I. Introduction

I.1. Sets

We take a set to be a collection of definite and distinguishable objects into a coherent whole.

- $x \in A \dots x$ is an element (or member) of the set A
- $x \notin A \dots x$ is not a member of the set A
- $A \subset B$ ... the set A is a subset of the set B (*inclusion*)
- $A = B$ ... the sets A and B have the same elements; the following holds: $A \subset B$ and $B \subset A$
- $\emptyset$ ... an empty set
- $A \cup B$ ... a union of the sets A and B
- $A \cap B$ ... an intersection of the sets A and B
- disjoint sets ... A and B are disjoint if $A \cap B = \emptyset$
- $A \setminus B = \{x \in A;\ x \notin B\}$ ... a difference of the sets A and B
- $A_1 \times \cdots \times A_m = \{[a_1, \dots, a_m];\ a_1 \in A_1, \dots, a_m \in A_m\}$ ... a Cartesian product

Let I be a non-empty set of indices and suppose we have a system of sets A_α, where the indices α run over I.

- $\bigcup\limits_{\alpha \in I} A_\alpha$ ... the set of all elements belonging to at least one of the sets A_α
- $\bigcap\limits_{\alpha \in I} A_\alpha$ ... the set of all elements belonging to every A_α

I.2. Logic, methods of proofs

A *statement* (or proposition) is a sentence which can be declared to be either true or false.

- $\neg$, also non ... *negation*
- $\&$ (also $\wedge$) ... *conjunction*, logical "and"
- $\vee$ ... *disjuction* (alternative), logical "or"
- $\Rightarrow$ ... *implication*
- $\Leftrightarrow$ ... *equivalence*; "if and only if"

Tautology is a compound statement, which is true independently of the truthness of its elementary statements.
Examples of tautologies:

- $A \vee \neg A$
- $\neg(A \ \& \ \neg A)$
- $\big((A \ \& \ B) \ \& \ C\big) \Leftrightarrow \big(A \ \& \ (B \ \& \ C)\big)$
- $\neg(A \ \& \ B) \Leftrightarrow (\neg A \vee \neg B)$
- $\neg(A \vee B) \Leftrightarrow (\neg A \ \& \ \neg B)$
- $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$
- $\neg(A \Rightarrow B) \Leftrightarrow (A \ \& \ \neg B)$
- $(A \Leftrightarrow B) \Leftrightarrow \big((A \Rightarrow B) \ \& \ (B \Rightarrow A)\big)$
- $(A \Rightarrow B) \Leftrightarrow (\neg A \vee B)$

A *predicate* (or propositional function) is an expression or sentence involving variables which becomes a statement once we substitute certain elements of a given set for the variables.

General form:

$$V(x), x \in M$$

$$V(x_1, \dots, x_n), x_1 \in M_1, \dots, x_n \in M_n$$

If $A(x)$, $x \in M$ is a predicate, then the statement "$A(x)$ holds for every x from M." is shortened to

$$\forall x \in M\colon A(x).$$

The statement "There exists x in M such that $A(x)$ holds." is shortened to

$$\exists x \in M\colon A(x).$$

The statement "There is only one x in M such that $A(x)$ holds." is shortened to

$$\exists! x \in M\colon A(x).$$

If $A(x)$, $x \in M$ and $B(x)$, $x \in M$ are predicates, then

$$\forall x \in M, B(x)\colon A(x) \quad \text{means} \quad \forall x \in M\colon (B(x) \Rightarrow A(x)),$$

$$\exists x \in M, B(x)\colon A(x) \quad \text{means} \quad \exists x \in M\colon (A(x) \ \& \ B(x)).$$

Negations of the statements with quantifiers:

$$\neg(\forall x \in M\colon A(x)) \quad \text{is the same as} \quad \exists x \in M\colon \neg A(x),$$

$$\neg(\exists x \in M\colon A(x)) \quad \text{is the same as} \quad \forall x \in M\colon \neg A(x).$$

Methods of proofs

- direct proof
- indirect proof
- proof by contradiction
- mathematical induction

Theorem 1 (de Morgan rules)**.** *Let S, A_α, $\alpha \in I$, where $I \neq \emptyset$, be sets. Then*

$$S \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (S \setminus A_\alpha) \qquad \textit{and} \qquad S \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (S \setminus A_\alpha).$$

Theorem 2 (Cauchy inequality)**.** *Let $a_1, \dots, a_n$, $b_1, \dots, b_n$ be real numbers. Then*

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right).$$

Example (irrationality of $\sqrt{2}$). If a real number x solves the equation $x^2 = 2$, then x is not rational.

I.3. Number sets

Rational numbers

- A set of natural numbers
$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$
- A set of integers
$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n;\ n \in \mathbb{N}\} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$
- A set of rational numbers
$$\mathbb{Q} = \left\{\frac{p}{q};\ p \in \mathbb{Z}, q \in \mathbb{N}\right\},$$
where $\frac{p_1}{q_1} = \frac{p_2}{q_2}$ if and only if $p_1 \cdot q_2 = p_2 \cdot q_1$.

Real numbers

By a set of real numbers $\mathbb{R}$ we will understand a set on which there are operations of *addition* and *multiplication* (denoted by $+$ and $\cdot$), and a relation of *ordering* (denoted by $\leq$), such that it has the following three groups of properties.

I. The properties of addition and multiplication and their relationships.

II. The relationships of the ordering and the operations of addition and multiplication.

III. The infimum axiom.

The properties of addition and multiplication and their relationships:

- $\forall x, y \in \mathbb{R}\colon x + y = y + x$ (*commutativity of addition*),
- $\forall x, y, z \in \mathbb{R}\colon x + (y + z) = (x + y) + z$ (*associativity*),
- There is an element in $\mathbb{R}$ (denoted by 0 and called a *zero element*), such that $x + 0 = x$ for every $x \in \mathbb{R}$,
- $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\colon x + y = 0$ (y is called the *negative* of x, such y is only one, denoted by $-x$),
- $\forall x, y \in \mathbb{R}\colon x \cdot y = y \cdot x$ (*commutativity*),
- $\forall x, y, z \in \mathbb{R}\colon x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (*associativity*),
- There is a non-zero element in $\mathbb{R}$ (called *identity*, denoted by 1), such that $1 \cdot x = x$ for every $x \in \mathbb{R}$,
- $\forall x \in \mathbb{R} \setminus \{0\}\ \exists y \in \mathbb{R}\colon x \cdot y = 1$ (such y is only one, denoted by x^{-1} or $\frac{1}{x}$),
- $\forall x, y, z \in \mathbb{R}\colon (x + y) \cdot z = x \cdot z + y \cdot z$ (*distributivity*).

The relationships of the ordering and the operations of addition and multiplication:

- $\forall x, y, z \in \mathbb{R}\colon (x \leq y\ \ \&\ \ y \leq z) \Rightarrow x \leq z$ (*transitivity*),
- $\forall x, y \in \mathbb{R}\colon (x \leq y\ \ \&\ \ y \leq x) \Rightarrow x = y$ (*weak antisymmetry*),
- $\forall x, y \in \mathbb{R}\colon x \leq y \vee y \leq x$,
- $\forall x, y, z \in \mathbb{R}\colon x \leq y \Rightarrow x + z \leq y + z$,
- $\forall x, y \in \mathbb{R}\colon (0 \leq x\ \ \&\ \ 0 \leq y) \Rightarrow 0 \leq x \cdot y$.

