a)
A
a

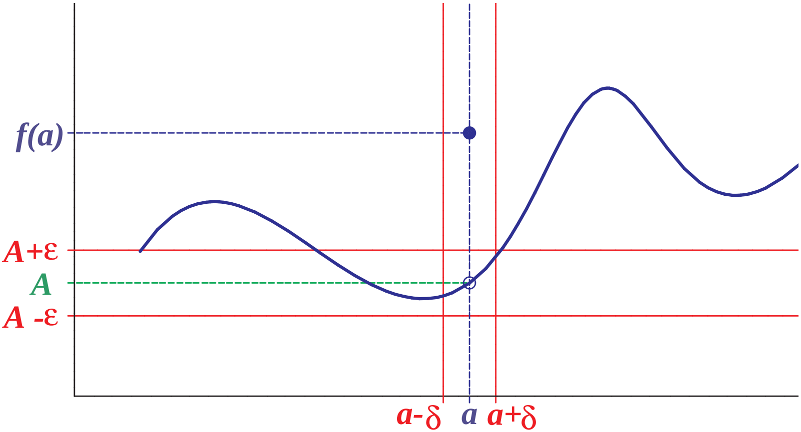

f(a)
A+ε
A
A -ε
a

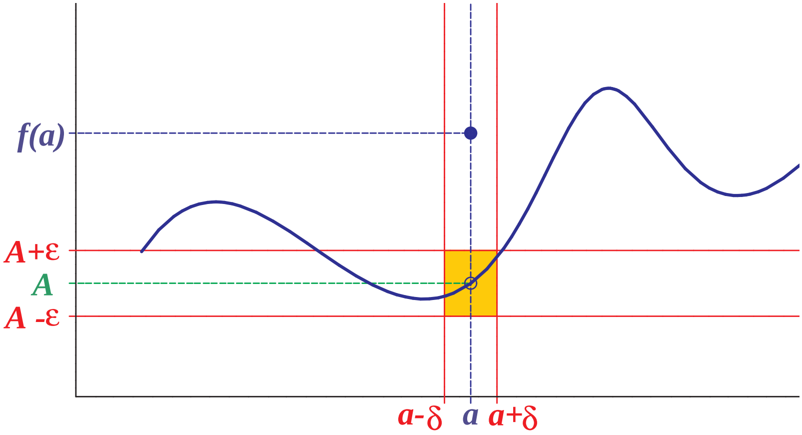

f(a)
A+ε
A
A -ε
a-δ
a
a+δ



f(a)
A+ε
A
A -ε
a-δ
a
a+δ


f(a)
A+ε
A
A -ε
a


f(a)
A+ε
A
A -ε
a-δ
a
a+δ


f(a)
A+ε
A
A -ε
a-δ
a
a+δ

Definition

We say that a function f is continuous at a point $c \in \mathbb{R}$ if

$$\lim_{x \to c} f(x) = f(c).$$

Definition

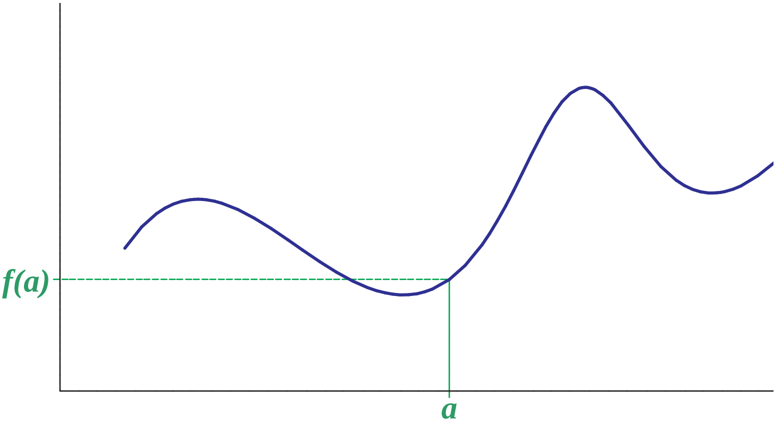
We say that a function f is continuous at a point $c \in \mathbb{R}$ if

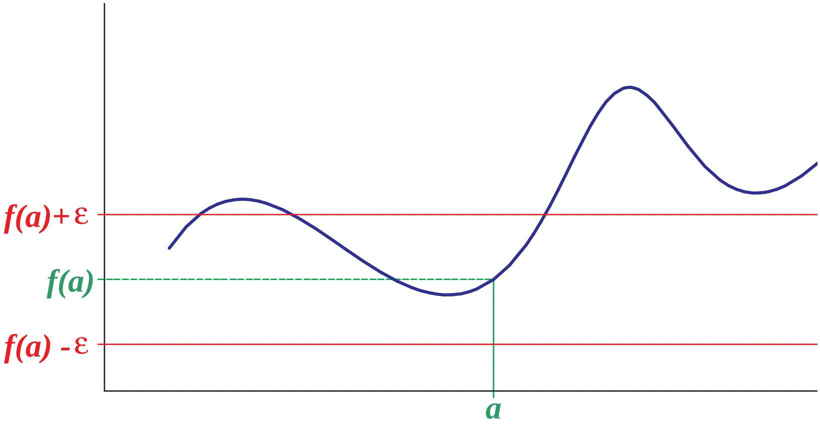
$$\lim_{x \to c} f(x) = f(c).$$

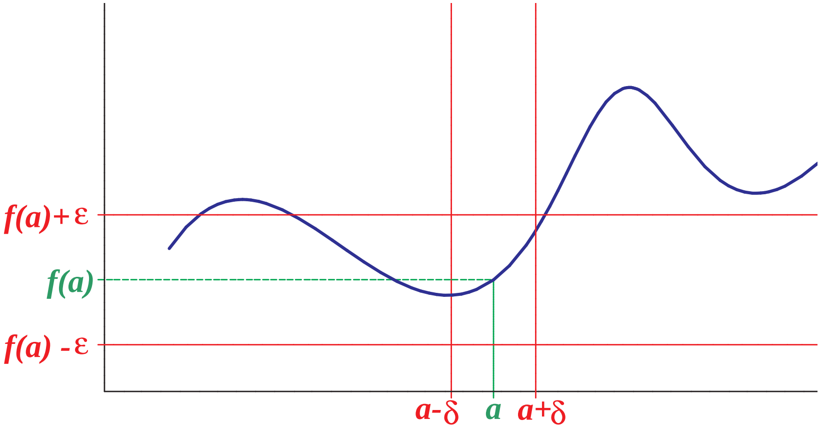
Remark

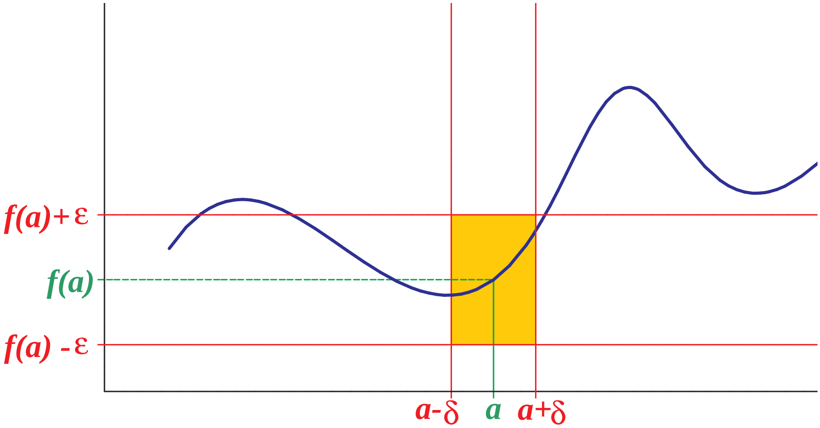
A function f is continuous at a point c if and only if

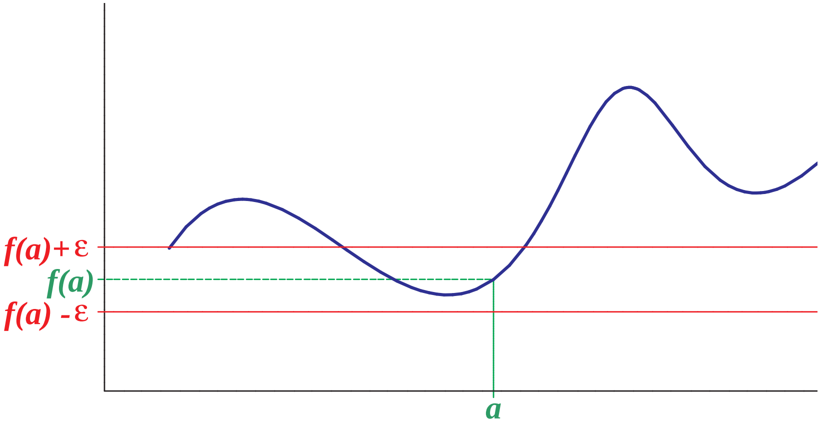
$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \ \exists \delta \in \mathbb{R}, \delta > 0 \ \forall x \in B(c, \delta) \colon f(x) \in B(f(c), \varepsilon).$$

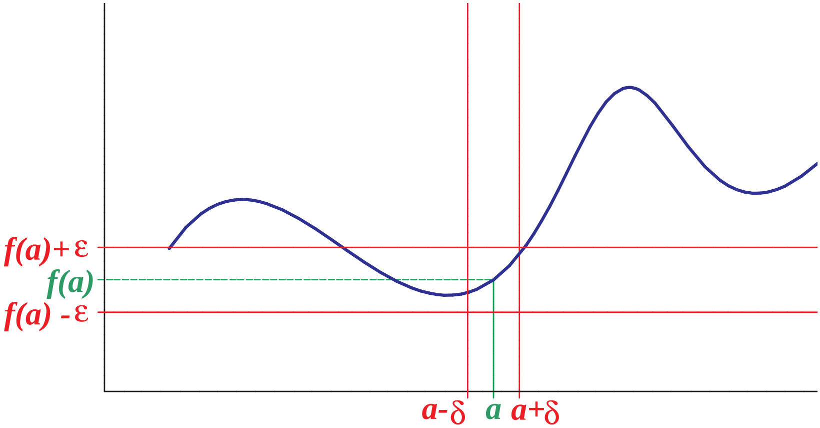

f(a)
a

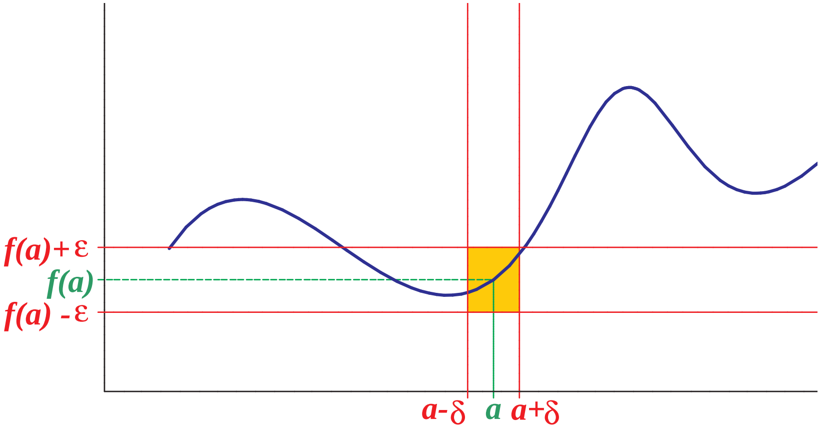

f(a)+ε
f(a)
f(a) -ε
a

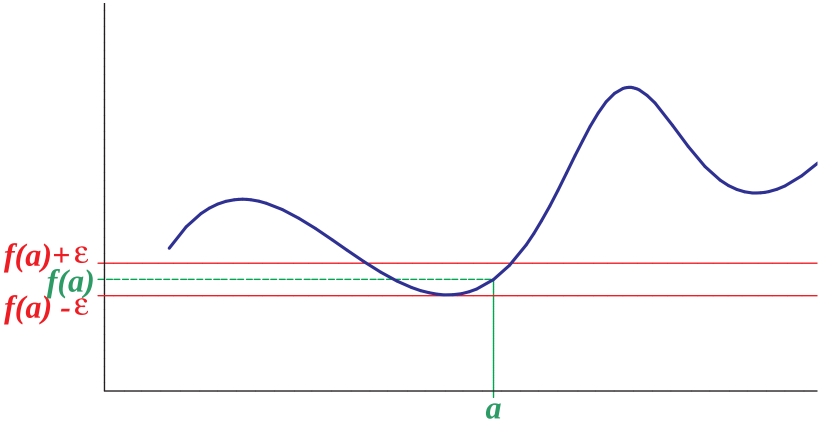

f(a)+ε
f(a)
f(a) -ε
a-δ
a
a+δ

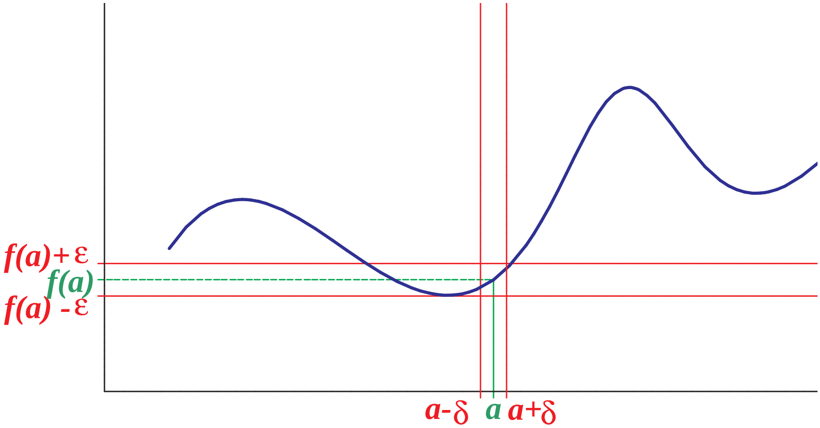

f(a)+ε
f(a)
f(a) -ε
a-δ
a
a+δ

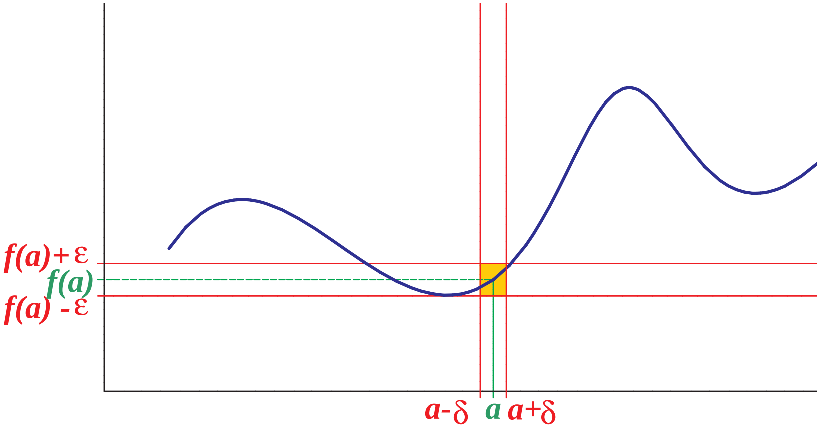

f(a)+ε
f(a)
f(a) -ε
a


f(a)+ε
f(a)
f(a) -ε
a-δ
a
a+δ


f(a)+ε
f(a)
f(a) -ε
a-δ
a
a+δ


f(a)+ε
f(a)
f(a) -ε
a


f(a)+ε
f(a)
f(a) -ε
a-δ
a
a+δ


f(a)+ε
f(a)
f(a) -ε
a-δ
a
a+δ

Definition

Let $\varepsilon > 0$. A neighbourhood and a punctured neighbourhood of $+\infty$ (resp. $-\infty$) is defined as follows:

$$P(+\infty, \varepsilon) = B(+\infty, \varepsilon) = (1/\varepsilon, +\infty),$$
$$P(-\infty, \varepsilon) = B(-\infty, \varepsilon) = (-\infty, -1/\varepsilon).$$

Definition

Let $\varepsilon > 0$. A neighbourhood and a punctured neighbourhood of $+\infty$ (resp. $-\infty$) is defined as follows:

$$P(+\infty, \varepsilon) = B(+\infty, \varepsilon) = (1/\varepsilon, +\infty),$$
$$P(-\infty, \varepsilon) = B(-\infty, \varepsilon) = (-\infty, -1/\varepsilon).$$

Definition

We say that $A \in \mathbb{R}^*$ is a limit of a function f at $c \in \mathbb{R}^*$ if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \ \exists \delta \in \mathbb{R}, \delta > 0 \ \forall x \in P(c, \delta) \colon f(x) \in B(A, \varepsilon).$$

Definition

Let $\varepsilon > 0$. A neighbourhood and a punctured neighbourhood of $+\infty$ (resp. $-\infty$) is defined as follows:

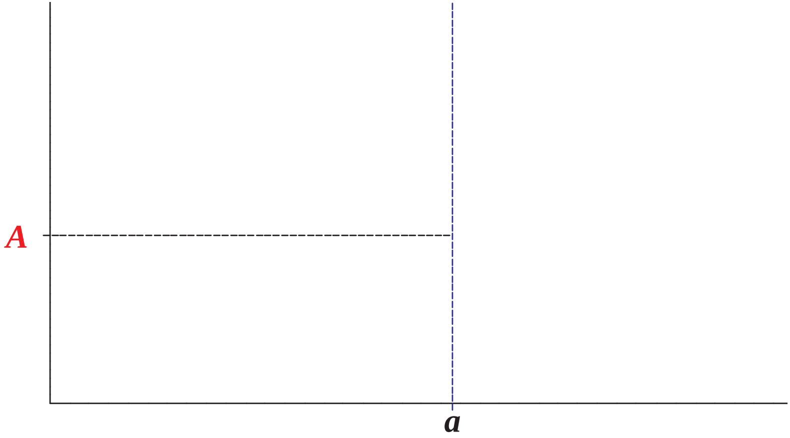
$$P(+\infty, \varepsilon) = B(+\infty, \varepsilon) = (1/\varepsilon, +\infty),$$
$$P(-\infty, \varepsilon) = B(-\infty, \varepsilon) = (-\infty, -1/\varepsilon).$$

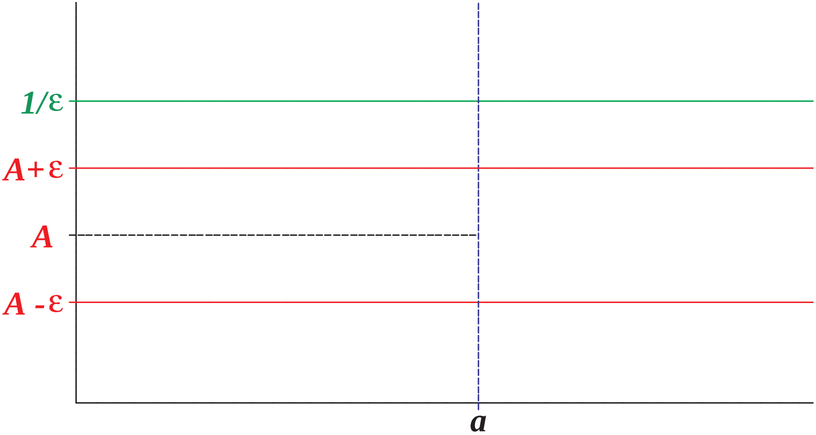
Definition

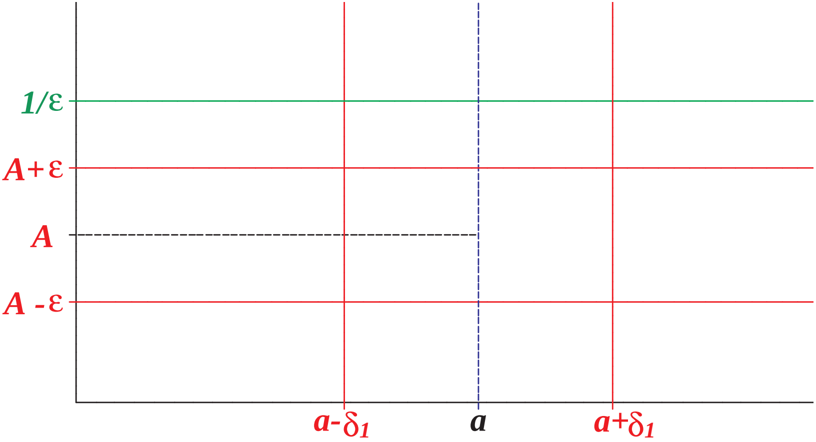
We say that $A \in \mathbb{R}^*$ is a limit of a function f at $c \in \mathbb{R}^*$ if

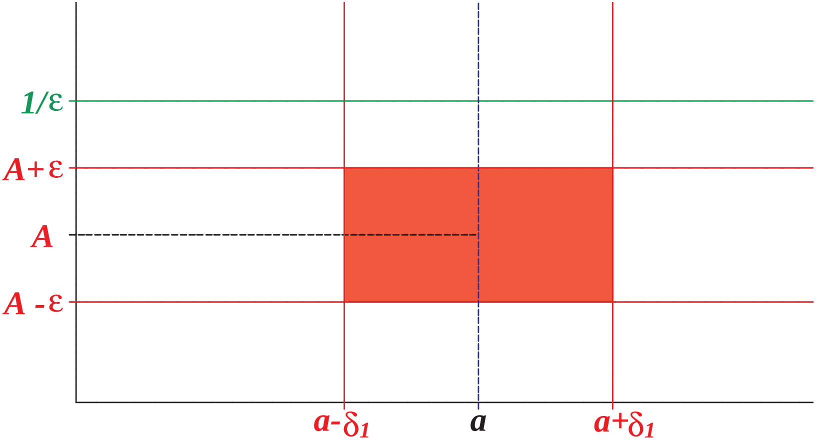
$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists \delta \in \mathbb{R}, \delta > 0\ \forall x \in P(c, \delta) \colon f(x) \in B(A, \varepsilon).$$

Theorem 20 holds also for $c \in \mathbb{R}^*$, $A \in \mathbb{R}^*$, so we can again use the notation $\lim_{x \to c} f(x) = A$.

