Cvičení 9 Téma: Laplaceova transformace

Laplaceova transformace funkce $f:(0,\infty)\to\mathbb{C}$

$$\mathcal{L}[f(t)](s)=F(s)=\int_0^\infty f(t)e^{-st}\,dt \quad, s\in\mathbb{C}$$

1) Nalezněte Laplaceovu transformaci funkce $f(t)$.

$\boxed{\begin{array}{l}\mathcal{L}[t\cdot f(t)](s)=-\frac{d}{ds}\mathcal{L}[f(t)](s)\\ \mathcal{L}[\sin(\omega t)](s)=\frac{\omega}{s^2+\omega^2}\end{array}}$

a) $f(t)=t\cdot\sin t$

$$\mathcal{L}[t\cdot\sin t](s)=-\frac{d}{ds}\mathcal{L}[\sin t](s)=-\frac{d}{ds}\frac{1}{s^2+1}=-\frac{-2s}{(s^2+1)^2}=\frac{2s}{(s^2+1)^2}$$

b) $f(t)=(1+e^t)^2$

$\boxed{\begin{array}{l}\mathcal{L}[t^n](s)=\frac{n!}{s^{n+1}}\\ \mathcal{L}[e^{at}](s)=\frac{1}{s-a}\\ \hline \mathcal{L}[e^{at}f(t)](s)=\mathcal{L}[f(t)](s-a)\\ \mathcal{L}[\cos(\omega t)](s)=\frac{s}{s^2+\omega^2}\end{array}}$

$$\mathcal{L}[(1+e^t)^2](s)=\mathcal{L}[1+2e^t+e^{2t}](s)=$$

$$=\mathcal{L}[1](s)+2\mathcal{L}[e^t](s)+\mathcal{L}[e^{2t}](s)=\frac{1}{s}+2\cdot\frac{1}{s-1}+\frac{1}{s-2}$$

c) $f(t)=t^2+e^t\cos 2t$

$$\mathcal{L}[t^2+e^t\cos 2t](s)=\mathcal{L}[t^2](s)+\mathcal{L}[e^t\cdot\cos 2t](s)=\frac{2}{s^3}+\mathcal{L}[\cos 2t](s-1)=$$

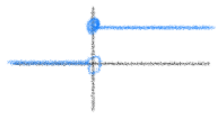
$$=\frac{2}{s^3}+\frac{s-1}{(s-1)^2+4}$$

d) $f(t)=\sin t\cdot\cos 3t$ (příklad navíc)

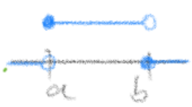
$$\mathcal{L}[\sin t\cdot\cos 3t](s)=\mathcal{L}\left[\frac{e^{it}-e^{-it}}{2i}\cos 3t\right](s)=\frac{1}{2i}\left(\mathcal{L}[e^{it}\cos 3t](s)-\mathcal{L}[e^{-it}\cos 3t](s)\right)$$

$$=\frac{1}{2i}\left(\mathcal{L}[\cos 3t](s-i)-\mathcal{L}[\cos 3t](s+i)\right)=\frac{1}{2i}\left(\frac{s-i}{(s-i)^2+9}-\frac{s+i}{(s+i)^2+9}\right).$$

Heavisideova funkce $\mathbb{1}(t)=\begin{cases}0, & t<0\\ 1, & t\geq 0\end{cases}$

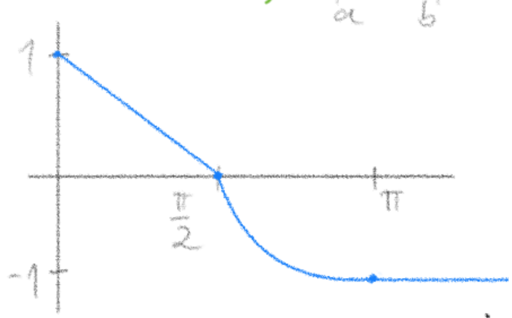


Pak charakteristická funkce intervalu $\chi_{[a,b)}(t)=\mathbb{1}(t-a)-\mathbb{1}(t-b)$.