Definition. We say that the set $M \subset \mathbb{R}$ is *bounded from below* if there exists a number $a \in \mathbb{R}$ such that for each $x \in M$ we have $x \geq a$. Such a number a is called a *lower bound* of the set M. Analogously we define the notions of a *set bounded from above* and an *upper bound*. We say that a set $M \subset \mathbb{R}$ is *bounded* if it is bounded from above and below.

The infimum axiom:

Let M be a non-empty set bounded from below. Then there exists a unique number $g \in \mathbb{R}$ such that

(i) $\forall x \in M\colon x \geq g$,

(ii) $\forall g' \in \mathbb{R}, g' > g\ \exists x \in M\colon x < g'$.

The number g is denoted by $\inf M$ and is called the *infimum* of the set M.

Remark.

- The infimum axiom says that every non-empty set bounded from below has an infimum.
- The infimum of the set M is its greatest lower bound.
- The real numbers exist and are uniquely determined by the properties I–III.

The following hold:

(i) $\forall x \in \mathbb{R}\colon x \cdot 0 = 0 \cdot x = 0$,

(ii) $\forall x \in \mathbb{R}\colon -x = (-1) \cdot x$,

(iii) $\forall x, y \in \mathbb{R}\colon xy = 0 \Rightarrow (x = 0 \vee y = 0)$,

(iv) $\forall x \in \mathbb{R}\ \forall n \in \mathbb{N}\colon x^{-n} = (x^{-1})^n$,

(v) $\forall x, y \in \mathbb{R} \colon (x > 0 \wedge y > 0) \Rightarrow xy > 0$,

(vi) $\forall x \in \mathbb{R}, x \geq 0\ \forall y \in \mathbb{R}, y \geq 0\ \forall n \in \mathbb{N} \colon x < y \Leftrightarrow x^n < y^n$.

Let $a, b \in \mathbb{R}$, $a \leq b$. We denote:

- An *open interval* $(a, b) = \{x \in \mathbb{R};\ a < x < b\}$,
- A *closed interval* $[a, b] = \{x \in \mathbb{R};\ a \leq x \leq b\}$,
- A *half-open interval* $[a, b) = \{x \in \mathbb{R};\ a \leq x < b\}$,
- A *half-open interval* $(a, b] = \{x \in \mathbb{R};\ a < x \leq b\}$.

The point a is called the *left endpoint of the interval*, The point b is called the *right endpoint of the interval*. A point in the interval which is not an endpoint is called an *inner point of the interval*.

Unbounded intervals:

$$(a, +\infty) = \{x \in \mathbb{R};\ a < x\}, \quad (-\infty, a) = \{x \in \mathbb{R};\ x < a\},$$

analogically $(-\infty, a]$, $[a, +\infty)$ and $(-\infty, +\infty)$. We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. If we transfer the addition and multiplication from $\mathbb{R}$ to the above sets, we obtain the usual operations on these sets.

A real number that is not rational is called *irrational*. The set $\mathbb{R} \setminus \mathbb{Q}$ is called the *set of irrational numbers*.

Complex numbers

By the set of *complex numbers* we mean the set of all expressions of the form $a + bi$, where $a, b \in \mathbb{R}$. The set of all complex numbers is denoted by $\mathbb{C}$. On $\mathbb{C}$ there are operations of addition and multiplication satisfying the group of properties I and moreover $i \cdot i = -1$.

Theorem ("fundamental theorem of algebra"). *Let $n \in \mathbb{N}$, $a_0, \dots, a_n \in \mathbb{C}$, $a_n \neq 0$. Then the equation*

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \dots + a_1 z + a_0 = 0$$

has at least one solution $z \in \mathbb{C}$.

Consequences of the infimum axiom

Definition. Let $M \subset \mathbb{R}$. A number $G \in \mathbb{R}$ satisfying

(i) $\forall x \in M \colon x \leq G$,

(ii) $\forall G' \in \mathbb{R}, G' < G\ \exists x \in M \colon x > G'$,

is called a *supremum* of the set M.

Theorem 3 (Supremum theorem). *Let $M \subset \mathbb{R}$ be a non-empty set bounded from above. Then there exists a unique supremum of the set M.*

The supremum of the set M is denoted by $\sup M$.

The following holds: $\sup M = -\inf(-M)$.

Definition. Let $M \subset \mathbb{R}$. We say that a is a *maximum* of the set M (denoted by $\max M$) if a is an upper bound of M and $a \in M$. Analogously we define a *minimum* of M, denoted by $\min M$.

Lemma 4 ("no holes"). *Let $M \subset \mathbb{R}$ and*

$$\forall x, y \in M\ \forall z \in \mathbb{R}, x < z < y \colon z \in M.$$

Then M is an interval.

Theorem 5 (Archimedean property). *For every $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ satisfying $n > x$.*

Theorem 6 (existence of an integer part). *For every $r \in \mathbb{R}$ there exists an integer part of r, i.e. a number $k \in \mathbb{Z}$ satisfying $k \leq r < k + 1$. The integer part of r is determined uniquely and it is denoted by $[r]$.*

Theorem 7 (nth root). *For every $x \in [0, +\infty)$ and every $n \in \mathbb{N}$ there exists a unique $y \in [0, +\infty)$ satisfying $y^n = x$.*

Theorem 8 (density of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$). *Let $a, b \in \mathbb{R}$, $a < b$. Then there exist $r \in \mathbb{Q}$ satisfying $a < r < b$ and $s \in \mathbb{R} \setminus \mathbb{Q}$ satisfying $a < s < b$.*

II. Limit of a sequence

II.1. Introduction

Definition. Suppose that to each natural number $n \in \mathbb{N}$ we assign a real number a_n. Then we say that $\{a_n\}_{n=1}^{\infty}$ is a *sequence* of real numbers. The number a_n is called the *nth member* of this sequence.

A sequence $\{a_n\}_{n=1}^{\infty}$ is equal to a sequence $\{b_n\}_{n=1}^{\infty}$ if $a_n = b_n$ holds for every $n \in \mathbb{N}$.

By the *set of all members of the sequence* $\{a_n\}_{n=1}^{\infty}$ we understand a set

$$\{x \in \mathbb{R};\ \exists n \in \mathbb{N} : a_n = x\}.$$

Definition. We say that a sequence $\{a_n\}$ is

- *bounded from above* if the set of all members of this sequence is bounded from above,
- *bounded from below* if the set of all members of this sequence is bounded from below,
- *bounded* if the set of all members of this sequence is bounded.

Definition. We say that a sequence $\{a_n\}$ is

- *increasing* if $a_n < a_{n+1}$ for every $n \in \mathbb{N}$,
- *decreasing* if $a_n > a_{n+1}$ for every $n \in \mathbb{N}$,
- *non-decreasing* if $a_n \le a_{n+1}$ for every $n \in \mathbb{N}$,
- *non-increasing* if $a_n \ge a_{n+1}$ for every $n \in \mathbb{N}$.

A sequence $\{a_n\}$ is *monotone* if it satisfies one of the conditions above. A sequence $\{a_n\}$ is *strictly monotone* if it is increasing or decreasing.

Definition. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.

- By the *sum of sequences* $\{a_n\}$ and $\{b_n\}$ we understand a sequence $\{a_n + b_n\}$.
- Analogously we define a *difference* and a *product of sequences*.
- Suppose all the members of the sequence $\{b_n\}$ are non-zero. Then by the *quotient of sequences* $\{a_n\}$ and $\{b_n\}$ we understand a sequence $\{\frac{a_n}{b_n}\}$.
- If $\lambda \in \mathbb{R}$, then by the λ-multiple of the sequence $\{a_n\}$ we understand a sequence $\{\lambda a_n\}$.

