Cvičení 4 Téma: Křivkový integrál

Derivace a integrál

$f: \mathbb{R} \to \mathbb{C}$. Pak $\operatorname{Re} f: \mathbb{R} \to \mathbb{R}$ a $\operatorname{Im} f: \mathbb{R} \to \mathbb{R}$ a

$f'(t) = (\operatorname{Re} f)'(t) + i\,(\operatorname{Im} f)'(t)$ & $\int f(t)\,dt = \int \operatorname{Re} f(t)\,dt + i \int \operatorname{Im} f(t)\,dt$

1) Je dána funkce $f(t) = \frac{1}{t-i}$, $t \in \mathbb{R}$. Vypočtěte $f'(t)$ a $\int_0^1 f(t)\,dt$.

Řešení

$$f(t) = \frac{1}{t-i} \cdot \frac{t+i}{t+i} = \frac{t+i}{t^2+1} \Rightarrow \operatorname{Re} f(t) = \frac{t}{t^2+1},\ \operatorname{Im} f(t) = \frac{1}{t^2+1}.$$

- $$(\operatorname{Re} f)'(t) = \frac{t^2+1-t\cdot 2t}{(t^2+1)^2} = \frac{1-t^2}{(t^2+1)^2},\quad (\operatorname{Im} f)'(t) = \frac{-2t}{(t^2+1)^2}$$

$$\Rightarrow f'(t) = \frac{1-t^2}{(t^2+1)^2} + i\,\frac{-2t}{(t^2+1)^2}$$

- $$\int_0^1 \operatorname{Re} f(t)\,dt = \int_0^1 \frac{t}{t^2+1}\,dt = \left[\begin{array}{c|c|c|c} \overbrace{t^2+1=s}^{\text{prostá na } (0,1)} & t & 0 & 1 \\ 2t\,dt = ds & s & 1 & 2 \end{array}\right] = \frac{1}{2}\int_1^2 \frac{1}{s}\,ds = \frac{1}{2}\Big[\ln|s|\Big]_1^2 = \frac{1}{2}\ln 2$$

$$\int_0^1 \operatorname{Im} f(t)\,dt = \int_0^1 \frac{1}{t^2+1}\,dt = \Big[\operatorname{arctg} t\Big]_0^1 = \operatorname{arctg} 1 - \operatorname{arctg} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\Rightarrow \int_0^1 f(t)\,dt = \frac{1}{2}\ln 2 + i\frac{\pi}{4} = \ln 2^{\frac{1}{2}} + i\frac{\pi}{4}$$

Věta: $\varphi: \mathbb{R} \to \mathbb{C}$, $f: \mathbb{C} \to \mathbb{C}$: $\exists \varphi'(t)$ & $\exists f'(\varphi(t)) \Rightarrow (f \circ \varphi)'(t) = f'(\varphi(t))\,\varphi'(t)$

Alternativní řešení - derivujeme komplexní funkci, i je konstanta

$$f'(t) = \left(\frac{1}{t-i}\right)' = \frac{-1}{(t-i)^2} \cdot \frac{(t+i)^2}{(t+i)^2} = \frac{-t^2-2ti+1}{(t+1)^2} \checkmark$$

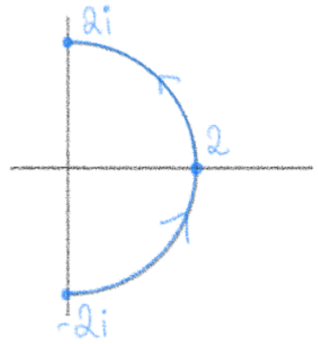
$$\int_0^1 f(t)\,dt = \int_0^1 \frac{1}{t-i}\,dt = \Big[\ln(t-i)\Big]_0^1 = \ln(1-i) - \ln(-i) =$$

$$= \ln\sqrt{2} + i\left(-\frac{\pi}{4}\right) - \left(\ln 1 + i\left(-\frac{\pi}{2}\right)\right) = \ln\sqrt{2} + i\frac{\pi}{4} \checkmark$$

Křivkový integrál: $$\int_C f(z)\,dz = \int_a^b f(\varphi(t))\,\varphi'(t)\,dt$$

kde C je křivka s parametrizací $\varphi: [a,b] \to \mathbb{C}$ a f je spojitá na C.