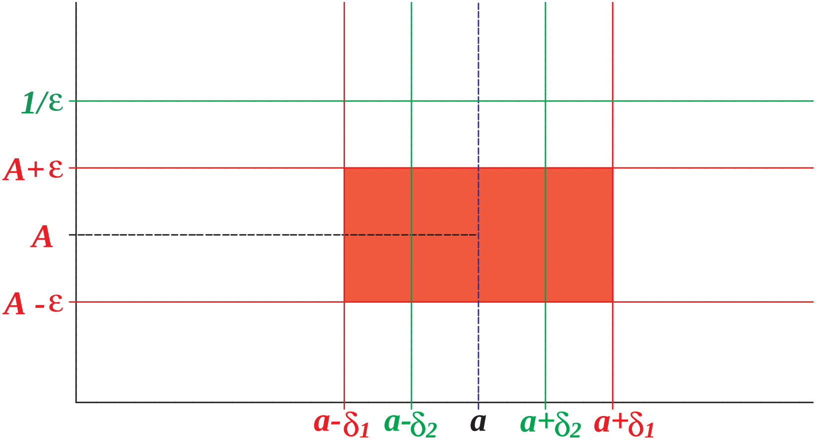

A
a

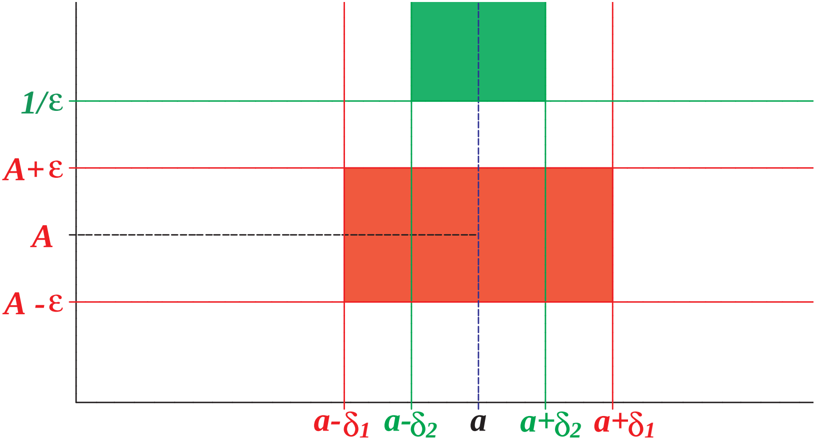

1/ε
A+ε
A
A -ε
a

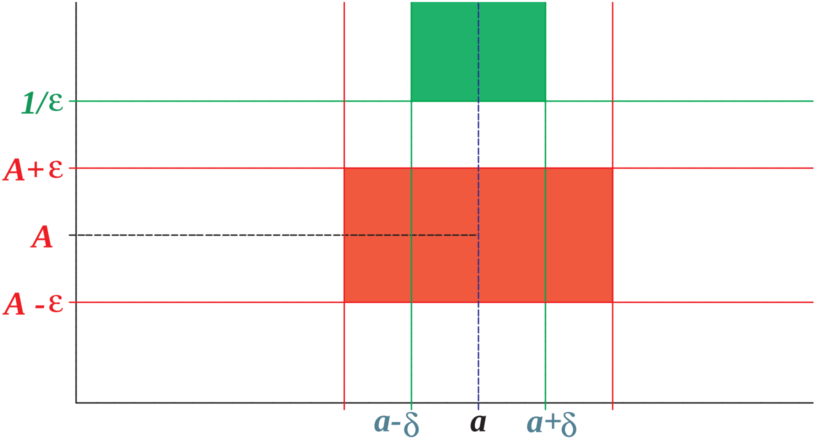

1/ε
A+ε
A
A -ε
a-δ1
a
a+δ1

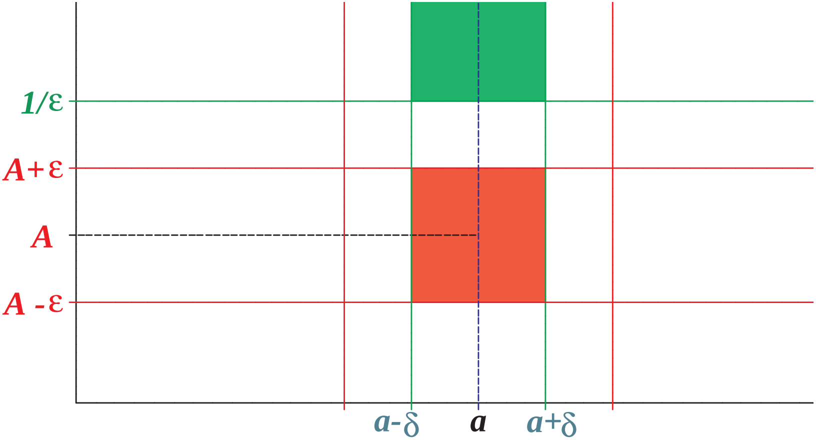


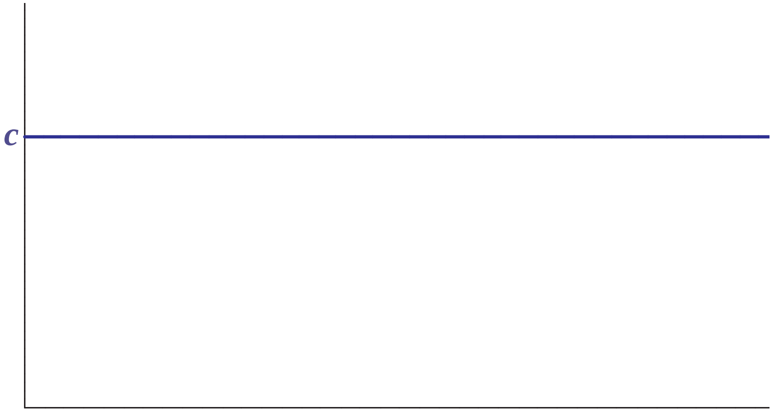
1/ε
A+ε
A
A -ε
a-δ1
a
a+δ1

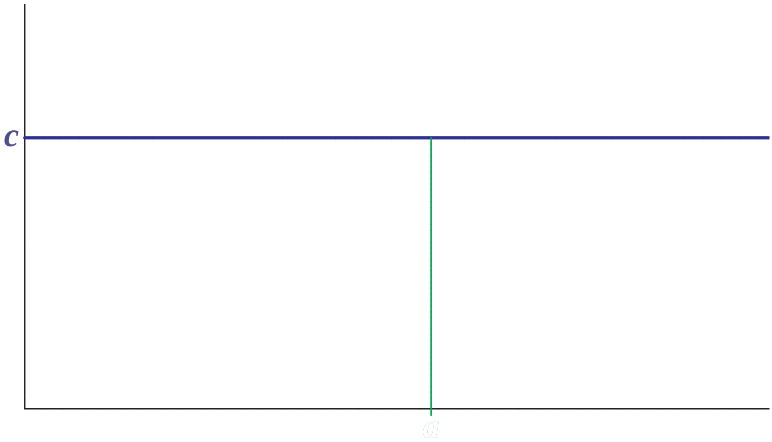

1/ε
A+ε
A
A -ε
a-δ1
a-δ2
a
a+δ2
a+δ1

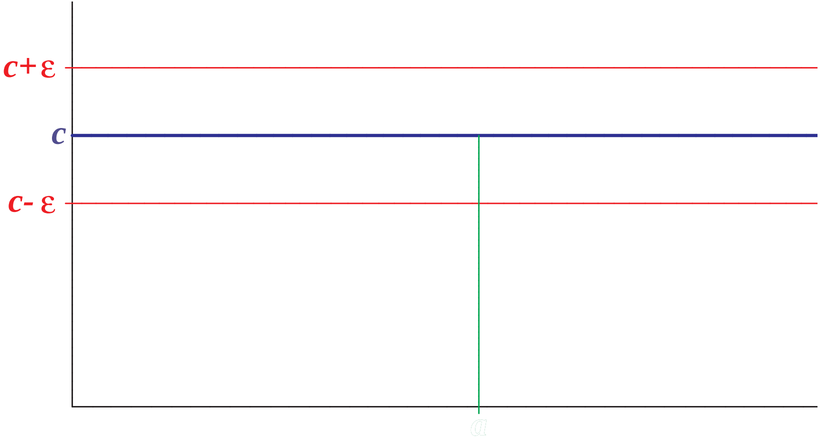

1/ε
A+ε
A
A -ε
a-δ1
a-δ2
a
a+δ2
a+δ1

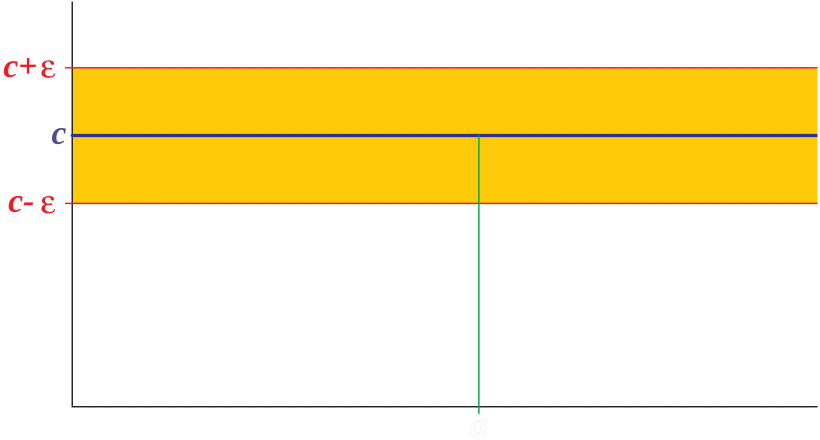

1/ε
A+ε
A
A -ε
a-δ
a
a+δ


1/ε
A+ε
A
A -ε
a-δ
a
a+δ

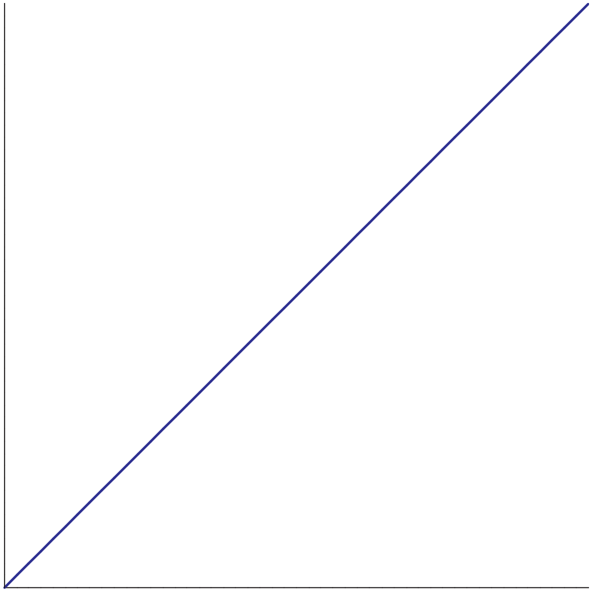

c

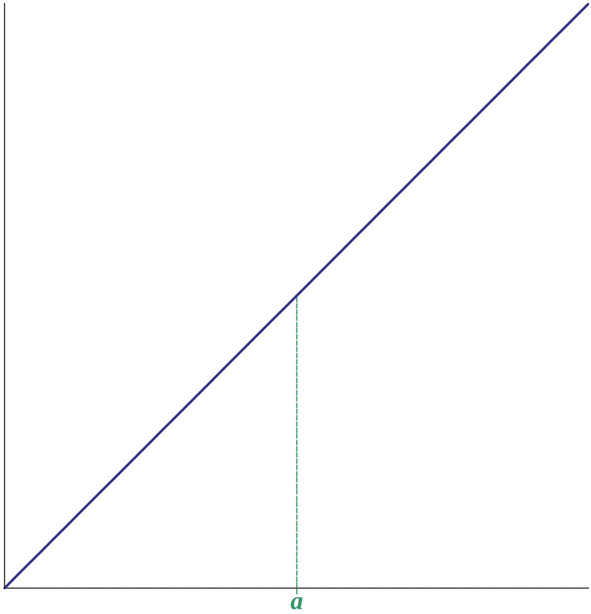

c
a

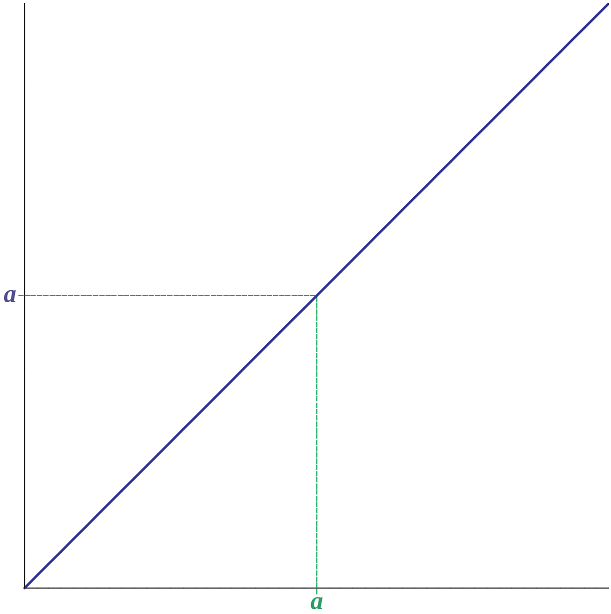

c+ ε
c
c- ε
a

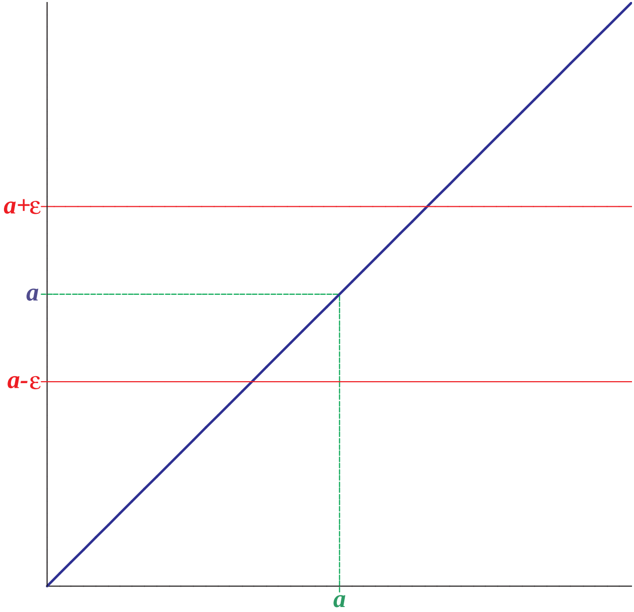


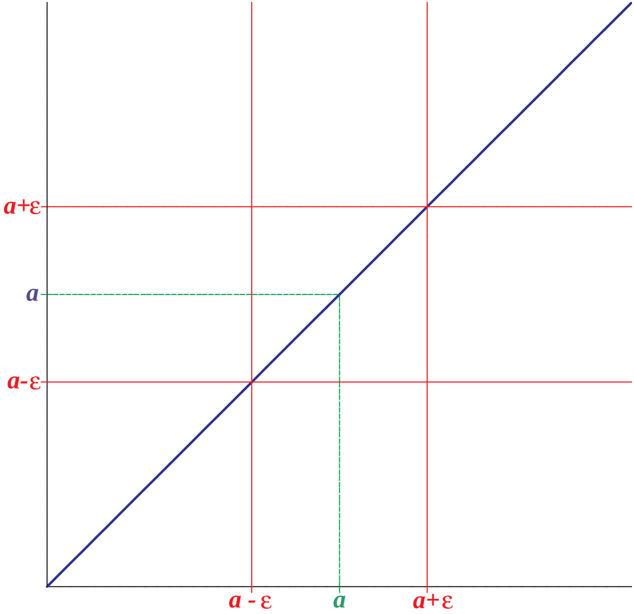
c+ ε
c
c- ε
a

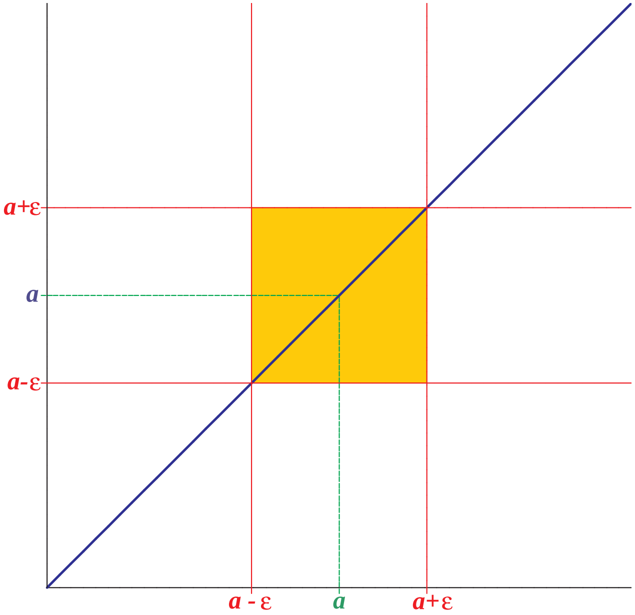



a


a
a


a+ε
a
a-ε
a


a+ε
a
a-ε
a - ε
a
a+ε


a+ε
a
a-ε
a - ε
a
a+ε

Definition

Let $c \in \mathbb{R}$ and $\varepsilon > 0$. We define

- a right neighbourhood of c by $B^+(c, \varepsilon) = [c, c + \varepsilon)$,

Definition

Let $c \in \mathbb{R}$ and $\varepsilon > 0$. We define

- a right neighbourhood of c by $B^{+}(c, \varepsilon) = [c, c + \varepsilon)$,
- a left neighbourhood of c by $B^{-}(c, \varepsilon) = (c - \varepsilon, c]$,

Definition

Let $c \in \mathbb{R}$ and $\varepsilon > 0$. We define

- a right neighbourhood of c by $B^+(c, \varepsilon) = [c, c + \varepsilon)$,
- a left neighbourhood of c by $B^-(c, \varepsilon) = (c - \varepsilon, c]$,
- a right punctured neighbourhood of c by $P^+(c, \varepsilon) = (c, c + \varepsilon)$,

Definition

Let $c \in \mathbb{R}$ and $\varepsilon > 0$. We define

- a right neighbourhood of c by $B^+(c, \varepsilon) = [c, c + \varepsilon)$,
- a left neighbourhood of c by $B^-(c, \varepsilon) = (c - \varepsilon, c]$,
- a right punctured neighbourhood of c by $P^+(c, \varepsilon) = (c, c + \varepsilon)$,
- a left punctured neighbourhood of c by $P^-(c, \varepsilon) = (c - \varepsilon, c)$,

Definition

Let $c \in \mathbb{R}$ and $\varepsilon > 0$. We define

- a right neighbourhood of c by $B^+(c, \varepsilon) = [c, c + \varepsilon)$,
- a left neighbourhood of c by $B^-(c, \varepsilon) = (c - \varepsilon, c]$,
- a right punctured neighbourhood of c by $P^+(c, \varepsilon) = (c, c + \varepsilon)$,
- a left punctured neighbourhood of c by $P^-(c, \varepsilon) = (c - \varepsilon, c)$,
- a left neighbourhood and left punctured neighbourhood of $+\infty$ by $B^-(+\infty, \varepsilon) = P^-(+\infty, \varepsilon) = (1/\varepsilon, +\infty)$,

Definition

Let $c \in \mathbb{R}$ and $\varepsilon > 0$. We define

- a right neighbourhood of c by $B^+(c, \varepsilon) = [c, c + \varepsilon)$,
- a left neighbourhood of c by $B^-(c, \varepsilon) = (c - \varepsilon, c]$,
- a right punctured neighbourhood of c by $P^+(c, \varepsilon) = (c, c + \varepsilon)$,
- a left punctured neighbourhood of c by $P^-(c, \varepsilon) = (c - \varepsilon, c)$,
- a left neighbourhood and left punctured neighbourhood of $+\infty$ by $B^-(+\infty, \varepsilon) = P^-(+\infty, \varepsilon) = (1/\varepsilon, +\infty)$,
- a right neighbourhood and right punctured neighbourhood of $-\infty$ by $B^+(-\infty, \varepsilon) = P^+(-\infty, \varepsilon) = (-\infty, -1/\varepsilon)$.

Definition

Let $A \in \mathbb{R}^*$, $c \in \mathbb{R} \cup \{-\infty\}$. We say that a function f has a limit from the right at c equal to $A \in \mathbb{R}^*$ (denoted by $\lim\limits_{x \to c+} f(x) = A$) if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \ \exists \delta \in \mathbb{R}, \delta > 0 \ \forall x \in P^+(c, \delta) \colon f(x) \in B(A, \varepsilon).$$

Analogously we define the notion of limit from the left at $c \in \mathbb{R} \cup \{+\infty\}$ and we use the notation $\lim\limits_{x \to c-} f(x)$.

Definition

Let $A \in \mathbb{R}^*$, $c \in \mathbb{R} \cup \{-\infty\}$. We say that a function f has a limit from the right at c equal to $A \in \mathbb{R}^*$ (denoted by $\lim\limits_{x \to c+} f(x) = A$) if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \ \exists \delta \in \mathbb{R}, \delta > 0 \ \forall x \in P^+(c, \delta) \colon f(x) \in B(A, \varepsilon).$$

Analogously we define the notion of limit from the left at $c \in \mathbb{R} \cup \{+\infty\}$ and we use the notation $\lim\limits_{x \to c-} f(x)$.