II.2. Convergence of sequences

Definition. We say that a sequence $\{a_n\}$ has a *limit* which equals to a number $A \in \mathbb{R}$ if to every positive real number ε there exists a natural number n_0 such that for every index $n \ge n_0$ we have $|a_n - A| < \varepsilon$, i.e.

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \ge n_0 \colon\ |a_n - A| < \varepsilon.$$

We say that a sequence $\{a_n\}$ is *convergent* if there exists $A \in \mathbb{R}$ which is a limit of $\{a_n\}$.

Theorem 9 (uniqueness of a limit)**.** *Every sequence has at most one limit.*

We use the notation $\lim\limits_{n\to\infty} a_n = A$ or simply $\lim a_n = A$.

Remark. Let $\{a_n\}$ be a sequence of real numbers and $A \in \mathbb{R}$. Then

$$\lim a_n = A \Leftrightarrow \lim(a_n - A) = 0 \Leftrightarrow \lim |a_n - A| = 0.$$

Theorem 10. *Every convergent sequence is bounded.*

Definition. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that a sequence $\{b_k\}_{k=1}^{\infty}$ is a *subsequence* of $\{a_n\}_{n=1}^{\infty}$ if there is an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that $b_k = a_{n_k}$ for every $k \in \mathbb{N}$.

Theorem 11 (limit of a subsequence)**.** *Let $\{b_k\}_{k=1}^{\infty}$ be a subsequence of $\{a_n\}_{n=1}^{\infty}$. If $\lim_{n\to\infty} a_n = A \in \mathbb{R}$, then also $\lim_{k\to\infty} b_k = A$.*

Remark. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, $A \in \mathbb{R}$, $K \in \mathbb{R}$, $K > 0$. If

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0\colon\ |a_n - A| < K\varepsilon,$$

then $\lim a_n = A$.

Theorem 12 (arithmetics of limits)**.** *Suppose that* $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$. *Then*

(i) $\lim(a_n + b_n) = A + B$*,*

(ii) $\lim(a_n \cdot b_n) = A \cdot B$*,*

(iii) if $B \neq 0$ *and* $b_n \neq 0$ *for all* $n \in \mathbb{N}$*, then* $\lim(a_n/b_n) = A/B$.

Theorem 13. *Suppose that* $\lim a_n = 0$ *and the sequence* $\{b_n\}$ *is bounded. Then* $\lim a_n b_n = 0$.

Theorem 14 (limits and ordering)**.** *Let* $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$.

(i) Suppose that there is $n_0 \in \mathbb{N}$ *such that* $a_n \geq b_n$ *for every* $n \geq n_0$. *Then* $A \geq B$.

(ii) Suppose that $A < B$. *Then there is* $n_0 \in \mathbb{N}$ *such that* $a_n < b_n$ *for every* $n \geq n_0$.

Theorem 15 (two policemen/sandwich theorem)**.** *Let* $\{a_n\}$*,* $\{b_n\}$ *be convergent sequences and let* $\{c_n\}$ *be a sequence such that*

(i) $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0\colon a_n \leq c_n \leq b_n$*,*

(ii) $\lim a_n = \lim b_n$.

Then $\lim c_n$ *exists and* $\lim c_n = \lim a_n$.

II.3. Infinite limits of sequences

Definition. We say that a sequence $\{a_n\}$ has a limit $+\infty$ (*plus infinity*) if

$$\forall L \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0\colon a_n > L.$$

We say that a sequence $\{a_n\}$ has a limit $-\infty$ (*minus infinity*) if

$$\forall K \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0\colon a_n < K.$$

Theorem 9 on the uniqueness of a limit holds also for the limits $+\infty$ and $-\infty$. If $\lim a_n = +\infty$, then we say that the sequence $\{a_n\}$ *diverges* to $+\infty$, similarly for $-\infty$. If $\lim a_n \in \mathbb{R}$, then we say that the limit is *finite*, if $\lim a_n = +\infty$ or $\lim a_n = -\infty$, then we say that the limit is *infinite*.

Theorem 10 does not hold for infinite limits. But:

Theorem 10'.

- *Suppose that* $\lim a_n = +\infty$. *Then the sequence* $\{a_n\}$ *is not bounded from above, but is bounded from below.*
- *Suppose that* $\lim a_n = -\infty$. *Then the sequence* $\{a_n\}$ *is not bounded from below, but is bounded from above.*

Theorem 11 (limit of a subsequence) holds also for infinite limits.

Definition. We define the *extended real line* by setting $\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\}$ with the following extension of operations and ordering from $\mathbb{R}$:

- $a < +\infty$ and $-\infty < a$ for $a \in \mathbb{R}$, $-\infty < +\infty$,
- $a + (+\infty) = (+\infty) + a = +\infty$ for $a \in \mathbb{R}^* \setminus \{-\infty\}$,
- $a + (-\infty) = (-\infty) + a = -\infty$ for $a \in \mathbb{R}^* \setminus \{+\infty\}$,
- $a \cdot (\pm\infty) = (\pm\infty) \cdot a = \pm\infty$ for $a \in \mathbb{R}^*$, $a > 0$,
- $a \cdot (\pm\infty) = (\pm\infty) \cdot a = \mp\infty$ for $a \in \mathbb{R}^*$, $a < 0$,
- $\frac{a}{\pm\infty} = 0$ pro $a \in \mathbb{R}$.

The following operations are not defined:

- $(-\infty) + (+\infty)$, $(+\infty) + (-\infty)$, $(+\infty) - (+\infty)$, $(-\infty) - (-\infty)$,

- $(+\infty)\cdot 0$, $0\cdot(+\infty)$, $(-\infty)\cdot 0$, $0\cdot(-\infty)$,
- $\frac{+\infty}{+\infty}$, $\frac{+\infty}{-\infty}$, $\frac{-\infty}{-\infty}$, $\frac{-\infty}{+\infty}$, $\frac{a}{0}$ for $a \in \mathbb{R}^*$.

Theorem 12' (arithmetics of limits)**.** *Suppose that* $\lim a_n = A \in \mathbb{R}^*$ *and* $\lim b_n = B \in \mathbb{R}^*$. *Then*

(i) $\lim(a_n \pm b_n) = A \pm B$ *if the right-hand side is defined,*

(ii) $\lim(a_n \cdot b_n) = A \cdot B$ *if the right-hand side is defined,*

(iii) $\lim a_n/b_n = A/B$ *if the right-hand side is defined.*

Theorem 16. *Suppose that* $\lim a_n = A \in \mathbb{R}^*$, $A > 0$, $\lim b_n = 0$ *and there is* $n_0 \in \mathbb{N}$ *such that we have* $b_n > 0$ *for every* $n \in \mathbb{N}$, $n \geq n_0$. *Then* $\lim a_n/b_n = +\infty$.

Theorem 14 (limits and ordering) and Theorem 15 (sandwich theorem) hold also for infinite limits. Even the following modification holds:

Theorem 15' (one policeman)**.** *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences.*

- *If* $\lim a_n = +\infty$ *and there is* $n_0 \in \mathbb{N}$ *such that* $b_n \geq a_n$ *for every* $n \in \mathbb{N}$, $n \geq n_0$, *then* $\lim b_n = +\infty$.
- *If* $\lim a_n = -\infty$ *and there is* $n_0 \in \mathbb{N}$ *such that* $b_n \leq a_n$ *for every* $n \in \mathbb{N}$, $n \geq n_0$, *then* $\lim b_n = -\infty$.