2) Z definice vypočtěte $\int_C \bar{z}\,dz$, kde C je jako na obrázku:



Řešení:

Parametrizace kružnice $k(S,R)$: $\varphi(t) = S + R\cdot e^{it},\ t\in[-\pi,\pi]$

$$\varphi(t) = 2\cdot e^{it} = 2\cdot(\cos t + i\sin t),\ t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

$$\varphi'(t) = 2ie^{it}$$

$$\int_C \bar z\,dz = \int_C (\mathrm{Re}\,z - i\,\mathrm{Im}\,z)\,dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\mathrm{Re}\,\varphi(t) - i\,\mathrm{Im}\,\varphi(t)\right)\varphi'(t)\,dz =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos t - i2\sin t)\,2ie^{it}\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cdot(\cos(-t) + i\sin(-t))\,2ie^{it}\,dt =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4ie^{-it}e^{it}\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4i\,dt = \Big[4it\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4i\frac{\pi}{2} - 4i\left(-\frac{\pi}{2}\right) = 2\cdot 2i\pi = \underline{4i\pi}.$$

3) Z definice vypočtěte $\int_C e^{z-\bar z}\,dz$, kde C je úsečka s počátečním bodem $1+i\frac{\pi}{2}$ a koncovým bodem 1.

Parametrizace úsečky $seg(a,b)$: $\varphi(t) = a + t(b-a),\ t\in[0,1]$

Řešení: $\varphi(t) = 1 + i\frac{\pi}{2} + t\left(1 - 1 - i\frac{\pi}{2}\right) = 1 + i\frac{\pi}{2} - ti\frac{\pi}{2},\ t\in[0,1].$

$$\varphi'(t) = -i\frac{\pi}{2}$$

$$z - \bar z = x + iy - x + iy = 2iy = 2i\,\mathrm{Im}\,z\ ;\quad \mathrm{Im}\,\varphi(t) = \frac{\pi}{2} - t\frac{\pi}{2}$$

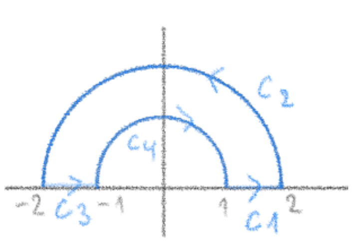
$$\int_C e^{z-\bar z}\,dz = \int_C e^{2i\,\mathrm{Im}\,z}\,dz = \int_0^1 e^{2i\,\mathrm{Im}\,\varphi(t)}\varphi'(t)\,dt = \int_0^1 e^{2i\left(\frac{\pi}{2} - t\frac{\pi}{2}\right)}\left(-i\frac{\pi}{2}\right)dt =$$

$$= -i\frac{\pi}{2}e^{2i\frac{\pi}{2}}\int_0^1 e^{-2it\frac{\pi}{2}}\,dt = -i\frac{\pi}{2}(-1)\cdot\int_0^1 e^{-it\pi}\,dt = \frac{\pi}{2}i\left[\frac{e^{-it\pi}}{-i\pi}\right]_0^1 =$$

$$= -\frac{1}{2}\left(e^{-i\pi} - e^0\right) = -\frac{1}{2}(-1-1) = \underline{1}.$$

4) Z definice vypočtěte $\int_C \frac{z+1}{z}\,dz$, kde C je kladně orientovaná hranice horního polomezikruží se středem v 0 a poloměry 1 a 2.

Řešení



Parametrizace

$C_1 \ldots\ \varphi_1(t) = 1 + t(2-1) = t+1,\ t\in[0,1];\ \varphi_1'(t) = 1$

$C_2 \ldots\ \varphi_2(t) = 0 + 2\cdot e^{it} = 2e^{it},\ t\in[0,\pi];\ \varphi_2'(t) = 2ie^{it}$

$C_3 \ldots\ \varphi_3(t) = -2 + t(-1-(-2)) = t-2,\ t\in[0,1];\ \varphi_3'(t) = 1$

$C_4 \ldots\ \varphi_4(t) = e^{-it},\ t\in[-\pi,0],\ \varphi_4'(t) = -ie^{-it}$ nebo e^{it} a vzít $-\int_{C_4}$

$$\int_C \frac{z+1}{z}\,dz = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = \int_0^1 \frac{t+2}{t+1}\cdot 1\,dt + \int_0^{\pi} \frac{2e^{it}+1}{\cancel{2e^{it}}}\cdot \cancel{2}i\cancel{e^{it}}\,dt + \int_0^1 \frac{t-1}{t-2}\cdot 1\,dt + \int_{-\pi}^{0} \frac{e^{-it}+1}{\cancel{e^{-it}}}\cdot(-i\cancel{e^{-it}})\,dt =$$

$$= \int_0^1 \left(1+\frac{1}{t+1}\right)dt + i\int_0^{\pi}(2e^{it}+1)\,dt + \int_0^1\left(1+\frac{1}{t-2}\right)dt - i\int_{-\pi}^{0}(e^{-it}+1)\,dt =$$