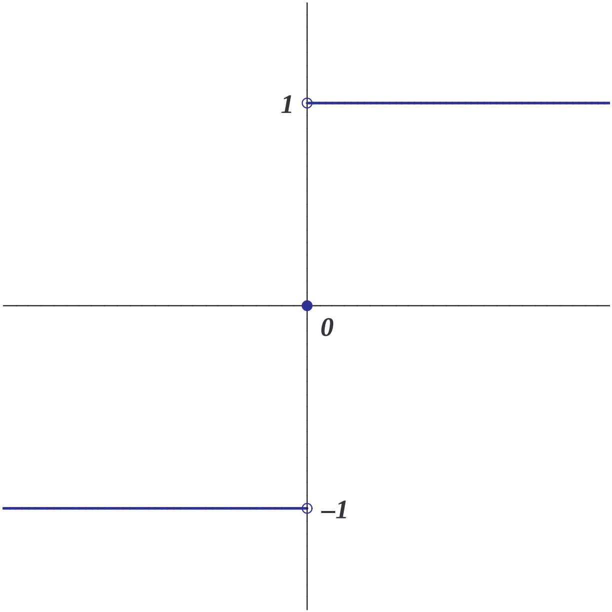
Remark

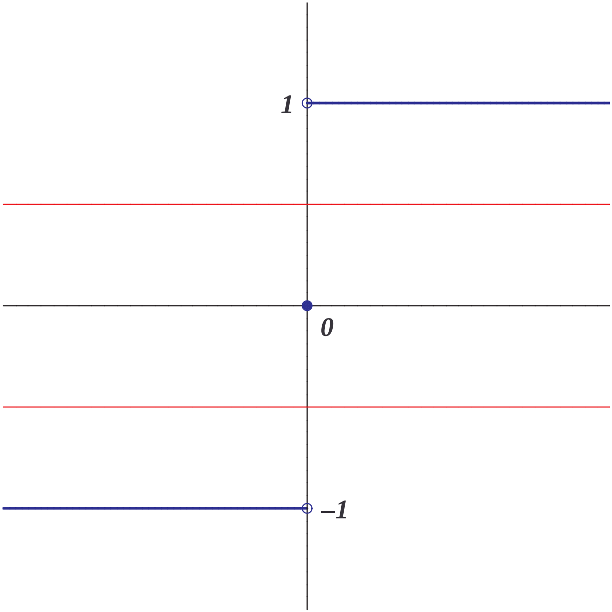
Let $c \in \mathbb{R}$, $A \in \mathbb{R}^*$. Then

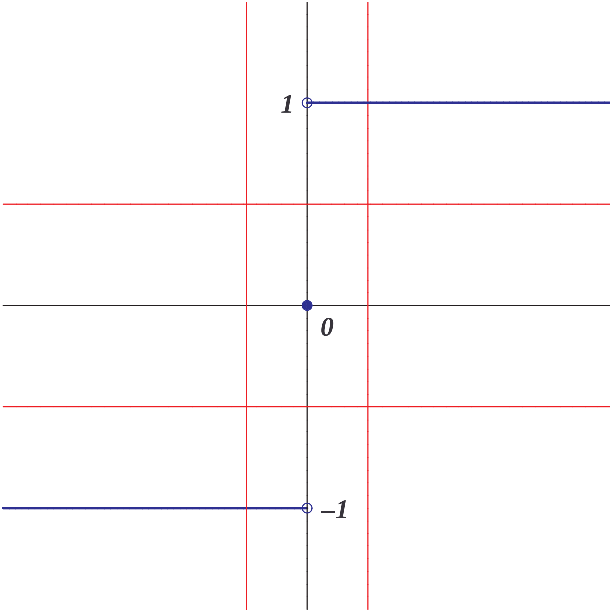
$$\lim_{x \to c} f(x) = A \Leftrightarrow \left(\lim_{x \to c+} f(x) = A \ \& \ \lim_{x \to c-} f(x) = A \right).$$

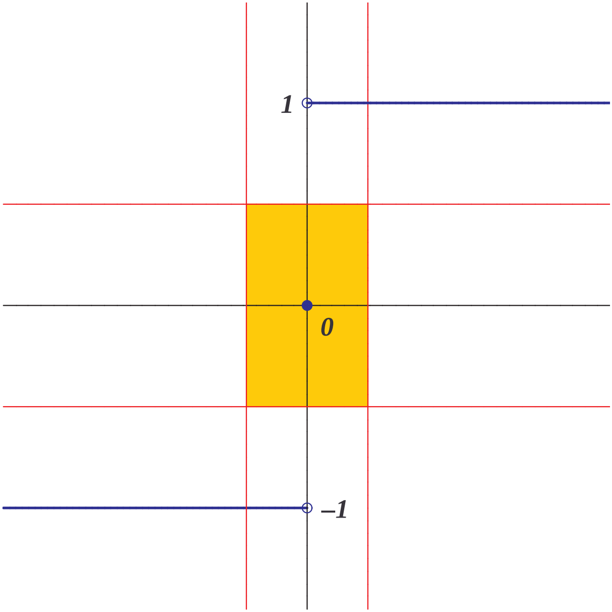
Definition

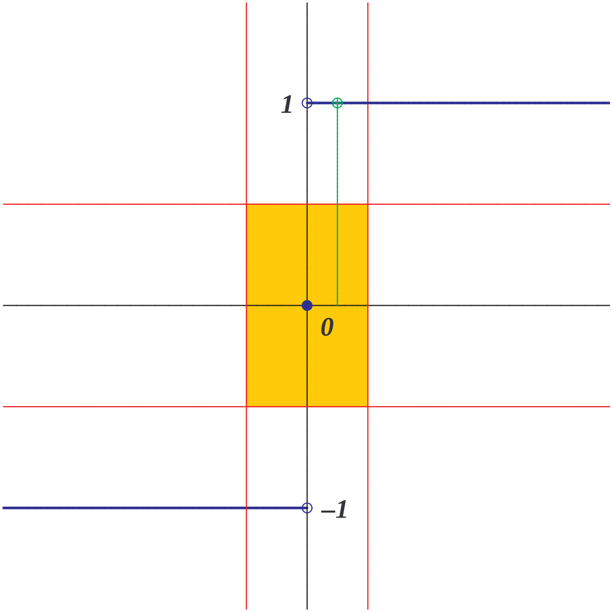
Let $c \in \mathbb{R}$. We say that a function f is continuous at c from the right (from the left, resp.) if $\lim_{x \to c+} f(x) = f(c)$ ($\lim_{x \to c-} f(x) = f(c)$, resp.).


1
0
–1


1
0
–1


1
0
–1


1
0
–1


1
0
–1

Theorem 21

Let f has a finite limit at c $\in \mathbb{R}^*$*. Then there exists* $\delta > 0$ *such that f is bounded on* $P(c, \delta)$.

Theorem 22 (arithmetics of limits)

Let $c \in \mathbb{R}^*$, $\lim_{x\to c} f(x) = A \in \mathbb{R}^*$ *and* $\lim_{x\to c} g(x) = B \in \mathbb{R}^*$. *Then*

(i) $\lim_{x\to c}(f(x) + g(x)) = A + B$ *if the expression* $A + B$ *is defined,*

(ii) $\lim_{x\to c} f(x)g(x) = AB$ *if the expression* AB *is defined,*

(iii) $\lim_{x\to c} f(x)/g(x) = A/B$ *if the expression* A/B *is defined.*

Theorem 22 (arithmetics of limits)

Let $c \in \mathbb{R}^*$, $\lim_{x\to c} f(x) = A \in \mathbb{R}^*$ and $\lim_{x\to c} g(x) = B \in \mathbb{R}^*$. Then

(i) $\lim_{x\to c}(f(x) + g(x)) = A + B$ *if the expression* $A + B$ *is defined,*

(ii) $\lim_{x\to c} f(x)g(x) = AB$ *if the expression* AB *is defined,*

(iii) $\lim_{x\to c} f(x)/g(x) = A/B$ *if the expression* A/B *is defined.*

Corollary

Suppose that the functions f and g are continuous at $c \in \mathbb{R}$. Then also the functions $f + g$ and fg are continuous at c. If moreover $g(c) \neq 0$, then also the function f/g is continuous at c.

Theorem 23

Let $c \in \mathbb{R}^*$, $\lim_{x \to c} g(x) = 0$, $\lim_{x \to c} f(x) = A \in \mathbb{R}^*$ and $A > 0$. If there exists $\eta > 0$ such that the function g is positive on $P(c, \eta)$, then $\lim_{x \to c} \big(f(x)/g(x)\big) = +\infty$.

Definition

A polynomial is a function P of the form

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad x \in \mathbb{R},$$

where $n \in \mathbb{N} \cup \{0\}$ and $a_0, a_1, \ldots, a_n \in \mathbb{R}$. The numbers $a_0, \ldots, a_n$ are called the coefficients of the polynomial P.

Definition

A polynomial is a function P of the form

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad x \in \mathbb{R},$$

where $n \in \mathbb{N} \cup \{0\}$ and $a_0, a_1, \dots, a_n \in \mathbb{R}$. The numbers $a_0, \dots, a_n$ are called the coefficients of the polynomial P.

Remark

Let $n, m \in \mathbb{N} \cup \{0\}$ and

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad x \in \mathbb{R},$$
$$Q(x) = b_0 + b_1 x + \cdots + b_m x^m, \quad x \in \mathbb{R},$$

where $a_0, a_1, \dots, a_n \in \mathbb{R}$, $a_n \neq 0$, $b_0, b_1, \dots, b_m \in \mathbb{R}$, $b_m \neq 0$. If the polynomials P and Q are equal (i.e. $P(x) = Q(x)$ for each $x \in \mathbb{R}$), then $n = m$ and $a_0 = b_0, \dots, a_n = b_n$.

Definition

Let P be a polynomial of the form

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad x \in \mathbb{R}.$$

We say that P is a polynomial of degree n if $a_n \neq 0$. The degree of a zero polynomial (i.e. a constant zero function defined on $\mathbb{R}$) is defined as -1.

Theorem 24 (limits and inequalities)

Suppose that $c \in \mathbb{R}^*$ *and* $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ *exist.*
(i) If $\lim_{x\to c} f(x) > \lim_{x\to c} g(x)$*, then there exists* $\delta > 0$ *such that*

$$\forall x \in P(c,\delta)\colon\ f(x) > g(x).$$

Theorem 24 (limits and inequalities)

Suppose that $c \in \mathbb{R}^*$ *and* $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ *exist.*
(i) If $\lim_{x\to c} f(x) > \lim_{x\to c} g(x)$, *then there exists* $\delta > 0$ *such that*

$$\forall x \in P(c,\delta)\colon\ f(x) > g(x).$$

(ii) If there exists $\delta > 0$ *such that* $\forall x \in P(c,\delta)\colon\ f(x) \leq g(x)$, *then*

$$\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x).$$

Theorem 24 (limits and inequalities)

Suppose that $c \in \mathbb{R}^*$ *and* $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ *exist.*
(i) If $\lim_{x\to c} f(x) > \lim_{x\to c} g(x)$, *then there exists* $\delta > 0$ *such that*

$$\forall x \in P(c,\delta)\colon\ f(x) > g(x).$$

(ii) If there exists $\delta > 0$ *such that* $\forall x \in P(c,\delta)\colon\ f(x) \leq g(x)$, *then*

$$\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x).$$

(iii) (two policemen/sandwich theorem) Suppose that there exists $\eta > 0$ *such that*

$$\forall x \in P(c,\eta)\colon\ f(x) \leq h(x) \leq g(x).$$

If moreover $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = A \in \mathbb{R}^*$, *then the limit* $\lim_{x\to c} h(x)$ *also exists and equals A.*

Corollary

Let $c \in \mathbb{R}^*$, $\lim_{x \to c} f(x) = 0$ and suppose there exists $\eta > 0$ such that g is bounded on $P(c, \eta)$. Then $\lim_{x \to c} \big(f(x)g(x)\big) = 0$.

Theorem 25 (limit of a composition)

Let $c, A, B \in \mathbb{R}^*$, $\lim_{x\to c} g(x) = A$, $\lim_{y\to A} f(y) = B$ and at least on of the following conditions is satisfied:

(I) $\exists \eta \in \mathbb{R}, \eta > 0\ \forall x \in P(c, \eta)\colon g(x) \neq A$,

(C) *the function f is continuous at A.*

Then

$$\lim_{x\to c} f\big(g(x)\big) = B.$$

Theorem 25 (limit of a composition)

Let $c, A, B \in \mathbb{R}^*$, $\lim_{x\to c} g(x) = A$, $\lim_{y\to A} f(y) = B$ and at least on of the following conditions is satisfied:

(I) $\exists \eta \in \mathbb{R}, \eta > 0\ \forall x \in P(c,\eta) \colon g(x) \neq A$,

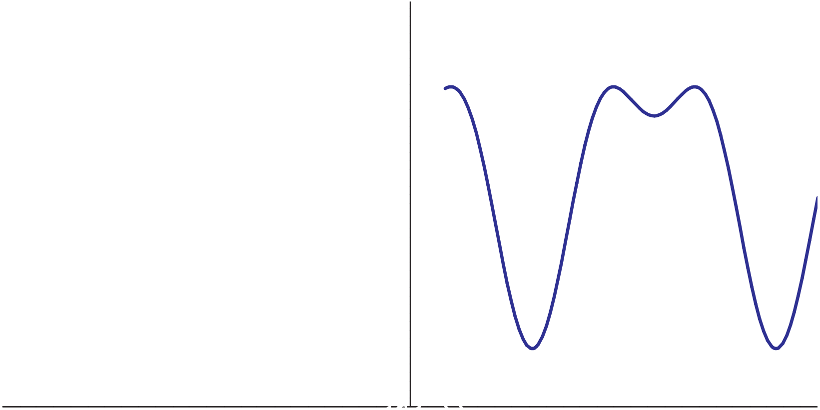
(C) *the function f is continuous at A.*

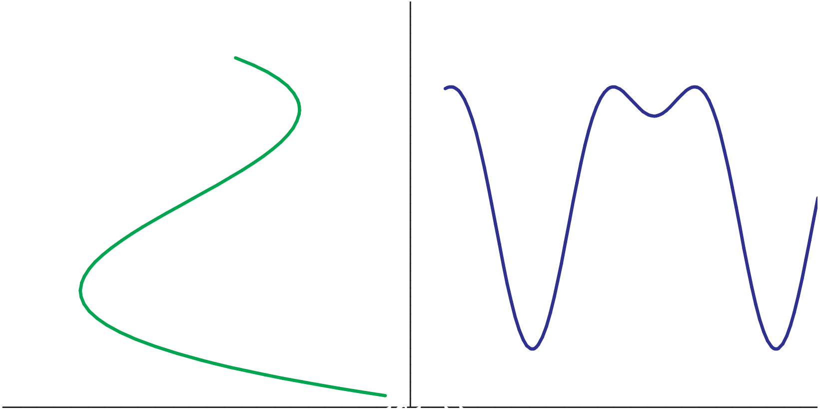
Then

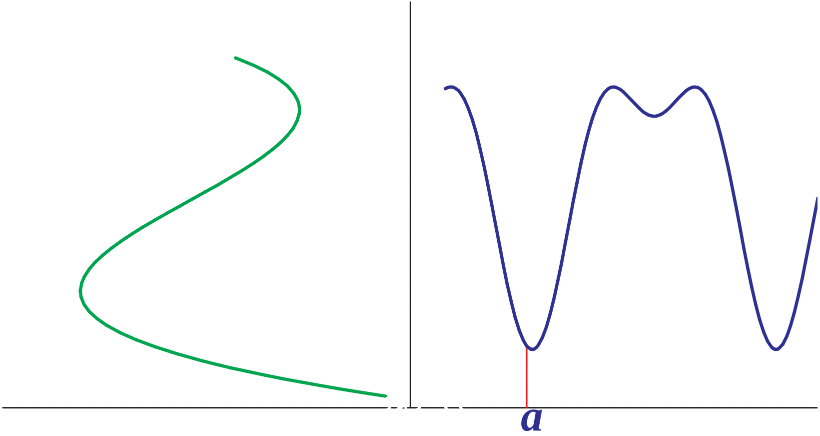
$$\lim_{x\to c} f\big(g(x)\big) = B.$$

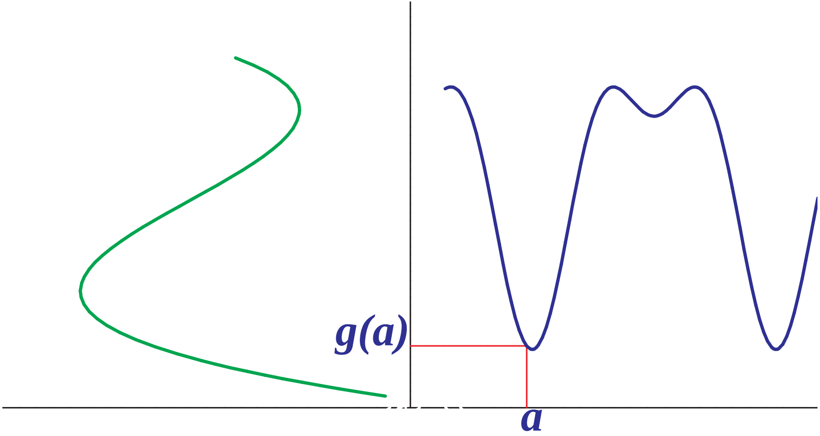
Corollary

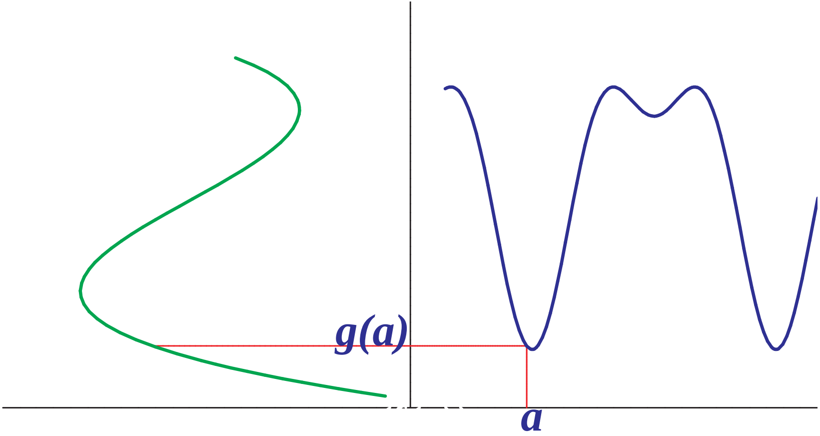
Suppose that the function g is continuous at $c \in \mathbb{R}$ and the function f is continuous at $g(c)$. Then the function $f \circ g$ is continuous at c.