Definition. Let $A \subset \mathbb{R}$ be non-empty. If A is not bounded from above, then we define $\sup A = +\infty$. If A is not bounded from below, then we define $\inf A = -\infty$.

Lemma 17. *Let* $M \subset \mathbb{R}$ *be non-empty and* $G \in \mathbb{R}^*$. *Then the following statements are equivalent:*

(i) $G = \sup M$.

(ii) The number G *is an upper bound of* M *and there exists a sequence* $\{x_n\}_{n=1}^{\infty}$ *of members of* M *such that* $\lim x_n = G$.

II.4. Deeper theorems on limits of sequences

Theorem 18 (limit of a monotone sequence)**.** *Every monotone sequence has a limit. If* $\{a_n\}$ *is non-decreasing, then* $\lim a_n = \sup\{a_n;\ n \in \mathbb{N}\}$. *If* $\{a_n\}$ *is non-increasing, then* $\lim a_n = \inf\{a_n;\ n \in \mathbb{N}\}$.

Theorem 19 (Bolzano-Weierstraß)**.** *Every bounded sequence contains a convergent subsequence.*

III. Mappings

Definition. Let A and B be sets. A *mapping* f *from* A *to* B is a rule which assigns to each member x of the set A a unique member y of the set B. This element y is denoted by the symbol $f(x)$. The element y is called an *image* of x and the element x is called a *pre-image* of y.

- By $f\colon A \to B$ we denote the fact that f is a mapping from A to B.
- By $f\colon x \mapsto f(x)$ we denote the fact that the mapping f assigns $f(x)$ to an element x.
- The set A from the definition of the mapping f is called the *domain* of f and it is denoted by D_f.

Definition. Let $f\colon A \to B$ be a mapping.

- The subset $G_f = \{[x, y] \in A \times B;\ x \in A, y = f(x)\}$ of the Cartesian product $A \times B$ is called the *graph of the mapping* f.
- The *image* of the set $M \subset A$ under the mapping f is the set

$$f(M) = \{y \in B;\ \exists x \in M\colon f(x) = y\} \quad (= \{f(x);\ x \in M\}).$$

- The set $f(A)$ is called the *range* of the mapping f, it is denoted by R_f.
- The *pre-image* of the set $W \subset B$ under the mapping f is the set

$$f_{-1}(W) = \{x \in A;\ f(x) \in W\}.$$

Remark. Let $f\colon A \to B$, $X, Y \subset A$, $U, V \subset B$. Then

- $f_{-1}(U \cup V) = f_{-1}(U) \cup f_{-1}(V)$,
- $f_{-1}(U \cap V) = f_{-1}(U) \cap f_{-1}(V)$,
- $f(X \cup Y) = f(X) \cup f(Y)$,
- $f(X \cap Y) \subset f(X) \cap f(Y)$.

Definition. Let A, B, C be sets, $C \subset A$ and $f: A \to B$. The mapping $\tilde{f}: C \to B$ given by the formula $\tilde{f}(x) = f(x)$ for each $x \in C$ is called the *restriction of the mapping f to the set C*. It is denoted by $f|_C$.

Definition. Let $f: A \to B$ and $g: B \to C$ be two mappings. The symbol $g \circ f$ denotes a mapping from A to C defined by

$$(g \circ f)(x) = g(f(x)).$$

This mapping is called a *compound mapping* or a *composition of the mapping f and the mapping g*.

Definition. We say that a mapping $f: A \to B$

- maps the set A *onto* the set B if $f(A) = B$, i.e. if to each $y \in B$ there exist $x \in A$ such that $f(x) = y$;
- is *one-to-one* (or *injective*) if images of different elements differ, i.e.

$$\forall x_1, x_2 \in A: x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2),$$

- is a *bijection of A onto B* (or a *bijective mapping*), if it is at the same time one-to-one and maps A onto B.

Definition. Let $f: A \to B$ be bijective (i.e. one-to-one and onto). An *inverse mapping* $f^{-1}: B \to A$ is a mapping that to each $y \in B$ assigns a (uniquely determined) element $x \in A$ satisfying $f(x) = y$.

IV. Functions of one real variable

IV.1. Basic notions

Definition. A *function f of one real variable* (or a *function* for short) is a mapping $f: M \to \mathbb{R}$, where M is a subset of real numbers.

Definition. A function $f: J \to \mathbb{R}$ is *increasing* on an interval J, if for each pair $x_1, x_2 \in J$, $x_1 < x_2$ the inequality $f(x_1) < f(x_2)$ holds. Analogously we define a function *decreasing* (*non-decreasing*, *non-increasing*) on an interval J.

Definition. A *monotone function* on an interval J is a function which is non-decreasing or non-increasing on J. A *strictly monotone function* on an interval J is a function which is increasing or decreasing on J.

Definition. Let f be a function and $M \subset D_f$. We say that f is

- *bounded from above* on M if there is $K \in \mathbb{R}$ such that $f(x) \leq K$ for all $x \in M$,
- *bounded from below* on M if there is $K \in \mathbb{R}$ such that $f(x) \geq K$ for all $x \in M$,
- *bounded* on M if there is $K \in \mathbb{R}$ such that $|f(x)| \leq K$ for all $x \in M$,
- *odd* if for each $x \in D_f$ we have $-x \in D_f$ and $f(-x) = -f(x)$,
- *even* if for each $x \in D_f$ we have $-x \in D_f$ and $f(-x) = f(x)$,
- *periodic with a period a*, where $a \in \mathbb{R}$, $a > 0$, if for each $x \in D_f$ we have $x + a \in D_f$, $x - a \in D_f$ and $f(x + a) = f(x - a) = f(x)$.

IV.2. Limit of a function

Definition. Let $c \in \mathbb{R}$ and $\varepsilon > 0$. We define

- a *neighbourhood of a point c* with radius ε by $B(c, \varepsilon) = (c - \varepsilon, c + \varepsilon)$,
- a *punctured neighbourhood of a point c* with radius ε by $P(c, \varepsilon) = (c - \varepsilon, c + \varepsilon) \setminus \{c\}$.

Definition. We say that $A \in \mathbb{R}$ is a *limit of a function f at a point $c \in \mathbb{R}$* if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists \delta \in \mathbb{R}, \delta > 0\ \forall x \in P(c, \delta): f(x) \in B(A, \varepsilon).$$

Theorem 20 (uniqueness of a limit)**.** *Let f be a function and $c \in \mathbb{R}$. Then f has a most one limit $A \in \mathbb{R}$ at c.*

The fact that f has a limit $A \in \mathbb{R}$ at $c \in \mathbb{R}$ is denoted by $\lim\limits_{x \to c} f(x) = A$.

Definition. We say that a function f is *continuous at a point* $c \in \mathbb{R}$ if

$$\lim_{x \to c} f(x) = f(c).$$

Remark. A function f is continuous at a point c if and only if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \ \exists \delta \in \mathbb{R}, \delta > 0 \ \forall x \in B(c, \delta)\colon \ f(x) \in B(f(c), \varepsilon).$$

Definition. Let $\varepsilon > 0$. A neighbourhood and a punctured neighbourhood of $+\infty$ (resp. $-\infty$) is defined as follows:

$$P(+\infty, \varepsilon) = B(+\infty, \varepsilon) = (1/\varepsilon, +\infty),$$
$$P(-\infty, \varepsilon) = B(-\infty, \varepsilon) = (-\infty, -1/\varepsilon).$$

Definition. We say that $A \in \mathbb{R}^*$ is a *limit of a function f at $c \in \mathbb{R}^*$* if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \ \exists \delta \in \mathbb{R}, \delta > 0 \ \forall x \in P(c, \delta)\colon \ f(x) \in B(A, \varepsilon).$$

Theorem 20 holds also for $c \in \mathbb{R}^*$, $A \in \mathbb{R}^*$, so we can again use the notation $\lim_{x \to c} f(x) = A$.