2) Je dána funkce

$$f(t)=\begin{cases}-\frac{2}{\pi}t+1, & t\in[0,\frac{\pi}{2})\\ \cos(t), & t\in[\frac{\pi}{2},\pi)\\ -1, & t\in[\pi,\infty).\end{cases}$$



a) Pomocí Heavisideovy funkce zapište $f(t)$. (Přenásobení χ "zapíná" funkci)

$$f(t)=\underbrace{\left(-\frac{2}{\pi}t+1\right)\cdot\left(\mathbb{1}(t)-\mathbb{1}(t-\tfrac{\pi}{2})\right)}_{f_1}+\underbrace{\cos(t)\left(\mathbb{1}(t-\tfrac{\pi}{2})-\mathbb{1}(t-\pi)\right)}_{f_2}+\underbrace{(-1)\left(\mathbb{1}(t-\pi)\right)}_{f_3}$$

b) Nalezněte $\mathcal{L}[f(t)](s)$.

$\boxed{\begin{array}{l}\mathcal{L}[f(t-a)\mathbb{1}(t-a)](s)=e^{-as}\mathcal{L}[f(t)](s)\\ \mathcal{L}[f(t)\mathbb{1}(t-a)](s)=e^{-as}\mathcal{L}[f(t+a)](s)\end{array}}$

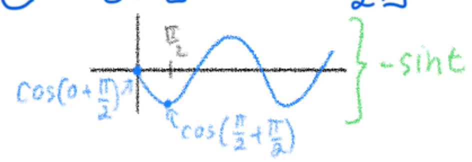
$$\mathcal{L}[f_1(t)](s)=\mathcal{L}[(-\tfrac{2}{\pi}t+1)\mathbb{1}(t)](s)-\mathcal{L}[(-\tfrac{2}{\pi}t+1)\mathbb{1}(t-\tfrac{\pi}{2})](s)=\mathcal{L}[-\tfrac{2}{\pi}t+1](s)-$$

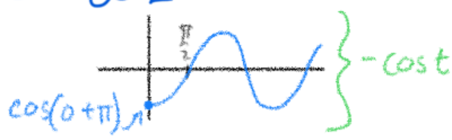
$$-\mathcal{L}\left[-\frac{2}{\pi}\left(t-\frac{\pi}{2}\right)\mathbb{1}\left(t-\frac{\pi}{2}\right)\right](s) = -\frac{2}{\pi}\mathcal{L}[t](s) + \mathcal{L}[1](s) + \frac{2}{\pi}\mathcal{L}\left[\left(t-\frac{\pi}{2}\right)\mathbb{1}\left(t-\frac{\pi}{2}\right)\right](s) =$$

$$= -\frac{2}{\pi}\cdot\frac{1}{s^2} + \frac{1}{s} + \frac{2}{\pi}e^{-\frac{\pi}{2}s}\mathcal{L}[t](s) = -\frac{2}{\pi s^2} + \frac{1}{s} + \frac{2e^{-\frac{\pi}{2}s}}{\pi}\cdot\frac{1}{s^2} = \frac{-2+\pi s+2e^{-\frac{\pi}{2}s}}{\pi s^2}$$

$$\mathcal{L}[f_2(t)](s) = \mathcal{L}\left[\cos(t)\cdot\mathbb{1}\left(t-\frac{\pi}{2}\right)\right](s) - \mathcal{L}[\cos(t)\cdot\mathbb{1}(t-\pi)](s) =$$

$$= e^{-\frac{\pi}{2}s}\mathcal{L}\left[\cos\left(t+\frac{\pi}{2}\right)\right](s) - e^{-\pi s}\mathcal{L}[\cos(t+\pi)](s) =$$





[nebo znalosti o posouvání, vztahy mezi sin a cos]

$$= e^{-\frac{\pi}{2}s}\mathcal{L}[-\sin t](s) - e^{-\pi s}\mathcal{L}[-\cos t](s) = -e^{-\frac{\pi}{2}s}\cdot\frac{1}{s^2+1} + e^{-\pi s}\frac{s}{s^2+1}$$

$$= \frac{se^{-\pi s} - e^{-\frac{\pi}{2}s}}{s^2+1}$$

$$\mathcal{L}[f_3(t)](s) = \mathcal{L}[-1\cdot\mathbb{1}(t-\pi)](s) = -\mathcal{L}[1\cdot\mathbb{1}(t-\pi)](s) = -e^{-\pi s}\mathcal{L}[1](s) = \frac{-e^{-\pi s}}{s}$$

$$\mathcal{L}[f(t)](s) = \frac{-2+\pi s+2e^{-\frac{\pi}{2}s}}{\pi s^2} + \frac{se^{-\pi s}-e^{-\frac{\pi}{2}s}}{s^2+1} - \frac{e^{-\pi s}}{s}.$$