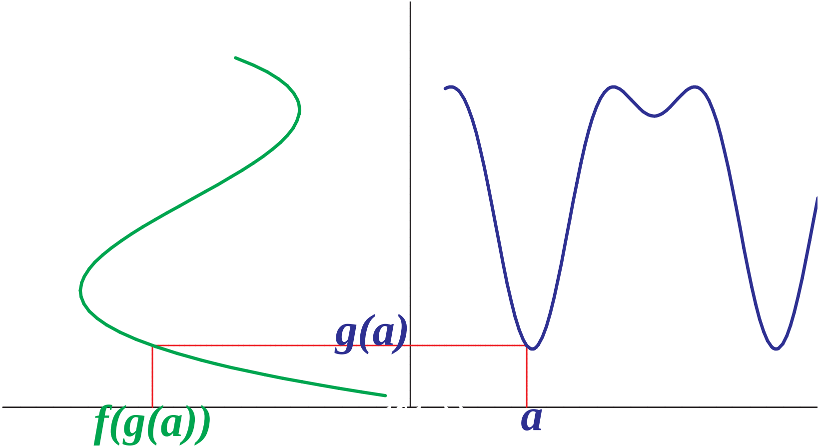


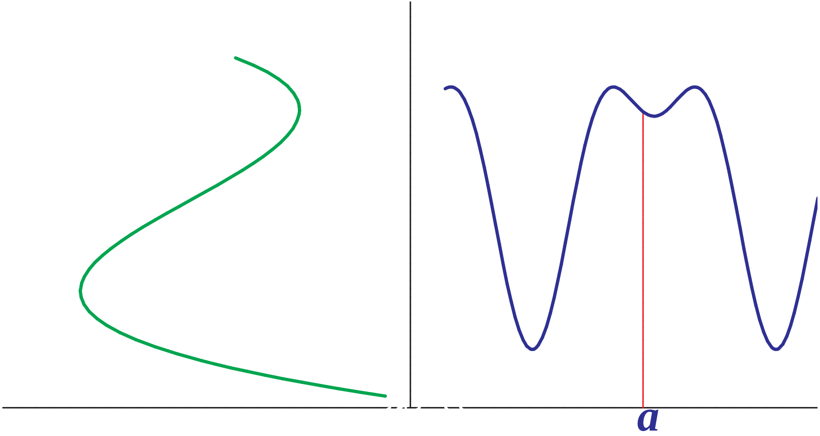


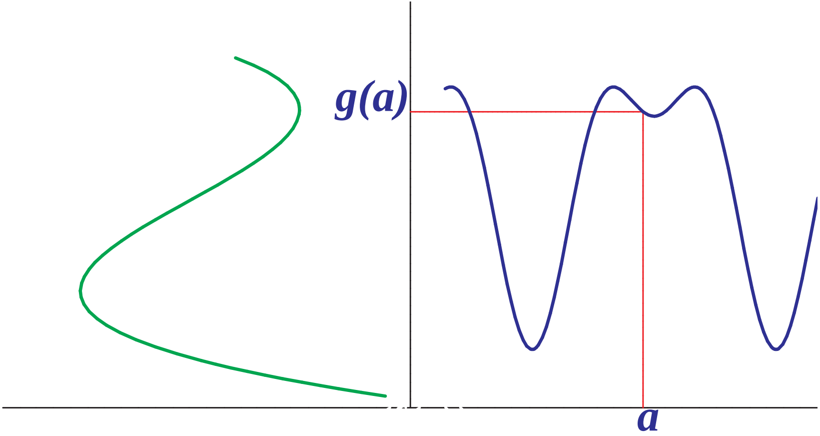

a

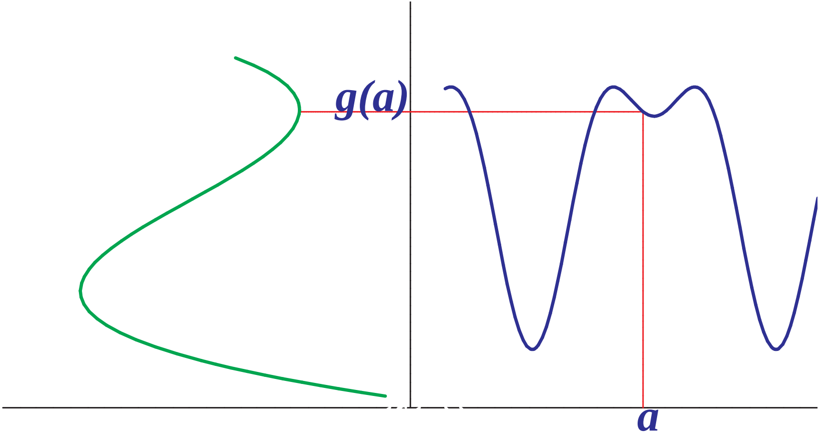

g(a)
a

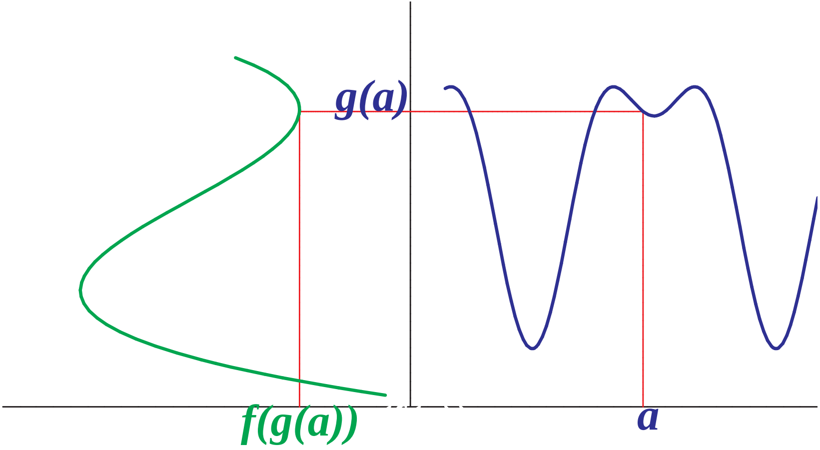

g(a)
a

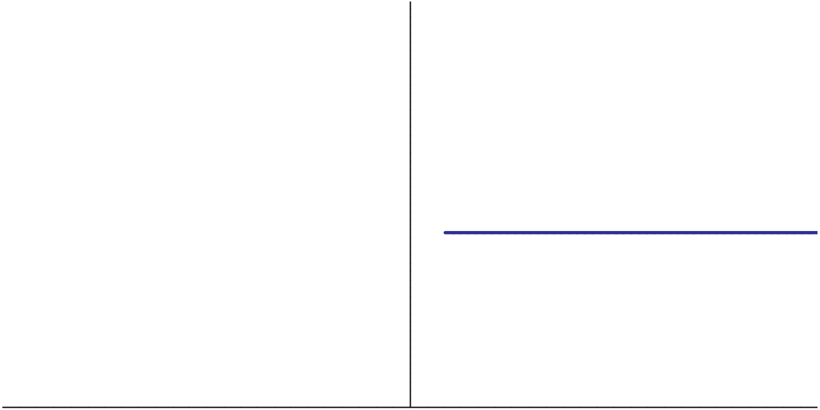

g(a)
f(g(a))
a

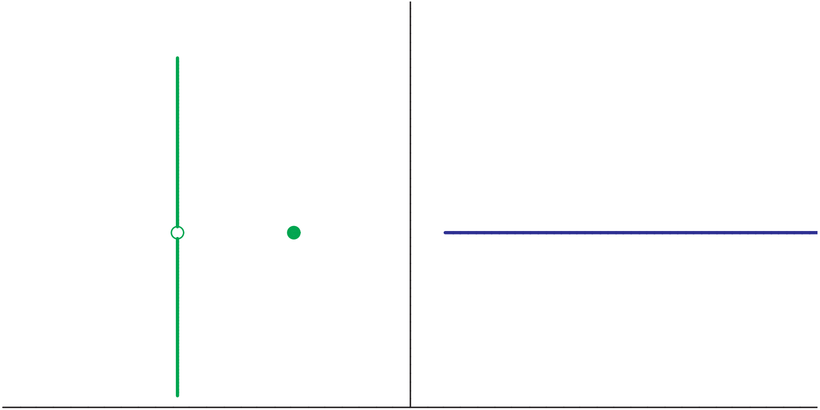

a

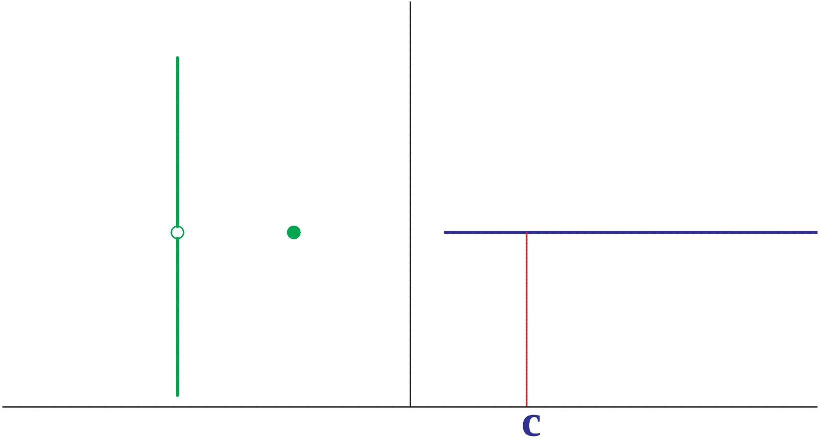

g(a)
a

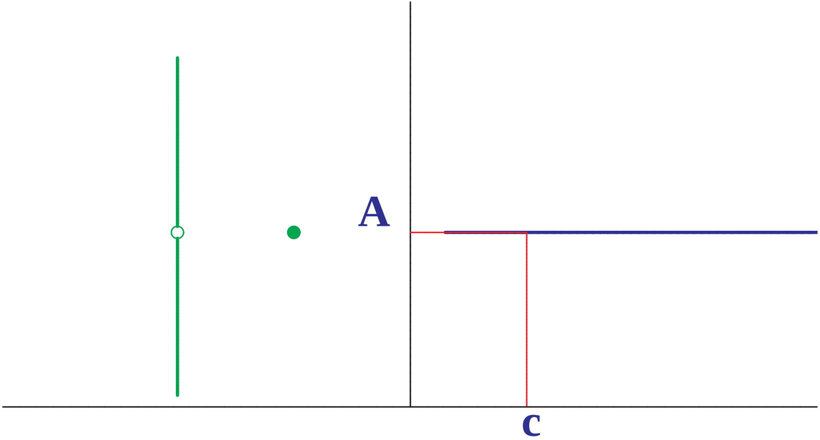

g(a)
a

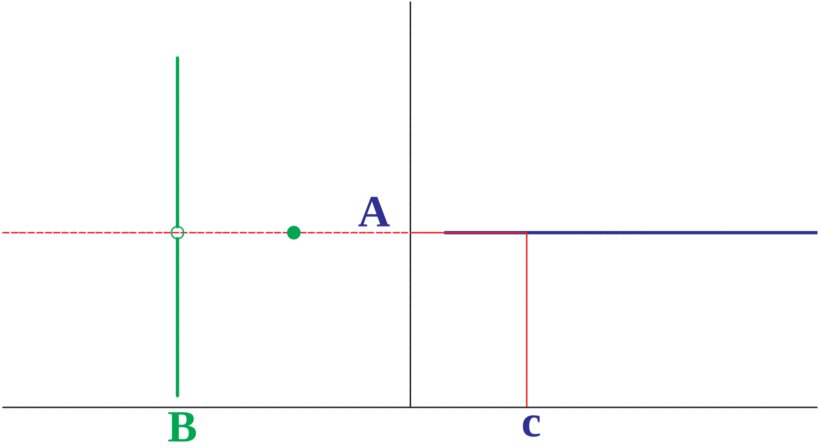

g(a)
f(g(a))
a

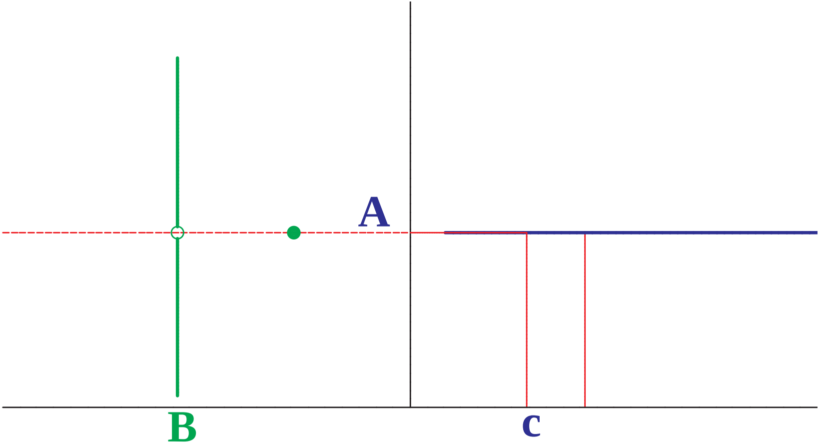


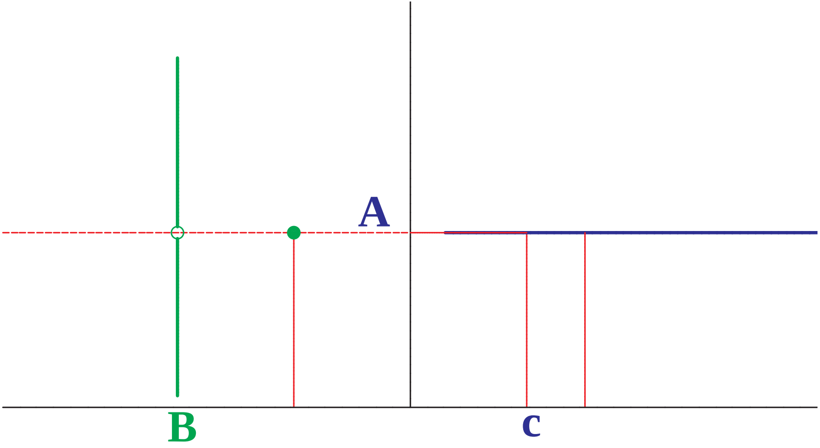


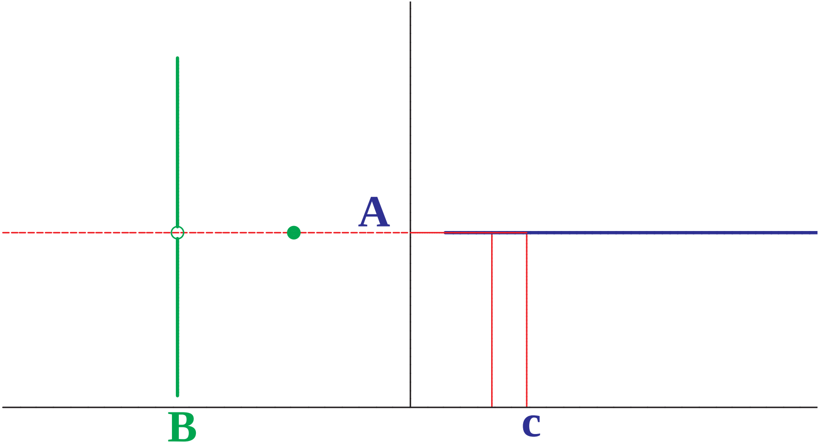

c

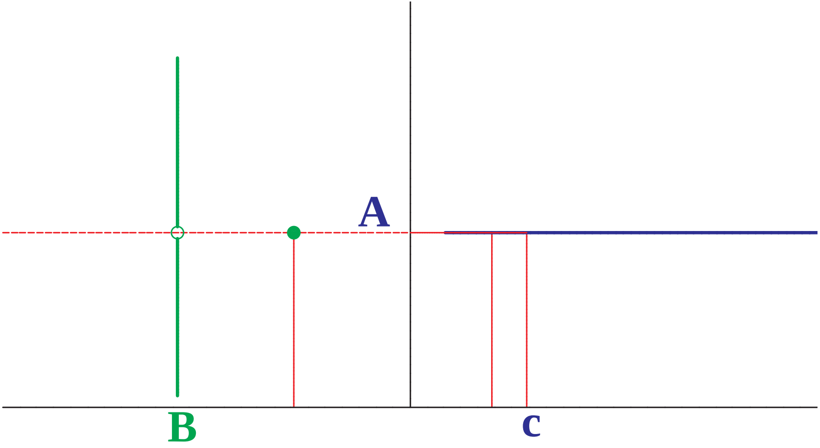

A
c

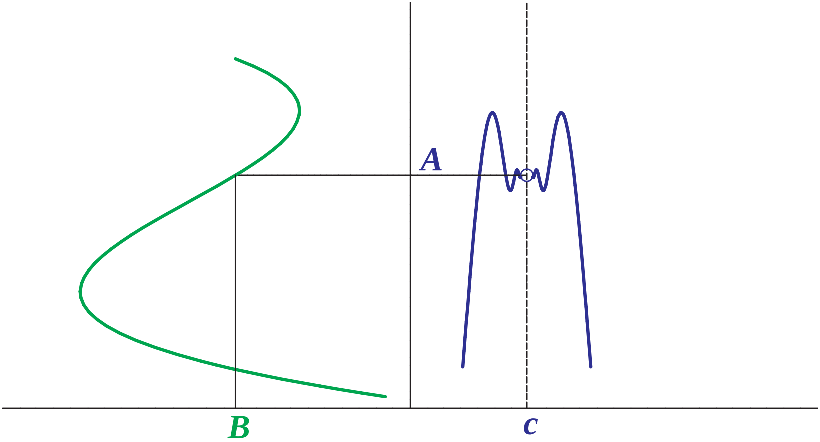

A
B
c

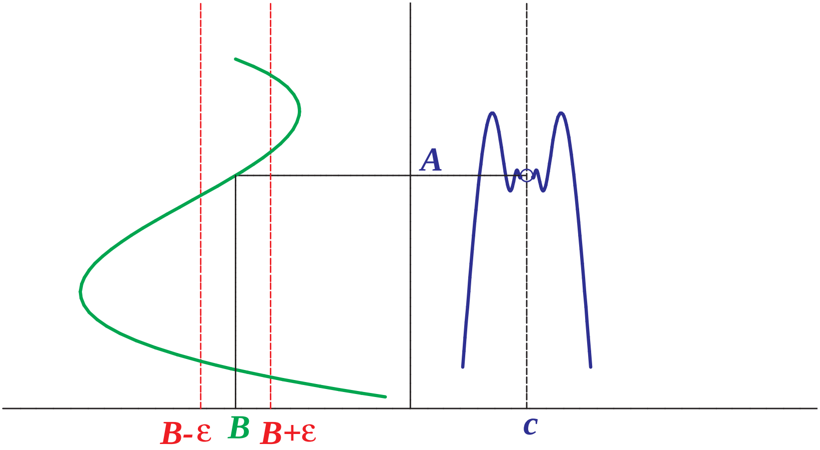

A
B
c

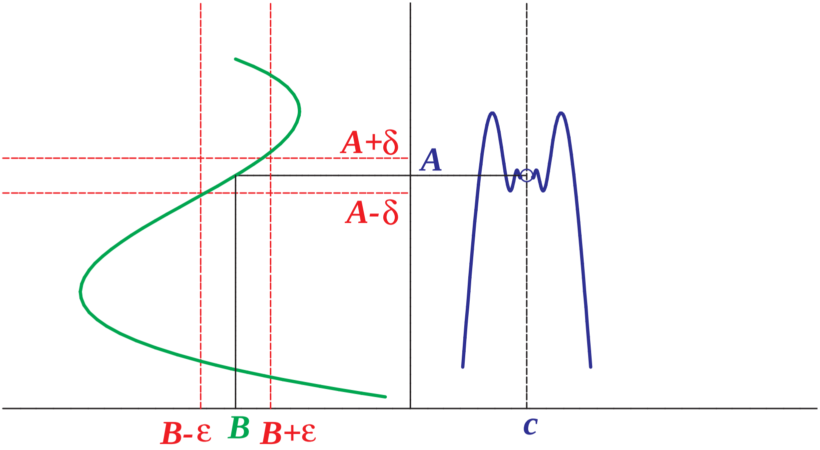

A
B
c

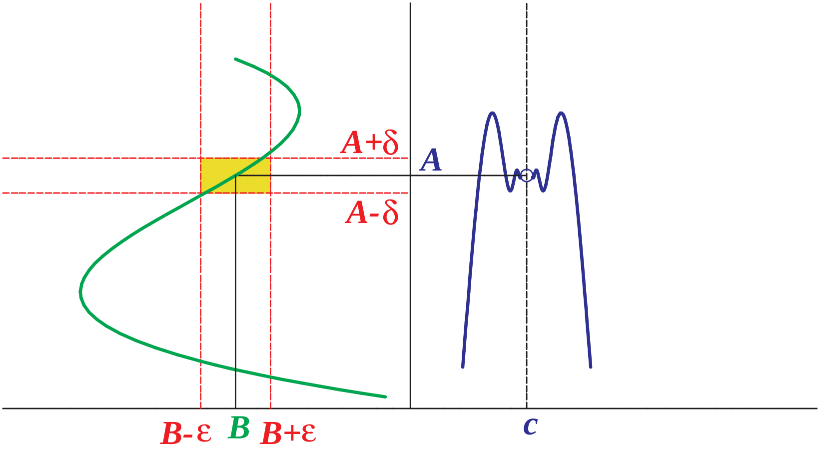

A
B
c

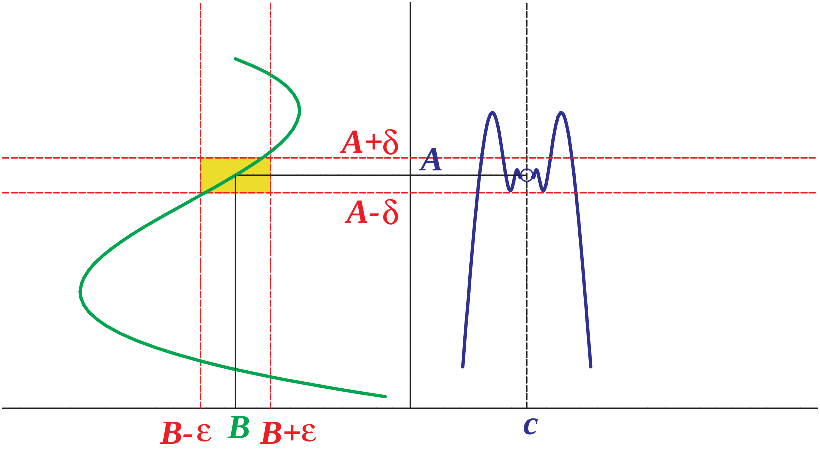

A
B
c

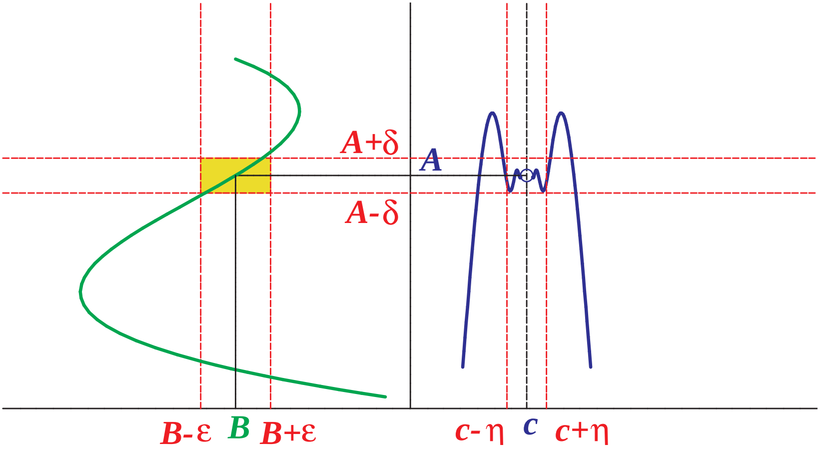

A
B
c

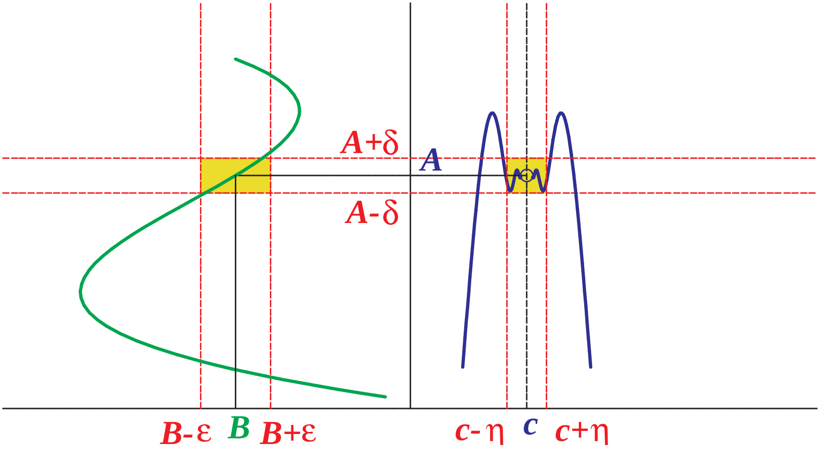

A
B-ε
B
B+ε
c

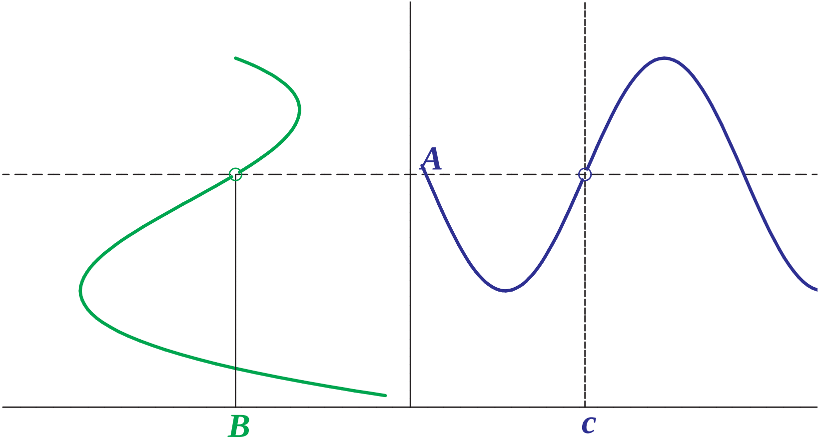

A+δ
A
A-δ
B-ε
B
B+ε
c

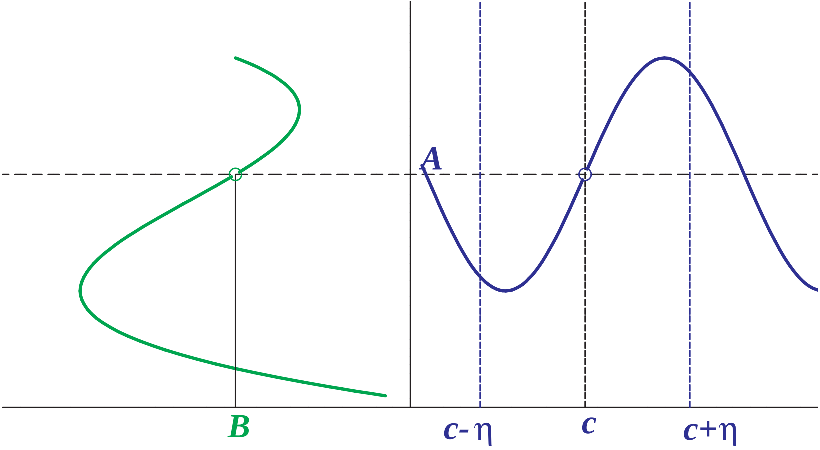

A+δ
A
A-δ
B-ε
B
B+ε
c

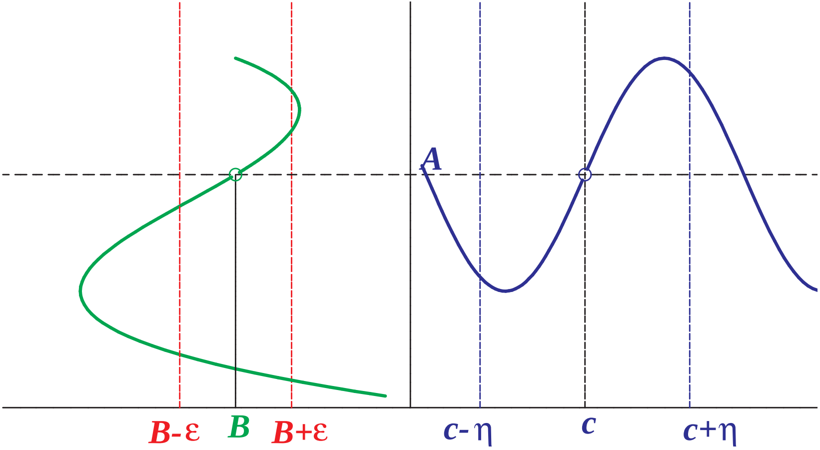

A+δ
A
A-δ
B-ε
B
B+ε
c

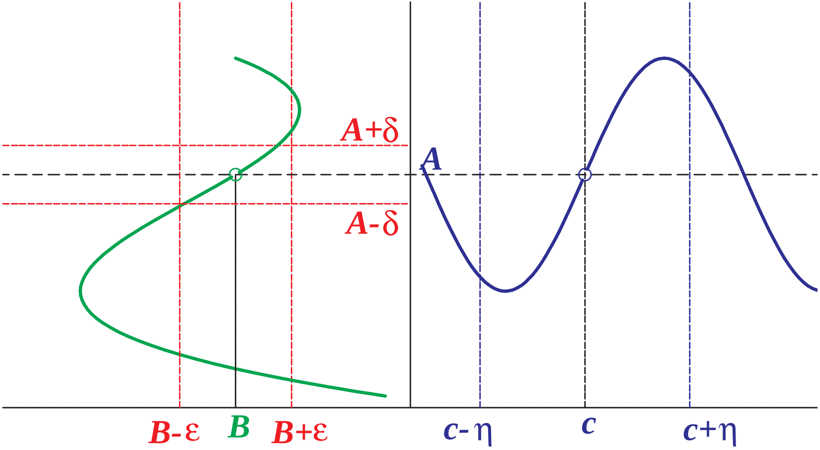

A+δ
A
A-δ
B-ε
B
B+ε
c-η
c
c+η

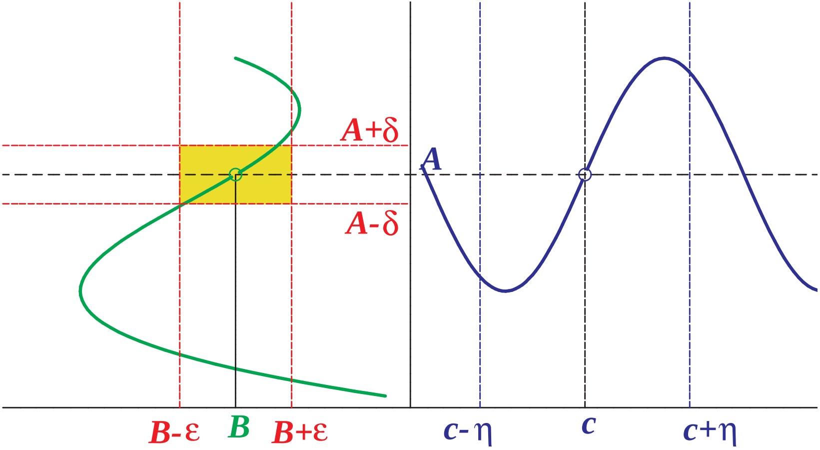

A+δ
A
A-δ
B-ε
B
B+ε
c- η
c
c+η

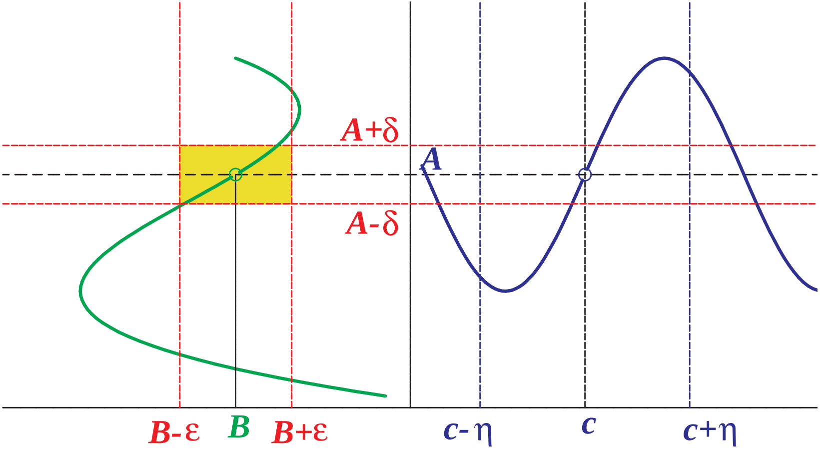

A
B
c

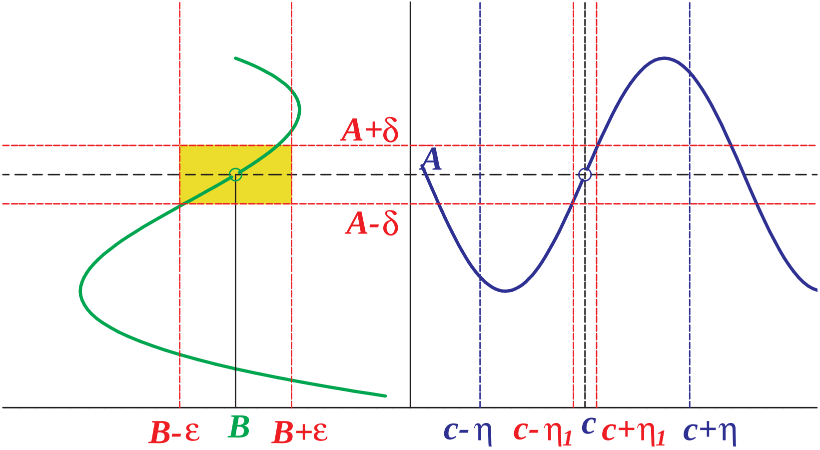

A
B
c- η
c
c+η

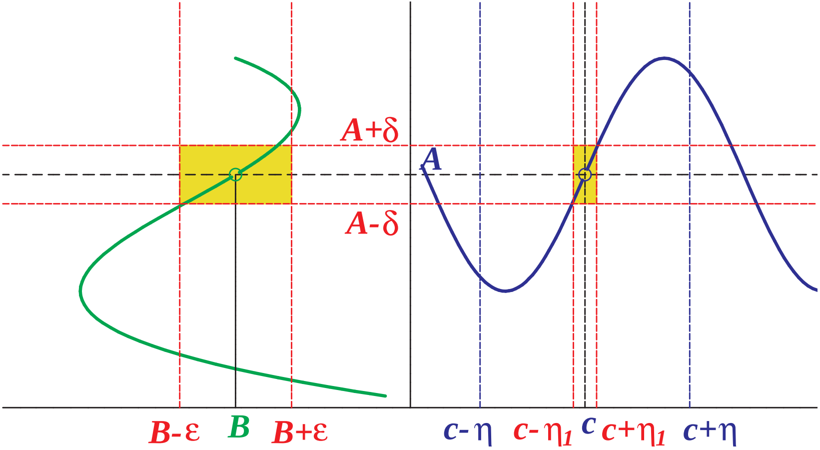

A
B-ε
B
B+ε
c-η
c
c+η


A+δ
A
A-δ
B-ε
B
B+ε
c-η
c
c+η

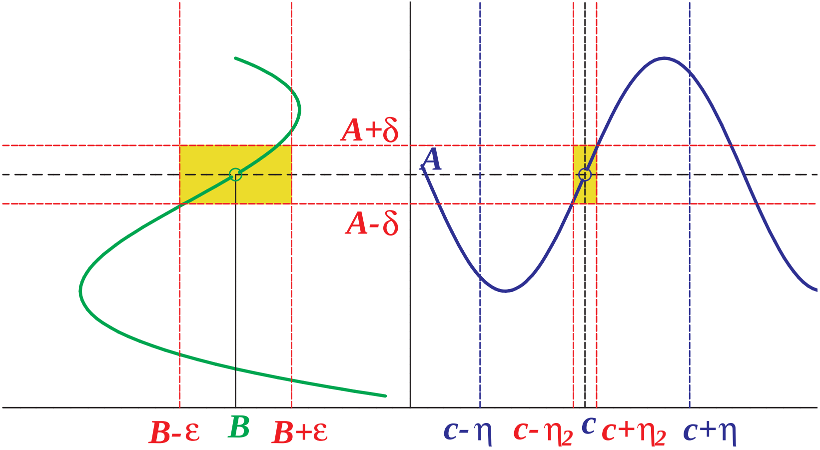

A+δ
A
A-δ
B-ε
B
B+ε
c-η
c
c+η


A+δ
A
A-δ
B-ε
B
B+ε
c-η
c
c+η


A+δ
A
A-δ
B-ε
B
B+ε
c-η
c-η1
c
c+η1
c+η



A+δ
A
A-δ
B-ε
B
B+ε
c-η
c-η1
c
c+η1
c+η



A+δ
A
A-δ
B-ε
B
B+ε
c- η
c- η2
c
c+η2
c+η

Theorem 26 (Heine)

Let $c \in \mathbb{R}^*$, $A \in \mathbb{R}^*$ *and the function* f *satisfies* $\lim_{x\to c} f(x) = A$. *If the sequence* $\{x_n\}$ *satisfies* $x_n \in D_f$, $x_n \neq c$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} x_n = c$, *then* $\lim_{n\to\infty} f(x_n) = A$.

Theorem 27 (limit of a monotone function)

Let $a, b \in \mathbb{R}^*$, $a < b$. Suppose that f is a function monotone on an interval (a, b). Then the limits $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ exist. Moreover,

- *if f is non-decreasing on (a, b), then $\lim_{x\to a+} f(x) = \inf f\big((a, b)\big)$ and $\lim_{x\to b-} f(x) = \sup f\big((a, b)\big)$;*
- *if f is non-increasing on (a, b), then $\lim_{x\to a+} f(x) = \sup f\big((a, b)\big)$ and $\lim_{x\to b-} f(x) = \inf f\big((a, b)\big)$.*

Definition

Let $J \subset \mathbb{R}$ be a non-degenerate interval (i.e. it contains infinitely many points). A function $f\colon J \to \mathbb{R}$ is continuous on the interval J if

- f is continuous at every inner point J,
- f is continuous from the right at the left endpoint of J if this point belongs to J,
- f is continuous from the left at the right endpoint of J if this point belongs to J.

Theorem 28 (continuity of the compound function on an interval)

Let I and J be intervals, $g\colon I \to J$, $f\colon J \to \mathbb{R}$, let g be continuous on I and let f be continuous on J. Then the function $f \circ g$ is continuous on I.

Theorem 29 (Bolzano, intermediate value theorem)

Let f be a function continuous on an interval $[a, b]$ and suppose that $f(a) < f(b)$. Then for each $C \in (f(a), f(b))$ there exists $\xi \in (a, b)$ satisfying $f(\xi) = C$.

Theorem 30 (an image of an interval under a continuous function)

Let J be an interval and let $f\colon J \to \mathbb{R}$ be a function continuous on J. Then $f(J)$ is an interval.

Definition

Let $M \subset \mathbb{R}$, $x \in M$ and a function f is defined at least on M (i.e. $M \subset D_f$). We say that f attains its maximum (resp. minimum) on M at $x \in M$ if

$$\forall y \in M\colon\ f(y) \leq f(x) \quad \text{(resp. } \forall y \in M\colon\ f(y) \geq f(x)\text{)}.$$

Definition

Let $M \subset \mathbb{R}$, $x \in M$ and a function f is defined at least on M (i.e. $M \subset D_f$). We say that f attains its maximum (resp. minimum) on M at $x \in M$ if

$$\forall y \in M\colon\ f(y) \leq f(x) \quad (\text{resp. } \forall y \in M\colon\ f(y) \geq f(x)).$$

The point x is called the point of maximum (resp. minimum) of the function f on M.

Definition

Let $M \subset \mathbb{R}$, $x \in M$ and a function f is defined at least on M (i.e. $M \subset D_f$). We say that f attains its maximum (resp. minimum) on M at $x \in M$ if

$$\forall y \in M\colon f(y) \leq f(x) \quad (\text{resp. } \forall y \in M\colon f(y) \geq f(x)).$$

The point x is called the point of maximum (resp. minimum) of the function f on M. The symbol $\max_M f$ (resp. $\min_M f$) denotes the maximal (resp. minimal) value of f on M (if such a value exists).

Definition

Let $M \subset \mathbb{R}$, $x \in M$ and a function f is defined at least on M (i.e. $M \subset D_f$). We say that f attains its maximum (resp. minimum) on M at $x \in M$ if

$$\forall y \in M\colon\ f(y) \le f(x) \quad (\text{resp. } \forall y \in M\colon\ f(y) \ge f(x)).$$

The point x is called the point of maximum (resp. minimum) of the function f on M. The symbol $\max_M f$ (resp. $\min_M f$) denotes the maximal (resp. minimal) value of f on M (if such a value exists). The points of maxima or minima are collectively called the points of extrema.

Definition

Let $M \subset \mathbb{R}$, $x \in M$ and a function f is defined at least on M (i.e. $M \subset D_f$). We say that the function f has at x

- a local maximum with respect to M if there exists $\delta > 0$ such that $\forall y \in B(x, \delta) \cap M\colon f(y) \leq f(x)$,

Definition

Let $M \subset \mathbb{R}$, $x \in M$ and a function f is defined at least on M (i.e. $M \subset D_f$). We say that the function f has at x

- a local maximum with respect to M if there exists $\delta > 0$ such that $\forall y \in B(x, \delta) \cap M\colon\ f(y) \leq f(x)$,
- a local minimum with respect to M if there exists $\delta > 0$ such that $\forall y \in B(x, \delta) \cap M\colon\ f(y) \geq f(x)$,

Definition

Let $M \subset \mathbb{R}$, $x \in M$ and a function f is defined at least on M (i.e. $M \subset D_f$). We say that the function f has at x

- a local maximum with respect to M if there exists $\delta > 0$ such that $\forall y \in B(x, \delta) \cap M\colon f(y) \leq f(x)$,