Definition. Let $c \in \mathbb{R}$ and $\varepsilon > 0$. We define

- a *right neighbourhood* of c by $B^+(c, \varepsilon) = [c, c + \varepsilon)$,
- a *left neighbourhood* of c by $B^-(c, \varepsilon) = (c - \varepsilon, c]$,
- a *right punctured neighbourhood* of c by $P^+(c, \varepsilon) = (c, c + \varepsilon)$,
- a *left punctured neighbourhood* of c by $P^-(c, \varepsilon) = (c - \varepsilon, c)$,
- a *left neighbourhood* and *left punctured neighbourhood* of $+\infty$ by $B^-(+\infty, \varepsilon) = P^-(+\infty, \varepsilon) = (1/\varepsilon, +\infty)$,
- a *right neighbourhood* and *right punctured neighbourhood* of $-\infty$ by $B^+(-\infty, \varepsilon) = P^+(-\infty, \varepsilon) = (-\infty, -1/\varepsilon)$.

Definition. Let $A \in \mathbb{R}^*$, $c \in \mathbb{R} \cup \{-\infty\}$. We say that a function f has a *limit from the right* at c equal to $A \in \mathbb{R}^*$ (denoted by $\lim\limits_{x \to c+} f(x) = A$) if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \ \exists \delta \in \mathbb{R}, \delta > 0 \ \forall x \in P^+(c, \delta)\colon \ f(x) \in B(A, \varepsilon).$$

Analogously we define the notion of *limit from the left* at $c \in \mathbb{R} \cup \{+\infty\}$ and we use the notation $\lim\limits_{x \to c-} f(x)$.

Remark. Let $c \in \mathbb{R}$, $A \in \mathbb{R}^*$. Then

$$\lim_{x \to c} f(x) = A \Leftrightarrow \left(\lim_{x \to c+} f(x) = A \ \ \& \ \lim_{x \to c-} f(x) = A \right).$$

Definition. Let $c \in \mathbb{R}$. We say that a function f is *continuous at c from the right* (*from the left*, resp.) if $\lim_{x \to c+} f(x) = f(c)$ ($\lim_{x \to c-} f(x) = f(c)$, resp.).

Theorem 21. *Let f has a finite limit at $c \in \mathbb{R}^*$. Then there exists $\delta > 0$ such that f is bounded on $P(c, \delta)$.*

Theorem 22 (arithmetics of limits)**.** *Let $c \in \mathbb{R}^*$, $\lim_{x \to c} f(x) = A \in \mathbb{R}^*$ and $\lim_{x \to c} g(x) = B \in \mathbb{R}^*$. Then*

(i) $\lim_{x \to c}(f(x) + g(x)) = A + B$ *if the expression $A + B$ is defined,*

(ii) $\lim_{x \to c} f(x)g(x) = AB$ *if the expression AB is defined,*

(iii) $\lim_{x \to c} f(x)/g(x) = A/B$ *if the expression A/B is defined.*

Corollary. *Suppose that the functions f and g are continuous at $c \in \mathbb{R}$. Then also the functions $f + g$ and fg are continuous at c. If moreover $g(c) \neq 0$, then also the function f/g is continuous at c.*

Theorem 23. *Let $c \in \mathbb{R}^*$, $\lim_{x \to c} g(x) = 0$, $\lim_{x \to c} f(x) = A \in \mathbb{R}^*$ and $A > 0$. If there exists $\eta > 0$ such that the function g is positive on $P(c, \eta)$, then $\lim_{x \to c}\big(f(x)/g(x)\big) = +\infty$.*

Definition. A *polynomial* is a function P of the form

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad x \in \mathbb{R},$$

where $n \in \mathbb{N} \cup \{0\}$ and $a_0, a_1, \dots, a_n \in \mathbb{R}$. The numbers $a_0, \dots, a_n$ are called the *coefficients of the polynomial* P.

Remark. Let $n, m \in \mathbb{N} \cup \{0\}$ and

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad x \in \mathbb{R},$$
$$Q(x) = b_0 + b_1 x + \cdots + b_m x^m, \quad x \in \mathbb{R},$$

where $a_0, a_1, \dots, a_n \in \mathbb{R}$, $a_n \neq 0$, $b_0, b_1, \dots, b_m \in \mathbb{R}$, $b_m \neq 0$. If the polynomials P and Q are equal (i.e. $P(x) = Q(x)$ for each $x \in \mathbb{R}$), then $n = m$ and $a_0 = b_0, \dots, a_n = b_n$.

Definition. Let P be a polynomial of the form

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad x \in \mathbb{R}.$$

We say that P is a polynomial of *degree n* if $a_n \neq 0$. The degree of a *zero polynomial* (i.e. a constant zero function defined on $\mathbb{R}$) is defined as -1.

Theorem 24 (limits and inequalities)**.** *Suppose that $c \in \mathbb{R}^*$ and $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ exist.*

(i) If $\lim_{x\to c} f(x) > \lim_{x\to c} g(x)$, then there exists $\delta > 0$ such that

$$\forall x \in P(c,\delta)\colon\ f(x) > g(x).$$

(ii) If there exists $\delta > 0$ such that $\forall x \in P(c,\delta)\colon\ f(x) \leq g(x)$, then

$$\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x).$$

(iii) (two policemen/sandwich theorem) Suppose that there exists $\eta > 0$ such that

$$\forall x \in P(c,\eta)\colon\ f(x) \leq h(x) \leq g(x).$$

If moreover $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = A \in \mathbb{R}^*$, then the limit $\lim_{x\to c} h(x)$ also exists and equals A.

Corollary. *Let $c \in \mathbb{R}^*$, $\lim_{x\to c} f(x) = 0$ and suppose there exists $\eta > 0$ such that g is bounded on $P(c,\eta)$. Then $\lim_{x\to c}\big(f(x)g(x)\big) =$* 0.

Theorem 25 (limit of a composition)**.** *Let $c, A, B \in \mathbb{R}^*$, $\lim_{x\to c} g(x) = A$, $\lim_{y\to A} f(y) = B$ and at least on of the following conditions is satisfied:*

(I) $\exists \eta \in \mathbb{R}, \eta > 0\ \forall x \in P(c,\eta)\colon g(x) \neq A$,

(C) the function f is continuous at A.

Then

$$\lim_{x\to c} f\big(g(x)\big) = B.$$

Corollary. *Suppose that the function g is continuous at $c \in \mathbb{R}$ and the function f is continuous at $g(c)$. Then the function $f \circ g$ is continuous at c.*

Theorem 26 (Heine)**.** *Let $c \in \mathbb{R}^*$, $A \in \mathbb{R}^*$ and the function f satisfies $\lim_{x\to c} f(x) = A$. If the sequence $\{x_n\}$ satisfies $x_n \in D_f$, $x_n \neq c$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = c$, then $\lim_{n\to\infty} f(x_n) = A$.*

Theorem 27 (limit of a monotone function)**.** *Let $a, b \in \mathbb{R}^*$, $a < b$. Suppose that f is a function monotone on an interval (a,b). Then the limits $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ exist. Moreover,*

- *if f is non-decreasing on (a,b), then $\lim_{x\to a+} f(x) = \inf f\big((a,b)\big)$ and $\lim_{x\to b-} f(x) = \sup f\big((a,b)\big)$;*
- *if f is non-increasing on (a,b), then $\lim_{x\to a+} f(x) = \sup f\big((a,b)\big)$ and $\lim_{x\to b-} f(x) = \inf f\big((a,b)\big)$.*

IV.3. Functions continuous on an interval

Definition. Let $J \subset \mathbb{R}$ be a non-degenerate interval (i.e. it contains infinitely many points). A function $f\colon J \to \mathbb{R}$ is *continuous on the interval J* if

- f is continuous at every inner point J,
- f is continuous from the right at the left endpoint of J if this point belongs to J,
- f is continuous from the left at the right endpoint of J if this point belongs to J.