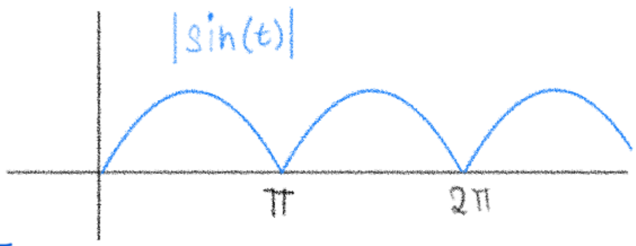
Laplaceova transformace T-periodické funkce

$$\mathcal{L}[f(t)](s) = \frac{\int_0^T f(t)e^{-st}\,dt}{1-e^{-sT}} = \frac{\mathcal{L}[f(t)\,(\mathbb{1}(t)-\mathbb{1}(t-T))](s)}{1-e^{-sT}}$$

3) Spočtěte Laplaceovu transformaci funkce $f(t)=|\sin(t)|$.

Řešení $T=\pi$

$$\mathcal{L}[|\sin(t)|](s) = \frac{\int_0^\pi \sin(t)e^{-st}\,dt}{1-e^{-s\pi}}$$



$$\int_0^\pi \sin(t)e^{-st}\,dt = \left[\sin(t)\cdot\frac{e^{-st}}{-s}\right]_0^\pi + \frac{1}{s}\int_0^\pi \cos(t)\cdot e^{-st}\,dt =$$

$\sin(t) \downarrow \cos(t)$, $\quad e^{-st} \uparrow \frac{e^{-st}}{-s}$ $\quad$ per partes $\quad$ $\cos(t)\downarrow -\sin(t)$, $\quad e^{-st}\uparrow \frac{e^{-st}}{-s}$

$$= 0 + \frac{1}{s}\left(\left[\cos(t)\frac{e^{-st}}{s}\right]_0^\pi - \frac{1}{s}\int_0^\pi \sin(t)\,e^{-st}\,dt\right) = \frac{1}{s}\left(-1\cdot\frac{e^{-s\pi}}{s} - 1\cdot\frac{1}{s}\right) - \frac{1}{s^2}\cdot\int_0^\pi \sin(t)e^{-st}\,dt$$

Řešíme rovnici

$$\int_0^\pi \sin(t)e^{-st}\,dt = \frac{-e^{-s\pi}-1}{s^2} - \frac{1}{s^2}\int_0^\pi \sin(t)e^{-st}\,dt$$

$$\left(\int_0^\pi \sin(t)e^{-st}\,dt\right)\left(1+\frac{1}{s^2}\right) = \frac{-e^{-s\pi}-1}{s^2} \qquad \Big|\cdot\frac{s^2}{s^2+1}$$

$$\int_0^\pi \sin(t)e^{-st}\,dt = -\frac{e^{-s\pi}+1}{s^2+1}$$

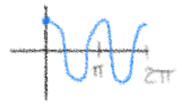
$$\mathcal{L}[|\sin(t)|](s) = \frac{-\frac{e^{-s\pi}+1}{s^2+1}}{1-e^{-s\pi}} = -\frac{e^{-s\pi}+1}{(1-e^{-s\pi})\cdot(s^2+1)}$$

4) Spočtěte Laplaceovu transformaci funkce $f(t)=e^t * [\sin(2t)\mathbb{1}(t-\frac{\pi}{4})]$.