- a local minimum with respect to M if there exists $\delta > 0$ such that $\forall y \in B(x, \delta) \cap M\colon f(y) \geq f(x)$,
- a strict local maximum with respect to M if there exists $\delta > 0$ such that $\forall y \in P(x, \delta) \cap M\colon f(y) < f(x)$,

Definition

Let $M \subset \mathbb{R}$, $x \in M$ and a function f is defined at least on M (i.e. $M \subset D_f$). We say that the function f has at x

- a local maximum with respect to M if there exists $\delta > 0$ such that $\forall y \in B(x, \delta) \cap M\colon f(y) \leq f(x)$,
- a local minimum with respect to M if there exists $\delta > 0$ such that $\forall y \in B(x, \delta) \cap M\colon f(y) \geq f(x)$,
- a strict local maximum with respect to M if there exists $\delta > 0$ such that $\forall y \in P(x, \delta) \cap M\colon f(y) < f(x)$,
- a strict local minimum with respect to M if there exists $\delta > 0$ such that $\forall y \in P(x, \delta) \cap M\colon f(y) > f(x)$.

Definition

Let $M \subset \mathbb{R}$, $x \in M$ and a function f is defined at least on M (i.e. $M \subset D_f$). We say that the function f has at x

- a local maximum with respect to M if there exists $\delta > 0$ such that $\forall y \in B(x, \delta) \cap M \colon f(y) \leq f(x)$,
- a local minimum with respect to M if there exists $\delta > 0$ such that $\forall y \in B(x, \delta) \cap M \colon f(y) \geq f(x)$,
- a strict local maximum with respect to M if there exists $\delta > 0$ such that $\forall y \in P(x, \delta) \cap M \colon f(y) < f(x)$,
- a strict local minimum with respect to M if there exists $\delta > 0$ such that $\forall y \in P(x, \delta) \cap M \colon f(y) > f(x)$.

The points of local maxima or minima are collectively called the points of local extrema.

Theorem 31 (Heine theorem for continuity on an interval)

Let f be a function continuous on an interval J and $c \in J$. Then $\lim f(x_n) = f(c)$ for each sequence $\{x_n\}_{n=1}^{\infty}$ of points in the interval J satisfying $\lim x_n = c$.

Theorem 32 (extrema of continuous functions)

Let f be a function continuous on an interval $[a, b]$. Then f attains its maximum and minimum on $[a, b]$.

Theorem 32 (extrema of continuous functions)

Let f be a function continuous on an interval $[a, b]$. Then f attains its maximum and minimum on $[a, b]$.

Corollary 33 (boundedness of a continuous function)

Let f be a function continuous on an interval $[a, b]$. Then f is bounded on $[a, b]$.

Theorem 34 (continuity of an inverse function)

Let f be a continuous function that is increasing (resp. decreasing) on an interval J. Then the function f^{-1} is continuous and increasing (resp. decreasing) on the interval $f(J)$.

Theorem 35 (logarithm)

There exist a unique function (denoted by log *and called the natural logarithm) with the following properties:*

(L1) $D_{\log} = (0, +\infty)$,

(L2) *the function* log *is increasing on* $(0, +\infty)$,

(L3) $\forall x, y \in (0, +\infty)\colon\ \log xy = \log x + \log y$,

(L4) $\lim\limits_{x \to 1} \frac{\log x}{x-1} = 1$.

Properties of the logarithm

Properties of the logarithm

- $\log 1 = 0$,

Properties of the logarithm

- $\log 1 = 0$,
- $\forall x \in (0, +\infty)\colon \ \log(1/x) = -\log x$,

Properties of the logarithm

- $\log 1 = 0$,
- $\forall x \in (0, +\infty)\colon\ \log(1/x) = -\log x$,
- $\forall n \in \mathbb{Z}\ \forall x \in (0, +\infty)\colon\ \log x^n = n \log x$,

Properties of the logarithm

- $\log 1 = 0$,
- $\forall x \in (0, +\infty)\colon\ \log(1/x) = -\log x$,
- $\forall n \in \mathbb{Z}\ \forall x \in (0, +\infty)\colon\ \log x^n = n \log x$,
- $\lim\limits_{x \to +\infty} \log x = +\infty$, $\lim\limits_{x \to 0+} \log x = -\infty$,

Properties of the logarithm

- $\log 1 = 0$,
- $\forall x \in (0, +\infty)\colon\ \log(1/x) = -\log x$,
- $\forall n \in \mathbb{Z}\ \forall x \in (0, +\infty)\colon\ \log x^n = n \log x$,
- $\lim_{x \to +\infty} \log x = +\infty$, $\lim_{x \to 0+} \log x = -\infty$,
- the function log is continuous on $(0, +\infty)$,

Properties of the logarithm

- $\log 1 = 0$,
- $\forall x \in (0, +\infty)\colon \ \log(1/x) = -\log x$,
- $\forall n \in \mathbb{Z}\ \forall x \in (0, +\infty)\colon \ \log x^n = n \log x$,
- $\lim\limits_{x\to+\infty} \log x = +\infty$, $\lim\limits_{x\to 0+} \log x = -\infty$,
- the function log is continuous on $(0, +\infty)$,
- $R_{\log} = \mathbb{R}$,

Properties of the logarithm

- $\log 1 = 0$,
- $\forall x \in (0, +\infty)\colon \; \log(1/x) = -\log x$,
- $\forall n \in \mathbb{Z}\; \forall x \in (0, +\infty)\colon \; \log x^n = n \log x$,
- $\lim_{x \to +\infty} \log x = +\infty$, $\lim_{x \to 0+} \log x = -\infty$,
- the function log is continuous on $(0, +\infty)$,
- $R_{\log} = \mathbb{R}$,
- there exists a unique number $e \in (0, +\infty)$ satisfying $\log e = 1$.

Definition

The exponential function (denoted by exp) is defined as an inverse function to the function log.

Definition

The exponential function (denoted by exp) is defined as an inverse function to the function log.

Properties of the exponential function

Definition

The exponential function (denoted by exp) is defined as an inverse function to the function log.

Properties of the exponential function

- $D_{\exp} = \mathbb{R}$, $R_{\exp} = (0, +\infty)$,

Definition

The exponential function (denoted by exp) is defined as an inverse function to the function log.

Properties of the exponential function

- $D_{\exp} = \mathbb{R}$, $R_{\exp} = (0, +\infty)$,
- the function exp is continuous and increasing on $\mathbb{R}$,

Definition

The exponential function (denoted by exp) is defined as an inverse function to the function log.

Properties of the exponential function

- $D_{\exp} = \mathbb{R}$, $R_{\exp} = (0, +\infty)$,
- the function exp is continuous and increasing on $\mathbb{R}$,
- $\exp 0 = 1$, $\exp 1 = e$,

Definition

The exponential function (denoted by exp) is defined as an inverse function to the function log.

Properties of the exponential function

- $D_{\exp} = \mathbb{R}$, $R_{\exp} = (0, +\infty)$,
- the function exp is continuous and increasing on $\mathbb{R}$,
- $\exp 0 = 1$, $\exp 1 = e$,
- $\forall x, y \in \mathbb{R}\colon\ \exp(x + y) = \exp(x)\exp(y)$,

Definition

The exponential function (denoted by exp) is defined as an inverse function to the function log.