Theorem 28 (continuity of the compound function on an interval)**.** *Let I and J be intervals, $g\colon I \to J$, $f\colon J \to \mathbb{R}$, let g be continuous on I and let f be continuous on J. Then the function $f \circ g$ is continuous on I.*

Theorem 29 (Bolzano, intermediate value theorem)**.** *Let f be a function continuous on an interval $[a, b]$ and suppose that $f(a) < f(b)$. Then for each $C \in (f(a), f(b))$ there exists $\xi \in (a, b)$ satisfying $f(\xi) = C$.*

Theorem 30 (an image of an interval under a continuous function)**.** *Let J be an interval and let $f\colon J \to \mathbb{R}$ be a function continuous on J. Then $f(J)$ is an interval.*

Definition. Let $M \subset \mathbb{R}$, $x \in M$ and a function f is defined at least on M (i.e. $M \subset D_f$). We say that f attains its *maximum* (resp. *minimum*) *on M* at $x \in M$ if

$$\forall y \in M\colon f(y) \leq f(x) \quad (\text{resp. } \forall y \in M\colon f(y) \geq f(x)).$$

The point x is called the *point of maximum* (resp. *minimum*) of the function f on M. The symbol $\max_M f$ (resp. $\min_M f$) denotes the maximal (resp. minimal) value of f on M (if such a value exists). The points of maxima or minima are collectively called the points of *extrema*.

Definition. Let $M \subset \mathbb{R}$, $x \in M$ and a function f is defined at least on M (i.e. $M \subset D_f$). We say that the function f has at x

- a *local maximum with respect to M* if there exists $\delta > 0$ such that $\forall y \in B(x, \delta) \cap M\colon f(y) \leq f(x)$,
- a *local minimum with respect to M* if there exists $\delta > 0$ such that $\forall y \in B(x, \delta) \cap M\colon f(y) \geq f(x)$,
- a *strict local maximum with respect to M* if there exists $\delta > 0$ such that $\forall y \in P(x, \delta) \cap M\colon f(y) < f(x)$,
- a *strict local minimum with respect to M* if there exists $\delta > 0$ such that $\forall y \in P(x, \delta) \cap M\colon f(y) > f(x)$.

The points of local maxima or minima are collectively called the points of *local extrema*.

Theorem 31 (Heine theorem for continuity on an interval)**.** *Let f be a function continuous on an interval J and $c \in J$. Then* $\lim f(x_n) = f(c)$ *for each sequence $\{x_n\}_{n=1}^{\infty}$ of points in the interval J satisfying* $\lim x_n = c$.

Theorem 32 (extrema of continuous functions)**.** *Let f be a function continuous on an interval $[a, b]$. Then f attains its maximum and minimum on $[a, b]$.*

Corollary 33 (boundedness of a continuous function)**.** *Let f be a function continuous on an interval $[a, b]$. Then f is bounded on $[a, b]$.*

Theorem 34 (continuity of an inverse function)**.** *Let f be a continuous function that is increasing (resp. decreasing) on an interval J. Then the function f^{-1} is continuous and increasing (resp. decreasing) on the interval $f(J)$.*

IV.4. Elementary functions

Theorem 35 (logarithm)**.** *There exist a unique function (denoted by* log *and called the natural logarithm) with the following properties:*

(L1) $D_{\log} = (0, +\infty)$,

(L2) the function log *is increasing on* $(0, +\infty)$,

(L3) $\forall x, y \in (0, +\infty)\colon \log xy = \log x + \log y$,

(L4) $\lim\limits_{x \to 1} \frac{\log x}{x-1} = 1$.

Properties of the logarithm

- $\log 1 = 0$,
- $\forall x \in (0, +\infty)\colon \log(1/x) = -\log x$,
- $\forall n \in \mathbb{Z}\ \forall x \in (0, +\infty)\colon \log x^n = n \log x$,
- $\lim\limits_{x \to +\infty} \log x = +\infty$, $\lim\limits_{x \to 0+} \log x = -\infty$,
- the function log is continuous on $(0, +\infty)$,
- $R_{\log} = \mathbb{R}$,
- there exists a unique number $e \in (0, +\infty)$ satisfying $\log e = 1$.

Definition. The *exponential function* (denoted by exp) is defined as an inverse function to the function log.

Properties of the exponential function

- $D_{\exp} = \mathbb{R}$, $R_{\exp} = (0, +\infty)$,
- the function exp is continuous and increasing on $\mathbb{R}$,
- $\exp 0 = 1$, $\exp 1 = e$,
- $\forall x, y \in \mathbb{R}\colon\ \exp(x + y) = \exp(x)\exp(y)$,
- $\forall x \in \mathbb{R}\colon\ \exp(-x) = 1/\exp x$,
- $\forall n \in \mathbb{Z}\ \forall x \in \mathbb{R}\colon\ \exp(nx) = (\exp x)^n$,
- $\lim\limits_{x\to+\infty} \exp x = +\infty$, $\lim\limits_{x\to-\infty} \exp x = 0$,
- $\lim\limits_{x\to 0} \frac{\exp(x)-1}{x} = 1$,
- $\forall r \in \mathbb{Q}\colon\ \exp r = e^r$.

Definition. Let $a, b \in \mathbb{R}$, $a > 0$. The *general power* a^b is defined by

$$a^b = \exp(b \log a).$$

Definition. Let $a, b \in (0, +\infty)$, $a \neq 1$. The *general logarithm* to base a is defined by

$$\log_a b = \frac{\log b}{\log a}.$$

Theorem 36 (the sine and the number π)**.** *There exists a unique positive real number (denoted by π) and a unique function sine (denoted by* sin*) with the following properties:*

(S1) $D_{\sin} = \mathbb{R}$,

(S2) sin *is increasing on* $[-\pi/2, \pi/2]$,

(S3) $\sin 0 = 0$,

(S4) $\forall x, y \in \mathbb{R}\colon\ \sin(x + y) = \sin x \cdot \sin(\frac{\pi}{2} - y) + \sin(\frac{\pi}{2} - x) \cdot \sin y$,

(S5) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$.

Definition. The function *cosine* is defined by $\cos x = \sin(\frac{\pi}{2} - x)$, $x \in \mathbb{R}$.

Properties of the sine and cosine

- The function cos is decreasing on $[0, \pi]$.
- $\cos\frac{\pi}{2} = 0$, $\cos 0 = \sin\frac{\pi}{2} = 1$, $\sin\pi = 0$, $\cos\pi = \sin(-\frac{\pi}{2}) = -1$, $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
- $\forall x \in \mathbb{R}\colon\ \sin(x + \pi) = -\sin x$
- The function cos is even, the function sin is odd.
- The functions sin and cos are 2π-periodic.
- $\forall x \in \mathbb{R}\colon\ \sin^2 x + \cos^2 x = 1$
- $\forall x \in \mathbb{R}\colon\ |\sin x| \leq 1, |\cos x| \leq 1$
- $\forall x, y \in \mathbb{R}\colon\ \sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$
- The functions sin and cos are continuous on $\mathbb{R}$.
- $R_{\sin} = R_{\cos} = [-1, 1]$
- The function sin is equal to zero exactly at the points of the set $\{k\pi;\ k \in \mathbb{Z}\}$, the function cos is equal to zero exactly et the points of the set $\{\frac{\pi}{2} + k\pi;\ k \in \mathbb{Z}\}$.