Řešení

$$\mathcal{L}\left[e^t * \left(\sin(2t)\cdot \mathbb{1}\left(t-\frac{\pi}{4}\right)\right)\right](s) = \mathcal{L}[e^t](s)\cdot\mathcal{L}\left[\sin(2t)\cdot\mathbb{1}\left(t-\frac{\pi}{4}\right)\right](s) =$$

$$= \frac{1}{s-1}\cdot e^{-\frac{\pi}{4}s}\,\mathcal{L}\left[\sin\left(2\left(t+\frac{\pi}{4}\right)\right)\right](s) = \frac{e^{-\frac{\pi}{4}s}}{s-1}\cdot\mathcal{L}\left[\sin\left(2t+\frac{\pi}{2}\right)\right](s)$$



$$= \frac{e^{-\frac{\pi}{4}s}}{s-1}\cdot \mathcal{L}[\cos(2t)](s) = \frac{e^{-\frac{\pi}{4}s}}{s-1}\cdot\frac{s}{s^2+4} = \frac{s\,e^{-\frac{\pi}{4}s}}{(s-1)(s^2+4)}$$

Inverzní Laplaceova transformace – metoda reziduí

Máme-li funkci $F(s)$ ve tvaru $\frac{P(s)}{Q(s)}$, $\operatorname{st} P < \operatorname{st} Q$, nebo $\frac{1}{1\pm e^{-sT}}\frac{P(s)}{Q(s)}$, $T>0$, pak

$$f(t) = \mathcal{L}^{-1}[F(s)](t) = \sum_{\omega \text{ póly } F} \operatorname{res}_\omega \left(F(s)\cdot e^{st}\right).$$

5) Stanovte Laplaceův vzor funkce $F(s)=\frac{1}{s(s-2)}$.

Řešení

a) pomocí metody reziduí   póly: 0, 2

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s-2)}\right](t) = \operatorname{res}_{s=0}\frac{1}{s(s-2)}\cdot e^{st} + \operatorname{res}_{s=2}\frac{1}{s(s-2)}e^{st} = \frac{e^{2t}-1}{2}$$

holomorfní na $U(0)$

$$\operatorname{res}_{s=0}\frac{1}{\underbrace{s}_{1\text{ nás}}(s-2)}\cdot e^{st} = \left.\frac{\frac{e^{st}}{s-2}}{1}\right|_{s=0} = \frac{1}{-2}$$

holom. na $U(2)$

$$\operatorname{res}_{s=2}\frac{e^{st}}{s\underbrace{(s-2)}_{1\text{ nás}}} = \left.\frac{\frac{e^{st}}{s}}{1}\right|_{s=2} = \frac{e^{2t}}{2}$$

b) pomocí konvoluce a známých obrazů

$$\left.\begin{aligned}\mathcal{L}[1](s) &= \frac{1}{s}\\ \mathcal{L}[e^{2t}](s) &= \frac{1}{s-2}\end{aligned}\right\} \Rightarrow F(s) = \frac{1}{s}\cdot\frac{1}{s-2} = \mathcal{L}[1](s)\cdot\mathcal{L}[e^{2t}](s) = \mathcal{L}[1*e^{2t}](s)$$

$$\Rightarrow f(t) = 1*e^{2t} = e^{2t}*1 = \int_0^t e^{2\tau}\cdot 1\,d\tau = \left[\frac{e^{2\tau}}{2}\right]_0^t = \frac{e^{2t}-1}{2}$$

6) Stanovte Laplaceův vzor funkce $F(s)=\dfrac{e^{-s}}{s^2-1}$.

Řešení (funkce není ve tvaru na metodu reziduí, ale lze využít posun)

$$F(s)=e^{-s}\cdot \underbrace{\frac{1}{s^2-1}}_{G(s)}$$

holom. na $U(1)$ holom. na $U(-1)$

$$g(t)=\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s-1)}\right](t)=\operatorname{res}_{s=-1}\frac{e^{st}}{\underset{1.\text{ nás}}{(s+1)}(s-1)}+\operatorname{res}_{s=1}\frac{e^{st}}{(s+1)\underset{1.\text{ nás}}{(s-1)}}=$$

$$=\frac{e^{st}}{s-1}\bigg|_{s=-1}+\frac{e^{st}}{s+1}\bigg|_{s=1}=\frac{e^{-t}}{-2}+\frac{e^{t}}{2}=\frac{e^{t}-e^{-t}}{2}$$

Odvození: $\mathcal{L}[\underbrace{g(t-1)\mathbb{1}(t-1)}_{f(t)}](s)=e^{-s}\cdot\mathcal{L}[g(t)](s)=\underbrace{e^{-s}\cdot G(s)}_{F(s)}$

$$f(t)=\frac{e^{(t-1)}-e^{-(t-1)}}{2}\cdot\mathbb{1}(t-1).$$