Properties of the exponential function

- $D_{\exp} = \mathbb{R}$, $R_{\exp} = (0, +\infty)$,
- the function exp is continuous and increasing on $\mathbb{R}$,
- $\exp 0 = 1$, $\exp 1 = e$,
- $\forall x, y \in \mathbb{R}\colon\ \exp(x+y) = \exp(x)\exp(y)$,
- $\forall x \in \mathbb{R}\colon\ \exp(-x) = 1/\exp x$,

Definition

The exponential function (denoted by exp) is defined as an inverse function to the function log.

Properties of the exponential function

- $D_{\exp} = \mathbb{R}$, $R_{\exp} = (0, +\infty)$,
- the function exp is continuous and increasing on $\mathbb{R}$,
- $\exp 0 = 1$, $\exp 1 = e$,
- $\forall x, y \in \mathbb{R}\colon\ \exp(x + y) = \exp(x)\exp(y)$,
- $\forall x \in \mathbb{R}\colon\ \exp(-x) = 1/\exp x$,
- $\forall n \in \mathbb{Z}\ \forall x \in \mathbb{R}\colon\ \exp(nx) = (\exp x)^n$,

Definition

The exponential function (denoted by exp) is defined as an inverse function to the function log.

Properties of the exponential function

- $D_{\exp} = \mathbb{R}$, $R_{\exp} = (0, +\infty)$,
- the function exp is continuous and increasing on $\mathbb{R}$,
- $\exp 0 = 1$, $\exp 1 = e$,
- $\forall x, y \in \mathbb{R}\colon\ \exp(x + y) = \exp(x)\exp(y)$,
- $\forall x \in \mathbb{R}\colon\ \exp(-x) = 1/\exp x$,
- $\forall n \in \mathbb{Z}\ \forall x \in \mathbb{R}\colon\ \exp(nx) = (\exp x)^n$,
- $\lim_{x\to+\infty} \exp x = +\infty$, $\lim_{x\to-\infty} \exp x = 0$,

Definition

The exponential function (denoted by exp) is defined as an inverse function to the function log.

Properties of the exponential function

- $D_{\exp} = \mathbb{R}$, $R_{\exp} = (0, +\infty)$,
- the function exp is continuous and increasing on $\mathbb{R}$,
- $\exp 0 = 1$, $\exp 1 = e$,
- $\forall x, y \in \mathbb{R}\colon\ \exp(x + y) = \exp(x)\exp(y)$,
- $\forall x \in \mathbb{R}\colon\ \exp(-x) = 1/\exp x$,
- $\forall n \in \mathbb{Z}\ \forall x \in \mathbb{R}\colon\ \exp(nx) = (\exp x)^n$,
- $\lim_{x\to+\infty} \exp x = +\infty$, $\lim_{x\to-\infty} \exp x = 0$,
- $\lim_{x\to 0} \frac{\exp(x)-1}{x} = 1$,

Definition

The exponential function (denoted by exp) is defined as an inverse function to the function log.

Properties of the exponential function

- $D_{\exp} = \mathbb{R}$, $R_{\exp} = (0, +\infty)$,
- the function exp is continuous and increasing on $\mathbb{R}$,
- $\exp 0 = 1$, $\exp 1 = e$,
- $\forall x, y \in \mathbb{R}\colon\ \exp(x+y) = \exp(x)\exp(y)$,
- $\forall x \in \mathbb{R}\colon\ \exp(-x) = 1/\exp x$,
- $\forall n \in \mathbb{Z}\ \forall x \in \mathbb{R}\colon\ \exp(nx) = (\exp x)^n$,
- $\lim\limits_{x\to+\infty} \exp x = +\infty$, $\lim\limits_{x\to-\infty} \exp x = 0$,
- $\lim\limits_{x\to 0} \frac{\exp(x)-1}{x} = 1$,
- $\forall r \in \mathbb{Q}\colon\ \exp r = e^r$.

Definition

Let $a, b \in \mathbb{R}$, $a > 0$. The general power a^b is defined by

$$a^b = \exp(b \log a).$$

Definition

Let $a, b \in \mathbb{R}$, $a > 0$. The general power a^b is defined by

$$a^b = \exp(b \log a).$$

Definition

Let $a, b \in (0, +\infty)$, $a \neq 1$. The general logarithm to base a is defined by

$$\log_a b = \frac{\log b}{\log a}.$$

Theorem 36 (the sine and the number π)

There exists a unique positive real number (denoted by π) and a unique function sine (denoted by sin*) with the following properties:*

(S1) $D_{\sin} = \mathbb{R}$,

(S2) sin *is increasing on* $[-\pi/2, \pi/2]$,

(S3) $\sin 0 = 0$,

(S4) $\forall x, y \in \mathbb{R}\colon\ \sin(x+y) = \sin x \cdot \sin(\frac{\pi}{2} - y) + \sin(\frac{\pi}{2} - x) \cdot \sin y$,

(S5) $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$.

Theorem 36 (the sine and the number π)

There exists a unique positive real number (denoted by π) and a unique function sine (denoted by $\sin$) with the following properties:

(S1) $D_{\sin} = \mathbb{R}$,

(S2) $\sin$ *is increasing on* $[-\pi/2, \pi/2]$,

(S3) $\sin 0 = 0$,

(S4) $\forall x, y \in \mathbb{R}\colon\ \sin(x+y) = \sin x \cdot \sin(\frac{\pi}{2} - y) + \sin(\frac{\pi}{2} - x) \cdot \sin y$,

(S5) $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$.

Definition

The function cosine is defined by $\cos x = \sin(\frac{\pi}{2} - x)$, $x \in \mathbb{R}$.

Properties of the sine and cosine

Properties of the sine and cosine

- The function cos is decreasing on $[0, \pi]$.

Properties of the sine and cosine

- The function cos is decreasing on $[0, \pi]$.
- $\cos\frac{\pi}{2} = 0$, $\cos 0 = \sin\frac{\pi}{2} = 1$, $\sin\pi = 0$, $\cos\pi = \sin(-\frac{\pi}{2}) = -1$, $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

Properties of the sine and cosine

- The function cos is decreasing on $[0, \pi]$.
- $\cos\frac{\pi}{2} = 0$, $\cos 0 = \sin\frac{\pi}{2} = 1$, $\sin\pi = 0$, $\cos\pi = \sin(-\frac{\pi}{2}) = -1$, $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
- $\forall x \in \mathbb{R}\colon\ \sin(x+\pi) = -\sin x$

Properties of the sine and cosine

- The function cos is decreasing on $[0, \pi]$.
- $\cos\frac{\pi}{2} = 0$, $\cos 0 = \sin\frac{\pi}{2} = 1$, $\sin\pi = 0$, $\cos\pi = \sin(-\frac{\pi}{2}) = -1$, $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
- $\forall x \in \mathbb{R}\colon\ \sin(x + \pi) = -\sin x$
- The function cos is even, the function sin is odd.

Properties of the sine and cosine

- The function cos is decreasing on $[0, \pi]$.
- $\cos\frac{\pi}{2} = 0$, $\cos 0 = \sin\frac{\pi}{2} = 1$, $\sin\pi = 0$, $\cos\pi = \sin(-\frac{\pi}{2}) = -1$, $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
- $\forall x \in \mathbb{R}\colon\ \sin(x + \pi) = -\sin x$
- The function cos is even, the function sin is odd.
- The functions sin and cos are 2π-periodic.

Properties of the sine and cosine

- The function cos is decreasing on $[0, \pi]$.
- $\cos\frac{\pi}{2} = 0$, $\cos 0 = \sin\frac{\pi}{2} = 1$, $\sin\pi = 0$, $\cos\pi = \sin(-\frac{\pi}{2}) = -1$, $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
- $\forall x \in \mathbb{R}\colon\ \sin(x+\pi) = -\sin x$
- The function cos is even, the function sin is odd.
- The functions sin and cos are 2π-periodic.
- $\forall x \in \mathbb{R}\colon\ \sin^2 x + \cos^2 x = 1$

Properties of the sine and cosine

- The function cos is decreasing on $[0, \pi]$.
- $\cos\frac{\pi}{2} = 0$, $\cos 0 = \sin\frac{\pi}{2} = 1$, $\sin\pi = 0$, $\cos\pi = \sin(-\frac{\pi}{2}) = -1$, $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
- $\forall x \in \mathbb{R}\colon\ \sin(x+\pi) = -\sin x$
- The function cos is even, the function sin is odd.
- The functions sin and cos are 2π-periodic.
- $\forall x \in \mathbb{R}\colon\ \sin^2 x + \cos^2 x = 1$
- $\forall x \in \mathbb{R}\colon\ |\sin x| \leq 1, |\cos x| \leq 1$

Properties of the sine and cosine

- The function cos is decreasing on $[0, \pi]$.
- $\cos\frac{\pi}{2} = 0$, $\cos 0 = \sin\frac{\pi}{2} = 1$, $\sin\pi = 0$, $\cos\pi = \sin(-\frac{\pi}{2}) = -1$, $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
- $\forall x \in \mathbb{R}\colon\ \sin(x+\pi) = -\sin x$
- The function cos is even, the function sin is odd.
- The functions sin and cos are 2π-periodic.
- $\forall x \in \mathbb{R}\colon\ \sin^2 x + \cos^2 x = 1$
- $\forall x \in \mathbb{R}\colon\ |\sin x| \leq 1, |\cos x| \leq 1$
- $\forall x, y \in \mathbb{R}\colon\ \sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$

Properties of the sine and cosine

- The function cos is decreasing on $[0, \pi]$.
- $\cos\frac{\pi}{2} = 0$, $\cos 0 = \sin\frac{\pi}{2} = 1$, $\sin\pi = 0$, $\cos\pi = \sin(-\frac{\pi}{2}) = -1$, $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
- $\forall x \in \mathbb{R}\colon\ \sin(x+\pi) = -\sin x$
- The function cos is even, the function sin is odd.
- The functions sin and cos are 2π-periodic.
- $\forall x \in \mathbb{R}\colon\ \sin^2 x + \cos^2 x = 1$
- $\forall x \in \mathbb{R}\colon\ |\sin x| \leq 1, |\cos x| \leq 1$
- $\forall x, y \in \mathbb{R}\colon\ \sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$
- The functions sin and cos are continuous on $\mathbb{R}$.

Properties of the sine and cosine

- The function cos is decreasing on $[0, \pi]$.
- $\cos\frac{\pi}{2} = 0$, $\cos 0 = \sin\frac{\pi}{2} = 1$, $\sin\pi = 0$, $\cos\pi = \sin(-\frac{\pi}{2}) = -1$, $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
- $\forall x \in \mathbb{R}\colon\ \sin(x+\pi) = -\sin x$
- The function cos is even, the function sin is odd.
- The functions sin and cos are 2π-periodic.
- $\forall x \in \mathbb{R}\colon\ \sin^2 x + \cos^2 x = 1$
- $\forall x \in \mathbb{R}\colon\ |\sin x| \le 1, |\cos x| \le 1$
- $\forall x, y \in \mathbb{R}\colon\ \sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$
- The functions sin and cos are continuous on $\mathbb{R}$.
- $R_{\sin} = R_{\cos} = [-1, 1]$

Properties of the sine and cosine

- The function cos is decreasing on $[0, \pi]$.
- $\cos \frac{\pi}{2} = 0$, $\cos 0 = \sin \frac{\pi}{2} = 1$, $\sin \pi = 0$, $\cos \pi = \sin(-\frac{\pi}{2}) = -1$, $\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$
- $\forall x \in \mathbb{R}\colon\ \sin(x + \pi) = -\sin x$
- The function cos is even, the function sin is odd.
- The functions sin and cos are 2π-periodic.
- $\forall x \in \mathbb{R}\colon\ \sin^2 x + \cos^2 x = 1$
- $\forall x \in \mathbb{R}\colon\ |\sin x| \leq 1, |\cos x| \leq 1$
- $\forall x, y \in \mathbb{R}\colon\ \sin x - \sin y = 2 \sin\left(\frac{x-y}{2}\right) \cos\left(\frac{x+y}{2}\right)$
- The functions sin and cos are continuous on $\mathbb{R}$.
- $R_{\sin} = R_{\cos} = [-1, 1]$
- The function sin is equal to zero exactly at the points of the set $\{k\pi;\ k \in \mathbb{Z}\}$, the function cos is equal to zero exactly et the points of the set $\{\frac{\pi}{2} + k\pi;\ k \in \mathbb{Z}\}$.

Definition

The function tangent is denoted by tg and defined by

$$\operatorname{tg} x = \frac{\sin x}{\cos x}$$

for every $x \in \mathbb{R}$ for which the fraction is defined, i.e.

$$D_{\operatorname{tg}} = \{x \in \mathbb{R};\ x \neq \pi/2 + k\pi, k \in \mathbb{Z}\}.$$

Definition

The function tangent is denoted by tg and defined by

$$\operatorname{tg} x = \frac{\sin x}{\cos x}$$

for every $x \in \mathbb{R}$ for which the fraction is defined, i.e.

$$D_{\mathrm{tg}} = \{x \in \mathbb{R};\ x \neq \pi/2 + k\pi, k \in \mathbb{Z}\}.$$

The function cotangent is denoted by cotg and defined on a set $D_{\mathrm{cotg}} = \{x \in \mathbb{R};\ x \neq k\pi, k \in \mathbb{Z}\}$ by

$$\operatorname{cotg} x = \frac{\cos x}{\sin x}.$$

Properties of the tangent and cotangent

Properties of the tangent and cotangent

- $\operatorname{tg}\frac{\pi}{4} = \operatorname{cotg}\frac{\pi}{4} = 1$

Properties of the tangent and cotangent

- $\operatorname{tg} \frac{\pi}{4} = \operatorname{cotg} \frac{\pi}{4} = 1$
- The functions tg and cotg are continuous at every point of their domains.

Properties of the tangent and cotangent

- $\operatorname{tg}\frac{\pi}{4} = \operatorname{cotg}\frac{\pi}{4} = 1$
- The functions tg and cotg are continuous at every point of their domains.
- The functions tg and cotg are odd.

Properties of the tangent and cotangent

- $\operatorname{tg} \frac{\pi}{4} = \operatorname{cotg} \frac{\pi}{4} = 1$
- The functions tg and cotg are continuous at every point of their domains.
- The functions tg and cotg are odd.
- The functions tg and cotg are π-periodic.

Properties of the tangent and cotangent

- $\operatorname{tg} \frac{\pi}{4} = \operatorname{cotg} \frac{\pi}{4} = 1$
- The functions tg and cotg are continuous at every point of their domains.
- The functions tg and cotg are odd.
- The functions tg and cotg are π-periodic.
- The function tg is increasing on $(-\pi/2, \pi/2)$, the function cotg is decreasing on $(0, \pi)$.

Properties of the tangent and cotangent

- $\operatorname{tg}\frac{\pi}{4} = \operatorname{cotg}\frac{\pi}{4} = 1$
- The functions tg and cotg are continuous at every point of their domains.
- The functions tg and cotg are odd.
- The functions tg and cotg are π-periodic.
- The function tg is increasing on $(-\pi/2, \pi/2)$, the function cotg is decreasing on $(0, \pi)$.
- $\lim\limits_{x \to \frac{\pi}{2}-} \operatorname{tg} x = +\infty$, $\lim\limits_{x \to -\frac{\pi}{2}+} \operatorname{tg} x = -\infty$,
  $\lim\limits_{x \to 0+} \operatorname{cotg} x = +\infty$, $\lim\limits_{x \to \pi-} \operatorname{cotg} x = -\infty$

Properties of the tangent and cotangent

- $\operatorname{tg} \frac{\pi}{4} = \operatorname{cotg} \frac{\pi}{4} = 1$
- The functions tg and cotg are continuous at every point of their domains.
- The functions tg and cotg are odd.
- The functions tg and cotg are π-periodic.
- The function tg is increasing on $(-\pi/2, \pi/2)$, the function cotg is decreasing on $(0, \pi)$.
- $\lim\limits_{x \to \frac{\pi}{2}-} \operatorname{tg} x = +\infty$, $\lim\limits_{x \to -\frac{\pi}{2}+} \operatorname{tg} x = -\infty$, $\lim\limits_{x \to 0+} \operatorname{cotg} x = +\infty$, $\lim\limits_{x \to \pi-} \operatorname{cotg} x = -\infty$
- $R_{\mathrm{tg}} = R_{\mathrm{cotg}} = \mathbb{R}$

Definition

- The function arcsine (denoted by arcsin) is an inverse function to the function $\sin|_{[-\frac{\pi}{2},\frac{\pi}{2}]}$.

Definition

- The function arcsine (denoted by arcsin) is an inverse function to the function $\sin|_{[-\frac{\pi}{2},\frac{\pi}{2}]}$.
- The function arccosine (denoted by arccos) is an inverse function to the function $\cos|_{[0,\pi]}$.

Definition

- The function arcsine (denoted by arcsin) is an inverse function to the function $\sin|_{[-\frac{\pi}{2},\frac{\pi}{2}]}$.
- The function arccosine (denoted by arccos) is an inverse function to the function $\cos|_{[0,\pi]}$.
- The function arctangent (denoted by arctg) is an inverse function to the function $\mathrm{tg}|_{(-\frac{\pi}{2},\frac{\pi}{2})}$.

Definition

- The function arcsine (denoted by arcsin) is an inverse function to the function $\sin|_{[-\frac{\pi}{2},\frac{\pi}{2}]}$.
- The function arccosine (denoted by arccos) is an inverse function to the function $\cos|_{[0,\pi]}$.
- The function arctangent (denoted by arctg) is an inverse function to the function $\mathrm{tg}|_{(-\frac{\pi}{2},\frac{\pi}{2})}$.
- The function arccotangent (denoted by arccotg) is an inverse function to the function $\mathrm{cotg}|_{(0,\pi)}$.

Properties of inverse trigonometric functions

Properties of inverse trigonometric functions

- $D_{\text{arcsin}} = D_{\text{arccos}} = [-1, 1]$, $D_{\text{arctg}} = D_{\text{arccotg}} = \mathbb{R}$

Properties of inverse trigonometric functions

- $D_{\text{arcsin}} = D_{\text{arccos}} = [-1, 1]$, $D_{\text{arctg}} = D_{\text{arccotg}} = \mathbb{R}$
- The functions arcsin and arctg are odd.

Properties of inverse trigonometric functions

- $D_{\text{arcsin}} = D_{\text{arccos}} = [-1, 1]$, $D_{\text{arctg}} = D_{\text{arccotg}} = \mathbb{R}$
- The functions arcsin and arctg are odd.
- The functions arcsin and arctg are increasing, the functions arccos and arccotg are decreasing (on their domains).

Properties of inverse trigonometric functions

- $D_{\text{arcsin}} = D_{\text{arccos}} = [-1, 1]$, $D_{\text{arctg}} = D_{\text{arccotg}} = \mathbb{R}$
- The functions arcsin and arctg are odd.
- The functions arcsin and arctg are increasing, the functions arccos and arccotg are decreasing (on their domains).
- The functions arcsin, arccos, arctg, and arccotg are continuous on their domains.

Properties of inverse trigonometric functions

- $D_{\text{arcsin}} = D_{\text{arccos}} = [-1, 1]$, $D_{\text{arctg}} = D_{\text{arccotg}} = \mathbb{R}$
- The functions arcsin and arctg are odd.
- The functions arcsin and arctg are increasing, the functions arccos and arccotg are decreasing (on their domains).
- The functions arcsin, arccos, arctg, and arccotg are continuous on their domains.
- $\operatorname{arctg} 0 = 0$, $\operatorname{arctg} 1 = \frac{\pi}{4}$, $\operatorname{arccotg} 0 = \frac{\pi}{2}$

Properties of inverse trigonometric functions

- $D_{\text{arcsin}} = D_{\text{arccos}} = [-1, 1]$, $D_{\text{arctg}} = D_{\text{arccotg}} = \mathbb{R}$
- The functions arcsin and arctg are odd.
- The functions arcsin and arctg are increasing, the functions arccos and arccotg are decreasing (on their domains).
- The functions arcsin, arccos, arctg, and arccotg are continuous on their domains.
- $\operatorname{arctg} 0 = 0$, $\operatorname{arctg} 1 = \frac{\pi}{4}$, $\operatorname{arccotg} 0 = \frac{\pi}{2}$
- $\lim_{x \to 0} \frac{\arcsin x}{x} = \lim_{x \to 0} \frac{\operatorname{arctg} x}{x} = 1$

Properties of inverse trigonometric functions

- $D_{\text{arcsin}} = D_{\text{arccos}} = [-1,1]$, $D_{\text{arctg}} = D_{\text{arccotg}} = \mathbb{R}$
- The functions arcsin and arctg are odd.
- The functions arcsin and arctg are increasing, the functions arccos and arccotg are decreasing (on their domains).
- The functions arcsin, arccos, arctg, and arccotg are continuous on their domains.
- $\operatorname{arctg} 0 = 0$, $\operatorname{arctg} 1 = \frac{\pi}{4}$, $\operatorname{arccotg} 0 = \frac{\pi}{2}$
- $\lim\limits_{x\to 0} \frac{\arcsin x}{x} = \lim\limits_{x\to 0} \frac{\operatorname{arctg} x}{x} = 1$
- $\forall x \in [-1,1]\colon\ \arcsin x + \arccos x = \frac{\pi}{2}$,
  $\forall x \in \mathbb{R}\colon\ \operatorname{arctg} x + \operatorname{arccotg} x = \frac{\pi}{2}$

Properties of inverse trigonometric functions

- $D_{\text{arcsin}} = D_{\text{arccos}} = [-1, 1]$, $D_{\text{arctg}} = D_{\text{arccotg}} = \mathbb{R}$
- The functions arcsin and arctg are odd.
- The functions arcsin and arctg are increasing, the functions arccos and arccotg are decreasing (on their domains).
- The functions arcsin, arccos, arctg, and arccotg are continuous on their domains.
- $\operatorname{arctg} 0 = 0$, $\operatorname{arctg} 1 = \frac{\pi}{4}$, $\operatorname{arccotg} 0 = \frac{\pi}{2}$
- $\lim\limits_{x\to 0} \frac{\arcsin x}{x} = \lim\limits_{x\to 0} \frac{\operatorname{arctg} x}{x} = 1$
- $\forall x \in [-1, 1]\colon\ \arcsin x + \arccos x = \frac{\pi}{2}$,
  $\forall x \in \mathbb{R}\colon\ \operatorname{arctg} x + \operatorname{arccotg} x = \frac{\pi}{2}$
- $\lim\limits_{x\to+\infty} \operatorname{arctg} x = \frac{\pi}{2}$, $\lim\limits_{x\to-\infty} \operatorname{arctg} x = -\frac{\pi}{2}$
  $\lim\limits_{x\to+\infty} \operatorname{arccotg} x = 0$, $\lim\limits_{x\to-\infty} \operatorname{arccotg} x = \pi$

Definition

Let f be a function and $a \in \mathbb{R}$. Then

- the derivative of the function f at the point a is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

- the derivative of f at a from the right is defined by

$$f'_{+}(a) = \lim_{h \to 0+} \frac{f(a+h) - f(a)}{h},$$

- the derivative of f at a from the left is defined by

$$f'_{-}(a) = \lim_{h \to 0-} \frac{f(a+h) - f(a)}{h},$$

Definition

Suppose that the function f has a finite derivative at a point $a \in \mathbb{R}$. The line

$$T_a = \left\{[x, y] \in \mathbb{R}^2;\ y = f(a) + f'(a)(x - a)\right\}.$$

is called the tangent to the graph of f at the point $[a, f(a)]$.