Definition. The function *tangent* is denoted by tg and defined by

$$\operatorname{tg} x = \frac{\sin x}{\cos x}$$

for every $x \in \mathbb{R}$ for which the fraction is defined, i.e.

$$D_{\operatorname{tg}} = \{x \in \mathbb{R};\ x \neq \pi/2 + k\pi, k \in \mathbb{Z}\}.$$

The function *cotangent* is denoted by cotg and defined on a set $D_{\operatorname{cotg}} = \{x \in \mathbb{R};\ x \neq k\pi, k \in \mathbb{Z}\}$ by

$$\operatorname{cotg} x = \frac{\cos x}{\sin x}.$$

Properties of the tangent and cotangent

- $\operatorname{tg} \frac{\pi}{4} = \operatorname{cotg} \frac{\pi}{4} = 1$
- The functions tg and cotg are continuous at every point of their domains.
- The functions tg and cotg are odd.
- The functions tg and cotg are π-periodic.
- The function tg is increasing on $(-\pi/2, \pi/2)$, the function cotg is decreasing on $(0, \pi)$.
- $\lim\limits_{x \to \frac{\pi}{2}-} \operatorname{tg} x = +\infty$, $\lim\limits_{x \to -\frac{\pi}{2}+} \operatorname{tg} x = -\infty$, $\lim\limits_{x \to 0+} \operatorname{cotg} x = +\infty$, $\lim\limits_{x \to \pi-} \operatorname{cotg} x = -\infty$
- $R_{\operatorname{tg}} = R_{\operatorname{cotg}} = \mathbb{R}$

Definition.

- The function *arcsine* (denoted by arcsin) is an inverse function to the function $\sin|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$.
- The function *arccosine* (denoted by arccos) is an inverse function to the function $\cos|_{[0,\pi]}$.
- The function *arctangent* (denoted by arctg) is an inverse function to the function $\operatorname{tg}|_{(-\frac{\pi}{2}, \frac{\pi}{2})}$.
- The function *arccotangent* (denoted by arccotg) is an inverse function to the function $\operatorname{cotg}|_{(0,\pi)}$.

Properties of inverse trigonometric functions

- $D_{\arcsin} = D_{\arccos} = [-1, 1]$, $D_{\operatorname{arctg}} = D_{\operatorname{arccotg}} = \mathbb{R}$
- The functions arcsin and arctg are odd.
- The functions arcsin and arctg are increasing, the functions arccos and arccotg are decreasing (on their domains).
- The functions arcsin, arccos, arctg, and arccotg are continuous on their domains.
- $\operatorname{arctg} 0 = 0$, $\operatorname{arctg} 1 = \frac{\pi}{4}$, $\operatorname{arccotg} 0 = \frac{\pi}{2}$
- $\lim\limits_{x \to 0} \frac{\arcsin x}{x} = \lim\limits_{x \to 0} \frac{\operatorname{arctg} x}{x} = 1$
- $\forall x \in [-1, 1]\colon\ \arcsin x + \arccos x = \frac{\pi}{2}$, $\forall x \in \mathbb{R}\colon\ \operatorname{arctg} x + \operatorname{arccotg} x = \frac{\pi}{2}$
- $\lim\limits_{x \to +\infty} \operatorname{arctg} x = \frac{\pi}{2}$, $\lim\limits_{x \to -\infty} \operatorname{arctg} x = -\frac{\pi}{2}$ $\lim\limits_{x \to +\infty} \operatorname{arccotg} x = 0$, $\lim\limits_{x \to -\infty} \operatorname{arccotg} x = \pi$

IV.5. Derivatives

Definition. Let f be a function and $a \in \mathbb{R}$. Then

- the *derivative of the function f at the point a* is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

- the *derivative of f at a from the right* is defined by

$$f'_+(a) = \lim_{h \to 0+} \frac{f(a+h) - f(a)}{h},$$

- the *derivative of f at a from the left* is defined by

$$f'_-(a) = \lim_{h\to 0-} \frac{f(a+h) - f(a)}{h},$$

if the respective limits exist.

Definition. Suppose that the function f has a finite derivative at a point $a \in \mathbb{R}$. The line

$$T_a = \left\{[x, y] \in \mathbb{R}^2;\ y = f(a) + f'(a)(x-a)\right\}.$$

is called the *tangent to the graph of f at the point* $[a, f(a)]$.

Theorem 37. *Suppose that the function f has a finite derivative at a point $a \in \mathbb{R}$. Then f is continuous at a.*

Theorem 38 (arithmetics of derivatives)**.** *Suppose that the functions f and g have finite derivatives at $a \in \mathbb{R}$ and let $\alpha \in \mathbb{R}$. Then*

(i) $(f+g)'(a) = f'(a) + g'(a)$,

(ii) $(\alpha f)'(a) = \alpha \cdot f'(a)$,

(iii) $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$,

(iv) if $g(a) \neq 0$, *then*

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}.$$

Theorem 39 (derivative of a compound function)**.** *Suppose that the function f has a finite derivative at $y_0 \in \mathbb{R}$, the function g has a finite derivative at $x_0 \in \mathbb{R}$, and $y_0 = g(x_0)$. Then*

$$(f \circ g)'(x_0) = f'(y_0) \cdot g'(x_0).$$

Theorem 40 (derivative of an inverse function)**.** *Let f be a function continuous and strictly monotone on an interval (a, b) and suppose that it has a finite and non-zero derivative $f'(x_0)$ at $x_0 \in (a, b)$. Then the function f^{-1} has a derivative at $y_0 = f(x_0)$ and*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}.$$

Derivatives of elementary functions

- $(\text{const.})' = 0$,
- $(x^n)' = nx^{n-1}$, $x \in \mathbb{R}, n \in \mathbb{N}$; $x \in \mathbb{R} \setminus \{0\}, n \in \mathbb{Z}, n < 0$,
- $(\log x)' = \frac{1}{x}$ for $x \in (0, +\infty)$,
- $(\exp x)' = \exp x$ for $x \in \mathbb{R}$,
- $(x^a)' = ax^{a-1}$ for $x \in (0, +\infty)$, $a \in \mathbb{R}$,
- $(a^x)' = a^x \log a$ for $x \in \mathbb{R}$, $a \in \mathbb{R}$, $a > 0$,
- $(\sin x)' = \cos x$ for $x \in \mathbb{R}$,
- $(\cos x)' = -\sin x$ for $x \in \mathbb{R}$,
- $(\operatorname{tg} x)' = \frac{1}{\cos^2 x}$ for $x \in D_{\text{tg}}$,
- $(\operatorname{cotg} x)' = -\frac{1}{\sin^2 x}$ for $x \in D_{\text{cotg}}$,
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$,
- $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$,
- $(\operatorname{arctg} x)' = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$,
- $(\operatorname{arccotg} x)' = -\frac{1}{1+x^2}$ for $x \in \mathbb{R}$.

Theorem 41 (necessary condition for a local extremum)**.** *Suppose that a function f has a local extremum at $x_0 \in \mathbb{R}$. If $f'(x_0)$ exists, then $f'(x_0) = 0$.*

IV.6. Deeper theorems on derivatives

Theorem 42 (Rolle)**.** *Suppose that $a, b \in \mathbb{R}$, $a < b$, and a function f has the following properties:*

(i) it is continuous on the interval $[a, b]$,

(ii) it has a derivative (finite or infinite) at every point of the open interval (a, b),

(iii) $f(a) = f(b)$.

Then there exists $\xi \in (a, b)$ satisfying $f'(\xi) = 0$.

Theorem 43 (Lagrange, mean value theorem)**.** *Suppose that $a, b \in \mathbb{R}$, $a < b$, a function f is continuous on an interval $[a, b]$ and has a derivative (finite or infinite) at every point of the interval (a, b). Then there is $\xi \in (a, b)$ satisfying*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 44 (sign of the derivative and monotonicity)**.** *Let $J \subset \mathbb{R}$ be a non-degenerate interval. Suppose that a function f is continuous on J and it has a derivative at every inner point of J (the set of all inner points of J is denoted by $\operatorname{Int} J$).*

(i) If $f'(x) > 0$ for all $x \in \operatorname{Int} J$, then f is increasing on J.

(ii) If $f'(x) < 0$ for all $x \in \operatorname{Int} J$, then f is decreasing on J.

(iii) If $f'(x) \geq 0$ for all $x \in \operatorname{Int} J$, then f in non-decreasing on J.

(iv) If $f'(x) \leq 0$ for all $x \in \operatorname{Int} J$, then f is non-increasing on J.

Theorem 45 (computation of a one-sided derivative)**.** *Suppose that a function f is continuous from the right at $a \in \mathbb{R}$ and the limit $\lim_{x \to a+} f'(x)$ exists. Then the derivative $f'_+(a)$ exists and*

$$f'_+(a) = \lim_{x \to a+} f'(x).$$

Theorem 46 (l'Hospital's rule)**.** *Suppose that functions f and g have finite derivatives on some punctured neighbourhood of $a \in \mathbb{R}^*$ and the limit $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exist. Suppose further that one of the following conditions hold:*

(i) $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$,

(ii) $\lim_{x \to a} |g(x)| = +\infty$.

Then the limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

IV.7. Convex and concave functions

Definition. We say that a function f is

- *convex* on an interval I if
$$f(\lambda x_1 + (1 - \lambda) x_2) \leq \lambda f(x_1) + (1 - \lambda) f(x_2),$$
for each $x_1, x_2 \in I$ and each $\lambda \in [0, 1]$;

- *concave* on an interval I if
$$f(\lambda x_1 + (1 - \lambda) x_2) \geq \lambda f(x_1) + (1 - \lambda) f(x_2),$$
for each $x_1, x_2 \in I$ and each $\lambda \in [0, 1]$;

- *strictly convex* on an interval I if
$$f(\lambda x_1 + (1 - \lambda) x_2) < \lambda f(x_1) + (1 - \lambda) f(x_2),$$
for each $x_1, x_2 \in I$, $x_1 \neq x_2$ and each $\lambda \in (0, 1)$;

- *strictly concave* on an interval I if
$$f(\lambda x_1 + (1 - \lambda) x_2) > \lambda f(x_1) + (1 - \lambda) f(x_2).$$
for each $x_1, x_2 \in I$, $x_1 \neq x_2$ and each $\lambda \in (0, 1)$.

Lemma 47. *A function f is convex on an interval I if and only if*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

for each three points $x_1, x_2, x_3 \in I$, $x_1 < x_2 < x_3$.

Definition. Suppose that a function f has a finite derivative on some neighbourhood of $a \in \mathbb{R}$. The *second derivative* of f at a is defined by

$$f''(a) = \lim_{h \to 0} \frac{f'(a+h) - f'(a)}{h}$$

if the limit exists.

Let $n \in \mathbb{N}$ and suppose that f has a finite nth derivative (denoted by $f^{(n)}$) on some neighbourhood of $a \in \mathbb{R}$. Then the $(n+1)$*th derivative* of f at a is defined by

$$f^{(n+1)}(a) = \lim_{h \to 0} \frac{f^{(n)}(a+h) - f^{(n)}(a)}{h}$$

if the limit exists.

Theorem 48 (second derivative and convexity)**.** *Let $a, b \in \mathbb{R}^*$, $a < b$, and suppose that a function f has a finite second derivative on the interval (a, b).*

(i) If $f''(x) > 0$ for each $x \in (a, b)$, then f is strictly convex on (a, b).

(ii) If $f''(x) < 0$ for each $x \in (a, b)$, then f is strictly concave on (a, b).

(iii) If $f''(x) \geq 0$ for each $x \in (a, b)$, then f is convex on (a, b).

(iv) If $f''(x) \leq 0$ for each $x \in (a, b)$, then f is concave on (a, b).

Definition. Suppose that a function f has a finite derivative at $a \in \mathbb{R}$ and let T_a denote the tangent to the graph of f at $[a, f(a)]$. We say that the point $[x, f(x)]$ *lies below the tangent T_a* if

$$f(x) < f(a) + f'(a) \cdot (x - a).$$

We say that the point $[x, f(x)]$ *lies above the tangent T_a* if the opposite inequality holds.

Definition. Suppose that a function f has a finite derivative at $a \in \mathbb{R}$ and let T_a denote the tangent to the graph of f at $[a, f(a)]$. We say that a is an *inflection point* of f if there is $\Delta > 0$ such that

(i) $\forall x \in (a - \Delta, a)\colon [x, f(x)]$ lies below the tangent T_a,

(ii) $\forall x \in (a, a + \Delta)\colon [x, f(x)]$ lies above the tangent T_a,

or

(i) $\forall x \in (a - \Delta, a)\colon [x, f(x)]$ lies above the tangent T_a,

(ii) $\forall x \in (a, a + \Delta)\colon [x, f(x)]$ lies below the tangent T_a.

Theorem 49 (necessary condition for inflection)**.** *Let $a \in \mathbb{R}$ be an inflection point of a function f. Then $f''(a)$ either does not exist or equals zero.*

Theorem 50 (sufficient condition for inflection)**.** *Suppose that a function f has a continuous first derivative on an interval (a, b) and $z \in (a, b)$. Suppose further that*

- $\forall x \in (a, z)\colon f''(x) > 0$,
- $\forall x \in (z, b)\colon f''(x) < 0$.

Then z is an inflection point of f.

IV.8. Investigation of functions

Definition. The line which is a graph of an affine function $x \mapsto kx + q$, $k, q \in \mathbb{R}$, is called an *asymptote* of the function f at $+\infty$ (resp. v $-\infty$) if

$$\lim_{x \to +\infty} (f(x) - kx - q) = 0, \quad (\text{resp. } \lim_{x \to -\infty} (f(x) - kx - q) = 0).$$

Proposition 51. *A function f has an asymptote at $+\infty$ given by the affine function $x \mapsto kx + q$ if and only if*

$$\lim_{x \to +\infty} \frac{f(x)}{x} = k \in \mathbb{R} \quad \textit{and} \quad \lim_{x \to +\infty} (f(x) - kx) = q \in \mathbb{R}.$$

Investigation of a function

1. Determine the domain and discuss the continuity of the function.
2. Find out symmetries: oddness, evenness, periodicity.
3. Find the limits at the “endpoints of the domain”.
4. Investigate the first derivative, find the intervals of monotonicity and local and global extrema. Determine the range.
5. Find the second derivative and determine the intervals where the function is concave or convex. Find the inflection points.
6. Find the asymptotes of the function.
7. Draw the graph of the function.