Definition

Suppose that the function f has a finite derivative at a point $a \in \mathbb{R}$. The line

$$T_a = \{[x, y] \in \mathbb{R}^2;\ y = f(a) + f'(a)(x - a)\}.$$

is called the tangent to the graph of f at the point $[a, f(a)]$.

Theorem 37

Suppose that the function f has a finite derivative at a point $a \in \mathbb{R}$. Then f is continuous at a.

Theorem 38 (arithmetics of derivatives)

Suppose that the functions f and g have finite derivatives at $a \in \mathbb{R}$ and let $\alpha \in \mathbb{R}$. Then

(i) $(f+g)'(a) = f'(a) + g'(a)$,

Theorem 38 (arithmetics of derivatives)

Suppose that the functions f and g have finite derivatives at $a \in \mathbb{R}$ and let $\alpha \in \mathbb{R}$. Then

(i) $(f+g)'(a) = f'(a) + g'(a)$,

(ii) $(\alpha f)'(a) = \alpha \cdot f'(a)$,

Theorem 38 (arithmetics of derivatives)

Suppose that the functions f and g have finite derivatives at $a \in \mathbb{R}$ and let $\alpha \in \mathbb{R}$. Then

(i) $(f+g)'(a) = f'(a) + g'(a)$,

(ii) $(\alpha f)'(a) = \alpha \cdot f'(a)$,

(iii) $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$,

Theorem 38 (arithmetics of derivatives)

Suppose that the functions f and g have finite derivatives at $a \in \mathbb{R}$ and let $\alpha \in \mathbb{R}$. Then

(i) $(f+g)'(a) = f'(a) + g'(a)$,

(ii) $(\alpha f)'(a) = \alpha \cdot f'(a)$,

(iii) $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$,

(iv) *if $g(a) \neq 0$, then*

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}.$$

Theorem 39 (derivative of a compound function)

Suppose that the function f has a finite derivative at $y_0 \in \mathbb{R}$, the function g has a finite derivative at $x_0 \in \mathbb{R}$, and $y_0 = g(x_0)$. Then

$$(f \circ g)'(x_0) = f'(y_0) \cdot g'(x_0).$$

Theorem 39 (derivative of a compound function)

Suppose that the function f has a finite derivative at $y_0 \in \mathbb{R}$, the function g has a finite derivative at $x_0 \in \mathbb{R}$, and $y_0 = g(x_0)$. Then

$$(f \circ g)'(x_0) = f'(y_0) \cdot g'(x_0).$$

Theorem 40 (derivative of an inverse function)

Let f be a function continuous and strictly monotone on an interval (a, b) and suppose that it has a finite and non-zero derivative $f'(x_0)$ at $x_0 \in (a, b)$. Then the function f^{-1} has a derivative at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}.$$

Derivatives of elementary functions

Derivatives of elementary functions

- $(\text{const.})' = 0$,

Derivatives of elementary functions

- $(\text{const.})' = 0$,
- $(x^n)' = nx^{n-1}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$; $x \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{Z}$, $n < 0$,

Derivatives of elementary functions

- $(\text{const.})' = 0$,
- $(x^n)' = nx^{n-1}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$; $x \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{Z}$, $n < 0$,
- $(\log x)' = \frac{1}{x}$ for $x \in (0, +\infty)$,

Derivatives of elementary functions

- $(\text{const.})' = 0$,
- $(x^n)' = nx^{n-1}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$; $x \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{Z}$, $n < 0$,
- $(\log x)' = \frac{1}{x}$ for $x \in (0, +\infty)$,
- $(\exp x)' = \exp x$ for $x \in \mathbb{R}$,

Derivatives of elementary functions

- $(\text{const.})' = 0$,
- $(x^n)' = nx^{n-1}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$; $x \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{Z}$, $n < 0$,
- $(\log x)' = \frac{1}{x}$ for $x \in (0, +\infty)$,
- $(\exp x)' = \exp x$ for $x \in \mathbb{R}$,
- $(x^a)' = ax^{a-1}$ for $x \in (0, +\infty)$, $a \in \mathbb{R}$,

Derivatives of elementary functions

- $(\text{const.})' = 0$,
- $(x^n)' = nx^{n-1}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$; $x \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{Z}$, $n < 0$,
- $(\log x)' = \frac{1}{x}$ for $x \in (0, +\infty)$,
- $(\exp x)' = \exp x$ for $x \in \mathbb{R}$,
- $(x^a)' = ax^{a-1}$ for $x \in (0, +\infty)$, $a \in \mathbb{R}$,
- $(a^x)' = a^x \log a$ for $x \in \mathbb{R}$, $a \in \mathbb{R}$, $a > 0$,

Derivatives of elementary functions

- $(\text{const.})' = 0$,
- $(x^n)' = nx^{n-1}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$; $x \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{Z}$, $n < 0$,
- $(\log x)' = \frac{1}{x}$ for $x \in (0, +\infty)$,
- $(\exp x)' = \exp x$ for $x \in \mathbb{R}$,
- $(x^a)' = ax^{a-1}$ for $x \in (0, +\infty)$, $a \in \mathbb{R}$,
- $(a^x)' = a^x \log a$ for $x \in \mathbb{R}$, $a \in \mathbb{R}$, $a > 0$,
- $(\sin x)' = \cos x$ for $x \in \mathbb{R}$,

Derivatives of elementary functions

- $(\text{const.})' = 0$,
- $(x^n)' = nx^{n-1}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$; $x \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{Z}$, $n < 0$,
- $(\log x)' = \frac{1}{x}$ for $x \in (0, +\infty)$,
- $(\exp x)' = \exp x$ for $x \in \mathbb{R}$,
- $(x^a)' = ax^{a-1}$ for $x \in (0, +\infty)$, $a \in \mathbb{R}$,
- $(a^x)' = a^x \log a$ for $x \in \mathbb{R}$, $a \in \mathbb{R}$, $a > 0$,
- $(\sin x)' = \cos x$ for $x \in \mathbb{R}$,
- $(\cos x)' = -\sin x$ for $x \in \mathbb{R}$,

Derivatives of elementary functions

- $(\text{const.})' = 0$,
- $(x^n)' = nx^{n-1}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$; $x \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{Z}$, $n < 0$,
- $(\log x)' = \frac{1}{x}$ for $x \in (0, +\infty)$,
- $(\exp x)' = \exp x$ for $x \in \mathbb{R}$,
- $(x^a)' = ax^{a-1}$ for $x \in (0, +\infty)$, $a \in \mathbb{R}$,
- $(a^x)' = a^x \log a$ for $x \in \mathbb{R}$, $a \in \mathbb{R}$, $a > 0$,
- $(\sin x)' = \cos x$ for $x \in \mathbb{R}$,
- $(\cos x)' = -\sin x$ for $x \in \mathbb{R}$,
- $(\operatorname{tg} x)' = \frac{1}{\cos^2 x}$ for $x \in D_{\operatorname{tg}}$,

Derivatives of elementary functions

- $(\text{const.})' = 0$,
- $(x^n)' = nx^{n-1}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$; $x \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{Z}$, $n < 0$,
- $(\log x)' = \frac{1}{x}$ for $x \in (0, +\infty)$,
- $(\exp x)' = \exp x$ for $x \in \mathbb{R}$,
- $(x^a)' = ax^{a-1}$ for $x \in (0, +\infty)$, $a \in \mathbb{R}$,
- $(a^x)' = a^x \log a$ for $x \in \mathbb{R}$, $a \in \mathbb{R}$, $a > 0$,
- $(\sin x)' = \cos x$ for $x \in \mathbb{R}$,
- $(\cos x)' = -\sin x$ for $x \in \mathbb{R}$,
- $(\operatorname{tg} x)' = \frac{1}{\cos^2 x}$ for $x \in D_{\operatorname{tg}}$,
- $(\operatorname{cotg} x)' = -\frac{1}{\sin^2 x}$ for $x \in D_{\operatorname{cotg}}$,

Derivatives of elementary functions

- $(\text{const.})' = 0$,
- $(x^n)' = nx^{n-1}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$; $x \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{Z}$, $n < 0$,
- $(\log x)' = \frac{1}{x}$ for $x \in (0, +\infty)$,
- $(\exp x)' = \exp x$ for $x \in \mathbb{R}$,
- $(x^a)' = ax^{a-1}$ for $x \in (0, +\infty)$, $a \in \mathbb{R}$,
- $(a^x)' = a^x \log a$ for $x \in \mathbb{R}$, $a \in \mathbb{R}$, $a > 0$,
- $(\sin x)' = \cos x$ for $x \in \mathbb{R}$,
- $(\cos x)' = -\sin x$ for $x \in \mathbb{R}$,
- $(\operatorname{tg} x)' = \frac{1}{\cos^2 x}$ for $x \in D_{\operatorname{tg}}$,
- $(\operatorname{cotg} x)' = -\frac{1}{\sin^2 x}$ for $x \in D_{\operatorname{cotg}}$,
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$,

Derivatives of elementary functions

- $(\text{const.})' = 0$,
- $(x^n)' = nx^{n-1}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$; $x \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{Z}$, $n < 0$,
- $(\log x)' = \frac{1}{x}$ for $x \in (0, +\infty)$,
- $(\exp x)' = \exp x$ for $x \in \mathbb{R}$,
- $(x^a)' = ax^{a-1}$ for $x \in (0, +\infty)$, $a \in \mathbb{R}$,
- $(a^x)' = a^x \log a$ for $x \in \mathbb{R}$, $a \in \mathbb{R}$, $a > 0$,
- $(\sin x)' = \cos x$ for $x \in \mathbb{R}$,
- $(\cos x)' = -\sin x$ for $x \in \mathbb{R}$,
- $(\operatorname{tg} x)' = \frac{1}{\cos^2 x}$ for $x \in D_{\operatorname{tg}}$,
- $(\operatorname{cotg} x)' = -\frac{1}{\sin^2 x}$ for $x \in D_{\operatorname{cotg}}$,
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$,
- $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$,

Derivatives of elementary functions

- $(\text{const.})' = 0$,
- $(x^n)' = nx^{n-1}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$; $x \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{Z}$, $n < 0$,
- $(\log x)' = \frac{1}{x}$ for $x \in (0, +\infty)$,
- $(\exp x)' = \exp x$ for $x \in \mathbb{R}$,
- $(x^a)' = ax^{a-1}$ for $x \in (0, +\infty)$, $a \in \mathbb{R}$,
- $(a^x)' = a^x \log a$ for $x \in \mathbb{R}$, $a \in \mathbb{R}$, $a > 0$,
- $(\sin x)' = \cos x$ for $x \in \mathbb{R}$,
- $(\cos x)' = -\sin x$ for $x \in \mathbb{R}$,
- $(\operatorname{tg} x)' = \frac{1}{\cos^2 x}$ for $x \in D_{\text{tg}}$,
- $(\operatorname{cotg} x)' = -\frac{1}{\sin^2 x}$ for $x \in D_{\text{cotg}}$,
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$,
- $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$,
- $(\operatorname{arctg} x)' = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$,

Derivatives of elementary functions

- $(\text{const.})' = 0$,
- $(x^n)' = nx^{n-1}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$; $x \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{Z}$, $n < 0$,
- $(\log x)' = \frac{1}{x}$ for $x \in (0, +\infty)$,
- $(\exp x)' = \exp x$ for $x \in \mathbb{R}$,
- $(x^a)' = ax^{a-1}$ for $x \in (0, +\infty)$, $a \in \mathbb{R}$,
- $(a^x)' = a^x \log a$ for $x \in \mathbb{R}$, $a \in \mathbb{R}$, $a > 0$,
- $(\sin x)' = \cos x$ for $x \in \mathbb{R}$,
- $(\cos x)' = -\sin x$ for $x \in \mathbb{R}$,
- $(\operatorname{tg} x)' = \frac{1}{\cos^2 x}$ for $x \in D_{\operatorname{tg}}$,
- $(\operatorname{cotg} x)' = -\frac{1}{\sin^2 x}$ for $x \in D_{\operatorname{cotg}}$,
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$,
- $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$,
- $(\operatorname{arctg} x)' = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$,
- $(\operatorname{arccotg} x)' = -\frac{1}{1+x^2}$ for $x \in \mathbb{R}$.

Theorem 41 (necessary condition for a local extremum)

Suppose that a function f has a local extremum at $x_0 \in \mathbb{R}$. If $f'(x_0)$ exists, then $f'(x_0) = 0$.

Theorem 42 (Rolle)

Suppose that $a, b \in \mathbb{R}$, $a < b$, and a function f has the following properties:

(i) *it is continuous on the interval $[a, b]$,*

(ii) *it has a derivative (finite or infinite) at every point of the open interval (a, b),*

(iii) $f(a) = f(b)$.

Then there exists $\xi \in (a, b)$ satisfying $f'(\xi) = 0$.

Theorem 42 (Rolle)

Suppose that $a, b \in \mathbb{R}$, $a < b$, and a function f has the following properties:

(i) *it is continuous on the interval $[a, b]$,*

(ii) *it has a derivative (finite or infinite) at every point of the open interval (a, b),*

(iii) *$f(a) = f(b)$.*

Then there exists $\xi \in (a, b)$ satisfying $f'(\xi) = 0$.

Theorem 43 (Lagrange, mean value theorem)

Suppose that $a, b \in \mathbb{R}$, $a < b$, a function f is continuous on an interval $[a, b]$ and has a derivative (finite or infinite) at every point of the interval (a, b). Then there is $\xi \in (a, b)$ satisfying

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 44 (sign of the derivative and monotonicity)

Let $J \subset \mathbb{R}$ be a non-degenerate interval. Suppose that a function f is continuous on J and it has a derivative at every inner point of J (the set of all inner points of J is denoted by $\operatorname{Int} J$).

(i) *If $f'(x) > 0$ for all $x \in \operatorname{Int} J$, then f is increasing on J.*

Theorem 44 (sign of the derivative and monotonicity)

Let $J \subset \mathbb{R}$ be a non-degenerate interval. Suppose that a function f is continuous on J and it has a derivative at every inner point of J (the set of all inner points of J is denoted by $\operatorname{Int} J$).

(i) *If $f'(x) > 0$ for all $x \in \operatorname{Int} J$, then f is increasing on J.*

(ii) *If $f'(x) < 0$ for all $x \in \operatorname{Int} J$, then f is decreasing on J.*

Theorem 44 (sign of the derivative and monotonicity)

Let $J \subset \mathbb{R}$ be a non-degenerate interval. Suppose that a function f is continuous on J and it has a derivative at every inner point of J (the set of all inner points of J is denoted by $\operatorname{Int} J$).

(i) *If $f'(x) > 0$ for all $x \in \operatorname{Int} J$, then f is increasing on J.*

(ii) *If $f'(x) < 0$ for all $x \in \operatorname{Int} J$, then f is decreasing on J.*

(iii) *If $f'(x) \geq 0$ for all $x \in \operatorname{Int} J$, then f in non-decreasing on J.*

Theorem 44 (sign of the derivative and monotonicity)

Let $J \subset \mathbb{R}$ be a non-degenerate interval. Suppose that a function f is continuous on J and it has a derivative at every inner point of J (the set of all inner points of J is denoted by $\operatorname{Int} J$).

(i) *If $f'(x) > 0$ for all $x \in \operatorname{Int} J$, then f is increasing on J.*

(ii) *If $f'(x) < 0$ for all $x \in \operatorname{Int} J$, then f is decreasing on J.*

(iii) *If $f'(x) \geq 0$ for all $x \in \operatorname{Int} J$, then f in non-decreasing on J.*

(iv) *If $f'(x) \leq 0$ for all $x \in \operatorname{Int} J$, then f is non-increasing on J.*

Theorem 45 (computation of a one-sided derivative)

Suppose that a function f is continuous from the right at $a \in \mathbb{R}$ and the limit $\lim_{x \to a+} f'(x)$ exists. Then the derivative $f'_+(a)$ exists and

$$f'_+(a) = \lim_{x \to a+} f'(x).$$

Theorem 46 (l'Hospital's rule)

Suppose that functions f and g have finite derivatives on some punctured neighbourhood of $a \in \mathbb{R}^*$ and the limit $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ exist. Suppose further that one of the following conditions hold:

(i) $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$,

Theorem 46 (l'Hospital's rule)

Suppose that functions f and g have finite derivatives on some punctured neighbourhood of $a \in \mathbb{R}^*$ and the limit $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exist. Suppose further that one of the following conditions hold:

(i) $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$,

(ii) $\lim_{x \to a} |g(x)| = +\infty$.

Theorem 46 (l'Hospital's rule)

Suppose that functions f and g have finite derivatives on some punctured neighbourhood of $a \in \mathbb{R}^*$ and the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exist. Suppose further that one of the following conditions hold:

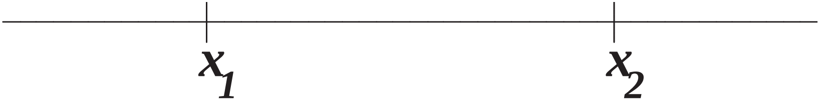
(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$,

(ii) $\lim_{x\to a} |g(x)| = +\infty$.

Then the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Convex combination



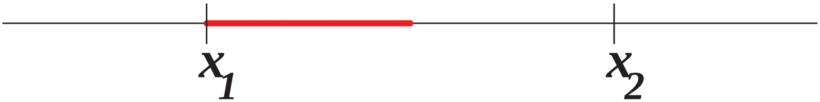
Convex combination

$$1 \cdot x_1 + 0 \cdot x_2 = x_1 + 0 \cdot (x_2 - x_1) = x_1$$

Convex combination

$$0 \cdot x_1 + 1 \cdot x_2 = x_1 + 1 \cdot (x_2 - x_1) = x_2$$

Convex combination



$$\frac{1}{2}x_1 + \frac{1}{2}x_2 = x_1 + \frac{1}{2}(x_2 - x_1)$$

Convex combination

$$\frac{3}{4}x_1 + \frac{1}{4}x_2 = x_1 + \frac{1}{4}(x_2 - x_1)$$

Convex combination

$$\frac{1}{4}x_1 + \frac{3}{4}x_2 = x_1 + \frac{3}{4}(x_2 - x_1)$$

Convex combination

$$\lambda x_1 + (1-\lambda)x_2 = x_1 + (1-\lambda)(x_2 - x_1), \quad \lambda \in [0, 1]$$

Definition

We say that a function f is

- convex on an interval I if
$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2),$$
for each $x_1, x_2 \in I$ and each $\lambda \in [0,1]$;

Definition

We say that a function f is

- convex on an interval I if
$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2),$$
for each $x_1, x_2 \in I$ and each $\lambda \in [0,1]$;
- concave on an interval I if
$$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2),$$
for each $x_1, x_2 \in I$ and each $\lambda \in [0,1]$;

Definition

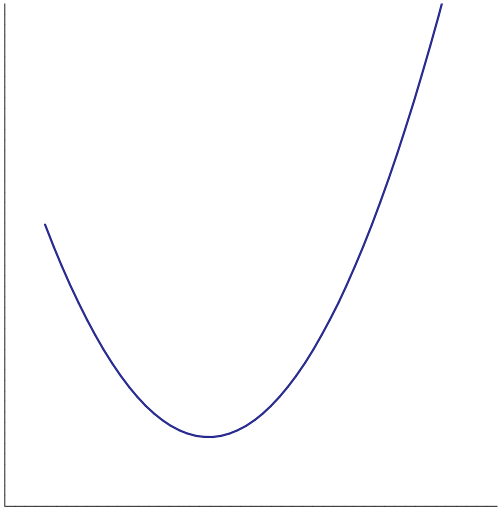
We say that a function f is

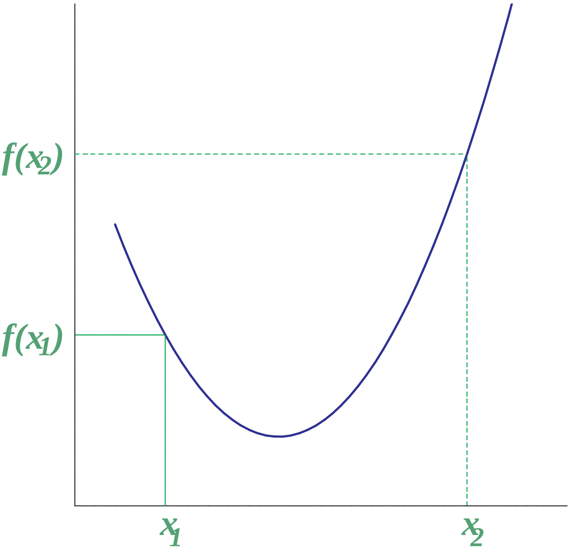
- convex on an interval I if
$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2),$$
for each $x_1, x_2 \in I$ and each $\lambda \in [0,1]$;
- concave on an interval I if
$$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2),$$
for each $x_1, x_2 \in I$ and each $\lambda \in [0,1]$;
- strictly convex on an interval I if
$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda)f(x_2),$$
for each $x_1, x_2 \in I$, $x_1 \neq x_2$ and each $\lambda \in (0,1)$;

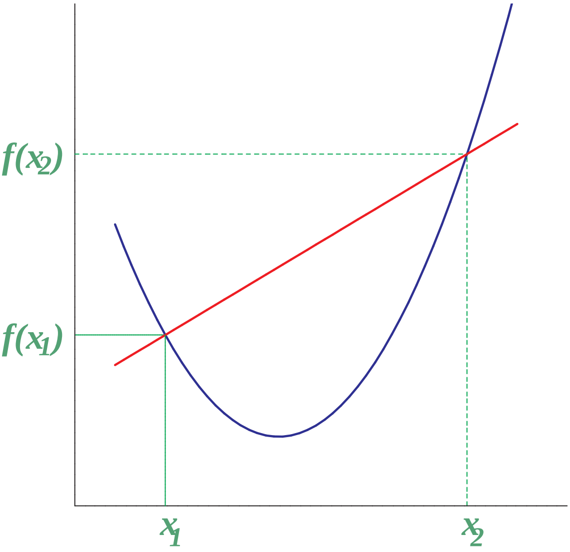
Definition

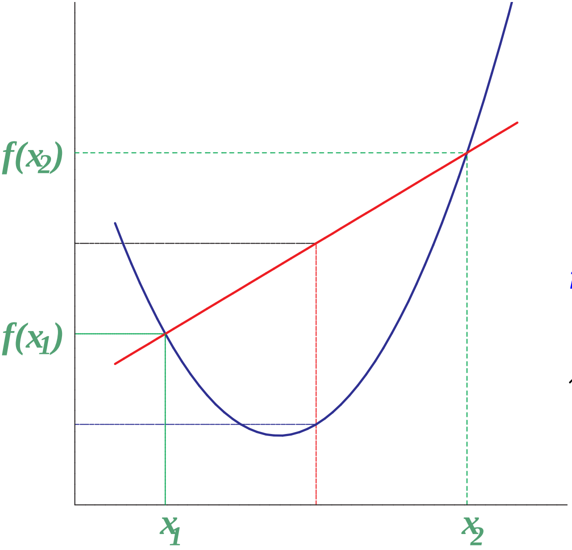
We say that a function f is

- convex on an interval I if
$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2),$$
for each $x_1, x_2 \in I$ and each $\lambda \in [0,1]$;
- concave on an interval I if
$$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2),$$
for each $x_1, x_2 \in I$ and each $\lambda \in [0,1]$;
- strictly convex on an interval I if
$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda)f(x_2),$$
for each $x_1, x_2 \in I$, $x_1 \neq x_2$ and each $\lambda \in (0,1)$;
- strictly concave on an interval I if
$$f(\lambda x_1 + (1-\lambda)x_2) > \lambda f(x_1) + (1-\lambda)f(x_2).$$
for each $x_1, x_2 \in I$, $x_1 \neq x_2$ and each $\lambda \in (0,1)$.




f(x2)
f(x1)
x1
x2

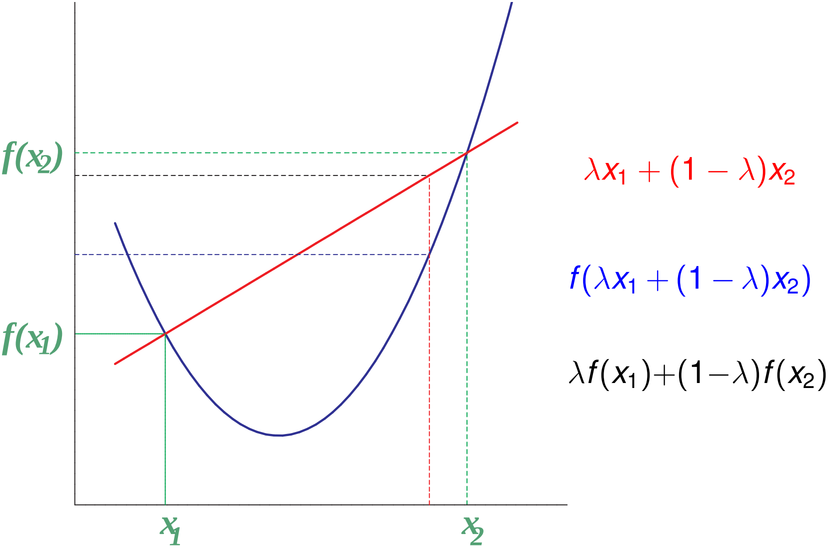

f(x2)
f(x1)
x1
x2



$\lambda x_1 + (1-\lambda)x_2$

$f(\lambda x_1 + (1-\lambda)x_2)$

$\lambda f(x_1) + (1-\lambda) f(x_2)$


f(x2)
f(x1)
x1
x2
$\lambda x_1 + (1-\lambda) x_2$
$f(\lambda x_1 + (1-\lambda) x_2)$
$\lambda f(x_1) + (1-\lambda) f(x_2)$

Lemma 47

A function f is convex on an interval I if and only if

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

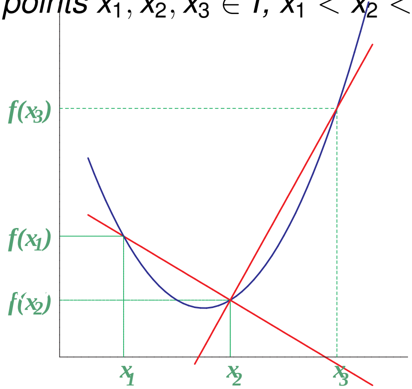
for each three points $x_1, x_2, x_3 \in I$, $x_1 < x_2 < x_3$.

Lemma 47

A function f is convex on an interval I if and only if

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

for each three points $x_1, x_2, x_3 \in I$, $x_1 < x_2 < x_3$.



Definition

Suppose that a function f has a finite derivative on some neighbourhood of $a \in \mathbb{R}$. The second derivative of f at a is defined by

$$f''(a) = \lim_{h \to 0} \frac{f'(a+h) - f'(a)}{h}$$

if the limit exists.

Definition

Suppose that a function f has a finite derivative on some neighbourhood of $a \in \mathbb{R}$. The second derivative of f at a is defined by

$$f''(a) = \lim_{h \to 0} \frac{f'(a+h) - f'(a)}{h}$$

if the limit exists.

Let $n \in \mathbb{N}$ and suppose that f has a finite nth derivative (denoted by $f^{(n)}$) on some neighbourhood of $a \in \mathbb{R}$. Then the $(n+1)$th derivative of f at a is defined by

$$f^{(n+1)}(a) = \lim_{h \to 0} \frac{f^{(n)}(a+h) - f^{(n)}(a)}{h}$$

if the limit exists.

Theorem 48 (second derivative and convexity)

Let $a, b \in \mathbb{R}^*$, $a < b$, and suppose that a function f has a finite second derivative on the interval (a, b).

(i) *If $f''(x) > 0$ for each $x \in (a, b)$, then f is strictly convex on (a, b).*

Theorem 48 (second derivative and convexity)

Let $a, b \in \mathbb{R}^*$, $a < b$, and suppose that a function f has a finite second derivative on the interval (a, b).

(i) *If $f''(x) > 0$ for each $x \in (a, b)$, then f is strictly convex on (a, b).*

(ii) *If $f''(x) < 0$ for each $x \in (a, b)$, then f is strictly concave on (a, b).*

Theorem 48 (second derivative and convexity)

Let $a, b \in \mathbb{R}^*$, $a < b$, and suppose that a function f has a finite second derivative on the interval (a, b).

(i) *If $f''(x) > 0$ for each $x \in (a, b)$, then f is strictly convex on (a, b).*

(ii) *If $f''(x) < 0$ for each $x \in (a, b)$, then f is strictly concave on (a, b).*

(iii) *If $f''(x) \geq 0$ for each $x \in (a, b)$, then f is convex on (a, b).*

Theorem 48 (second derivative and convexity)

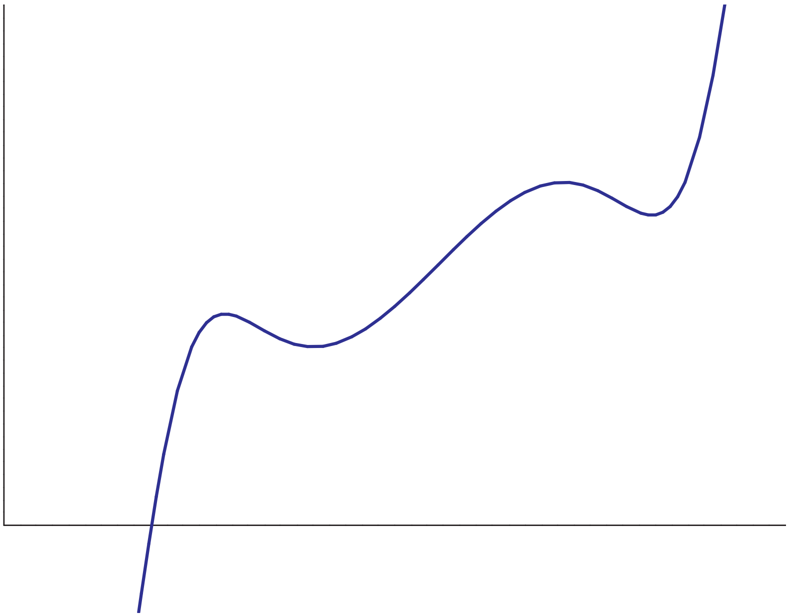
Let $a, b \in \mathbb{R}^*$, $a < b$, and suppose that a function f has a finite second derivative on the interval (a, b).

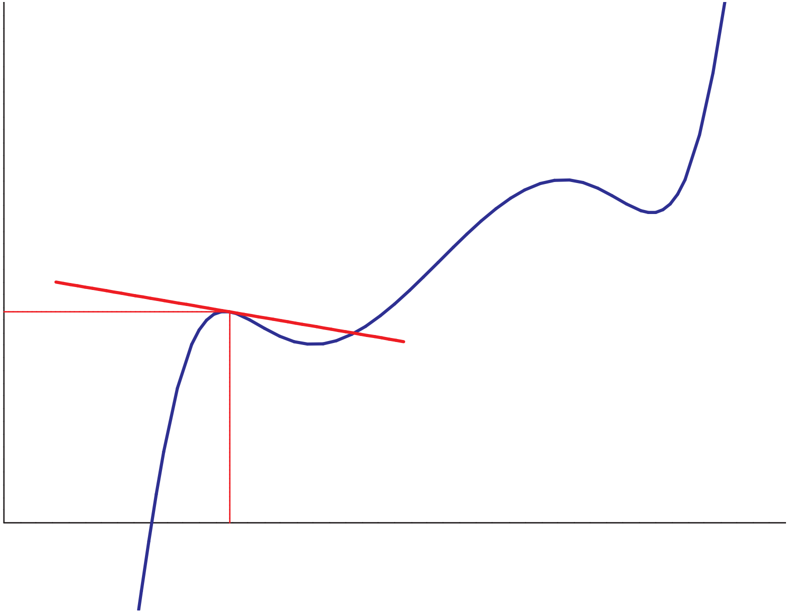
(i) *If $f''(x) > 0$ for each $x \in (a, b)$, then f is strictly convex on (a, b).*

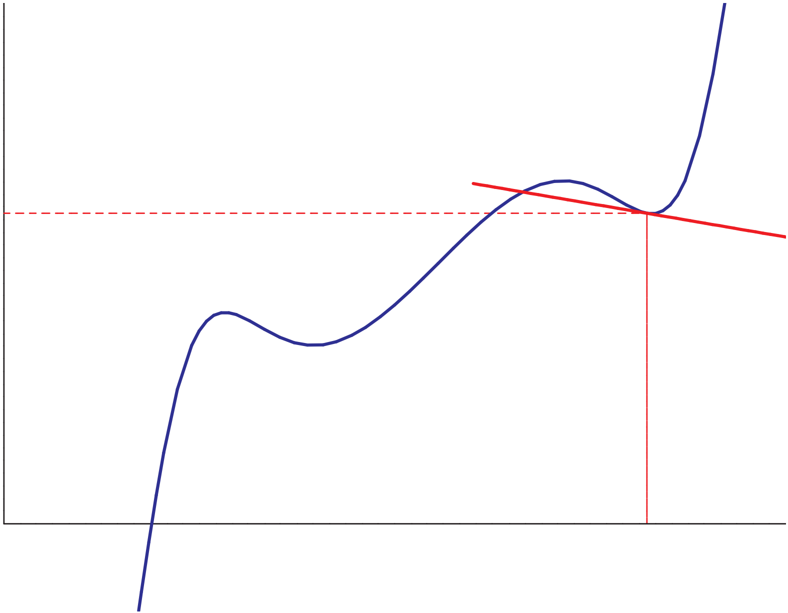
(ii) *If $f''(x) < 0$ for each $x \in (a, b)$, then f is strictly concave on (a, b).*

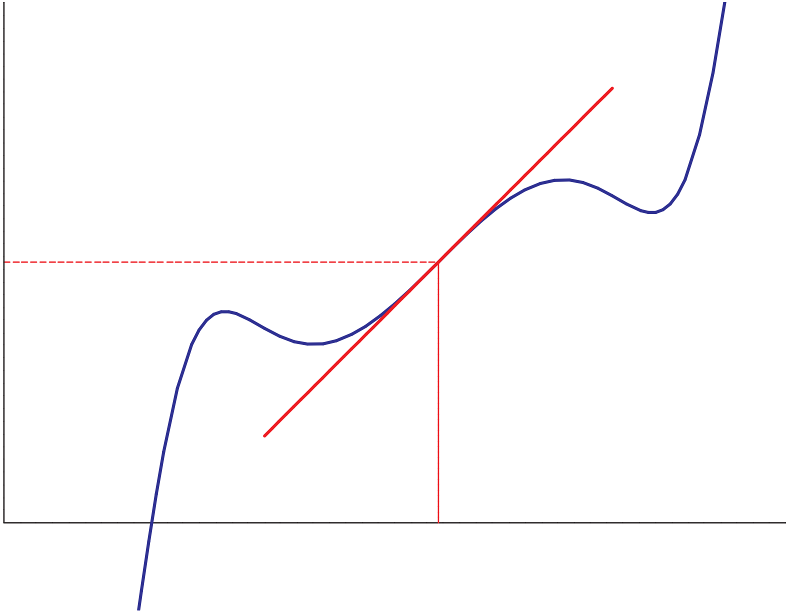
(iii) *If $f''(x) \geq 0$ for each $x \in (a, b)$, then f is convex on (a, b).*

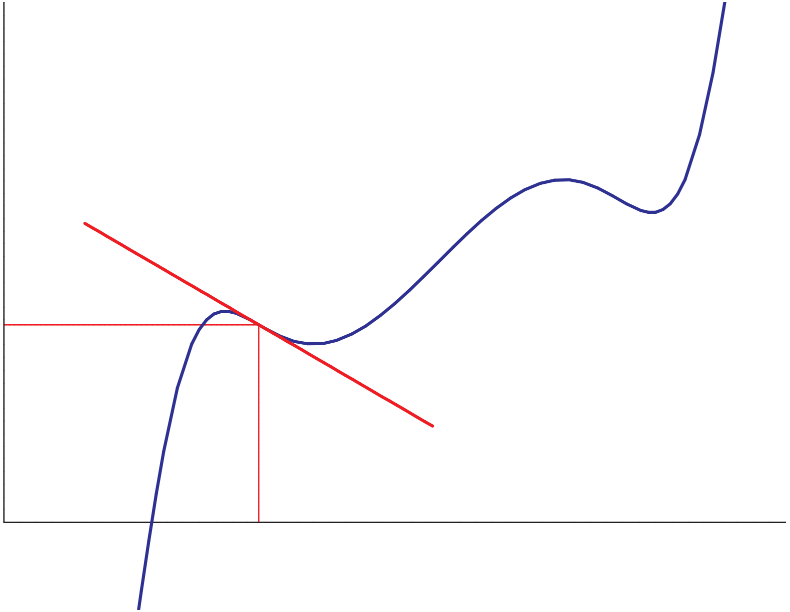
(iv) *If $f''(x) \leq 0$ for each $x \in (a, b)$, then f is concave on (a, b).*

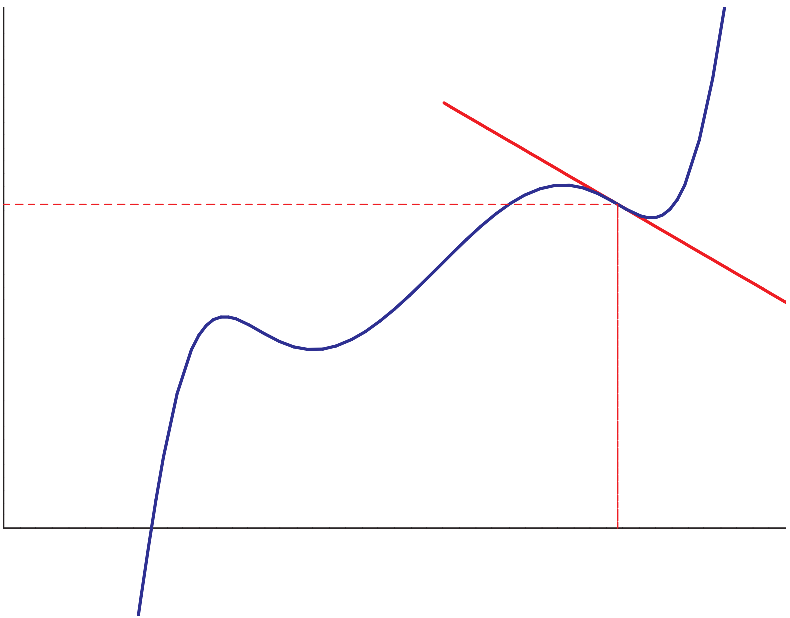












Definition

Suppose that a function f has a finite derivative at $a \in \mathbb{R}$ and let T_a denote the tangent to the graph of f at $[a, f(a)]$. We say that the point $[x, f(x)]$ lies below the tangent T_a if

$$f(x) < f(a) + f'(a) \cdot (x - a).$$

We say that the point $[x, f(x)]$ lies above the tangent T_a if the opposite inequality holds.

Definition

Suppose that a function f has a finite derivative at $a \in \mathbb{R}$ and let T_a denote the tangent to the graph of f at $[a, f(a)]$. We say that a is an inflection point of f if there is $\Delta > 0$ such that

(i) $\forall x \in (a - \Delta, a)\colon [x, f(x)]$ lies below the tangent T_a,

(ii) $\forall x \in (a, a + \Delta)\colon [x, f(x)]$ lies above the tangent T_a,

Definition

Suppose that a function f has a finite derivative at $a \in \mathbb{R}$ and let T_a denote the tangent to the graph of f at $[a, f(a)]$. We say that a is an inflection point of f if there is $\Delta > 0$ such that

(i) $\forall x \in (a - \Delta, a)$: $[x, f(x)]$ lies below the tangent T_a,

(ii) $\forall x \in (a, a + \Delta)$: $[x, f(x)]$ lies above the tangent T_a,

or

(i) $\forall x \in (a - \Delta, a)$: $[x, f(x)]$ lies above the tangent T_a,

(ii) $\forall x \in (a, a + \Delta)$: $[x, f(x)]$ lies below the tangent T_a.

Theorem 49 (necessary condition for inflection)

Let $a \in \mathbb{R}$ be an inflection point of a function f. Then $f''(a)$ either does not exist or equals zero.

Theorem 49 (necessary condition for inflection)

Let $a \in \mathbb{R}$ be an inflection point of a function f. Then $f''(a)$ either does not exist or equals zero.

Theorem 50 (sufficient condition for inflection)

Suppose that a function f has a continuous first derivative on an interval (a, b) and $z \in (a, b)$. Suppose further that

- $\forall x \in (a, z)\colon f''(x) > 0$,
- $\forall x \in (z, b)\colon f''(x) < 0$.

Then z is an inflection point of f.

Definition

The line which is a graph of an affine function $x \mapsto kx + q$, $k, q \in \mathbb{R}$, is called an asymptote of the function f at $+\infty$ (resp. v $-\infty$) if

$$\lim_{x\to+\infty}(f(x)-kx-q) = 0, \quad \text{(resp. } \lim_{x\to-\infty}(f(x)-kx-q) = 0\text{).}$$

Definition

The line which is a graph of an affine function $x \mapsto kx + q$, $k, q \in \mathbb{R}$, is called an asymptote of the function f at $+\infty$ (resp. v $-\infty$) if

$$\lim_{x\to+\infty}(f(x)-kx-q) = 0, \quad \text{(resp. } \lim_{x\to-\infty}(f(x)-kx-q) = 0).$$

Proposition 51

A function f has an asymptote at $+\infty$ given by the affine function $x \mapsto kx + q$ if and only if

$$\lim_{x\to+\infty}\frac{f(x)}{x} = k \in \mathbb{R} \quad \textit{and} \quad \lim_{x\to+\infty}(f(x) - kx) = q \in \mathbb{R}.$$

Investigation of a function

1. Determine the domain and discuss the continuity of the function.

Investigation of a function

1. Determine the domain and discuss the continuity of the function.
2. Find out symmetries: oddness, evenness, periodicity.

Investigation of a function

1. Determine the domain and discuss the continuity of the function.
2. Find out symmetries: oddness, evenness, periodicity.
3. Find the limits at the “endpoints of the domain”.

Investigation of a function

1. Determine the domain and discuss the continuity of the function.
2. Find out symmetries: oddness, evenness, periodicity.
3. Find the limits at the “endpoints of the domain”.
4. Investigate the first derivative, find the intervals of monotonicity and local and global extrema. Determine the range.

Investigation of a function

1. Determine the domain and discuss the continuity of the function.
2. Find out symmetries: oddness, evenness, periodicity.
3. Find the limits at the “endpoints of the domain”.
4. Investigate the first derivative, find the intervals of monotonicity and local and global extrema. Determine the range.
5. Find the second derivative and determine the intervals where the function is concave or convex. Find the inflection points.

Investigation of a function

1. Determine the domain and discuss the continuity of the function.
2. Find out symmetries: oddness, evenness, periodicity.
3. Find the limits at the “endpoints of the domain”.
4. Investigate the first derivative, find the intervals of monotonicity and local and global extrema. Determine the range.
5. Find the second derivative and determine the intervals where the function is concave or convex. Find the inflection points.
6. Find the asymptotes of the function.

Investigation of a function

1. Determine the domain and discuss the continuity of the function.
2. Find out symmetries: oddness, evenness, periodicity.
3. Find the limits at the “endpoints of the domain”.
4. Investigate the first derivative, find the intervals of monotonicity and local and global extrema. Determine the range.
5. Find the second derivative and determine the intervals where the function is concave or convex. Find the inflection points.
6. Find the asymptotes of the function.
7. Draw the graph of the function.