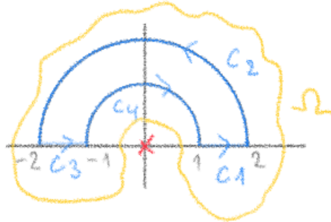
$$= \Big[t+\ln(t+1)\Big]_0^1 + i\left[\frac{2e^{it}}{i}+t\right]_0^{\pi} + \Big[t+\underbrace{\ln|t-2|}_{\text{reálné!}}\Big]_0^1 - i\left[\frac{e^{-it}}{-i}+t\right]_{-\pi}^{0} =$$

$$= 1+\cancel{\ln 2}-0-0+\underbrace{2e^{i\pi}}_{-2}+\cancel{i\pi}\underbrace{-2e^{i0}}_{-2}-0+1+\underbrace{\ln|-1|}_{0}-0-\cancel{\ln|-2|}+\underbrace{e^{-i0}}_{1}+0\underbrace{-e^{i\pi}}_{1}-\cancel{i\pi}=$$

$$= 1-2-2+1+1+1 = 0$$

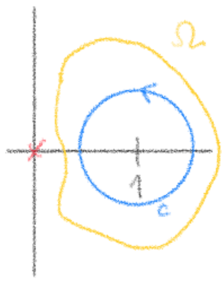
Poznámka: dalo by se provést z Cauchyovy věty

na Ω je $\frac{z+1}{z}$ holomorfní



5) Vypočtěte $\int_C \left(\frac{\sin(z^2)}{z^{50}} + \operatorname{Re} z\right)dz$, kde C je kladně orientovaná kružnice o rovnici $|z-1| = \frac{1}{2}$.

Řešení



$K(1, \frac{1}{2}) \ldots \varphi(t) = 1+\frac{1}{2}e^{it},\ t\in[-\pi,\pi]$; $\varphi'(t) = \frac{1}{2}ie^{it}$

$\operatorname{Re} z$ ... není diferencovatelná na $\mathbb{C}$ => z definice

$\frac{\sin(z^2)}{z^{50}}$ ... je diferencovatelná na $\mathbb{C}\setminus\{0\}$, speciálně na jednoduše souvislé oblasti Ω

=> z Cauchyovy věty $\int_C \frac{\sin(z^2)}{z^{50}}\,dz = 0$

$\boxed{\operatorname{Re} z = \frac{z+\bar z}{2},\ \overline{e^z} = e^{\bar z}}$

$$\int_C\left(\frac{\sin(z^2)}{z^{50}}+\operatorname{Re} z\right)dz = \int_C \frac{\sin(z^2)}{z^{50}}\,dz + \int_C \operatorname{Re} z\,dz = 0 + \int_C \operatorname{Re} z\,dz =$$

$$= \int_C \frac{z+\bar z}{2}\,dz = \int_{-\pi}^{\pi} \frac{1+\frac{1}{2}e^{it}+1+\frac{1}{2}e^{-it}}{2}\cdot\frac{1}{2}ie^{it}\,dt = \frac{i}{4}\int_{-\pi}^{\pi}\left(2e^{it}+\frac{1}{2}e^{2it}+\frac{1}{2}\right)dt =$$

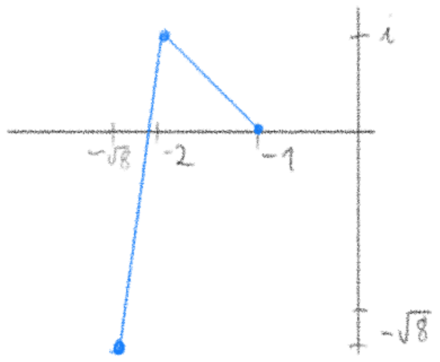
$$= \frac{i}{4}\left[\frac{2e^{it}}{i}+\frac{1}{2}\,\frac{e^{2it}}{2i}+\frac{t}{2}\right]_{-\pi}^{\pi} = \frac{i}{4}\left(\frac{2e^{i\pi}}{i}+\frac{e^{2i\pi}}{4i}+\frac{\pi}{2}-\frac{2e^{-i\pi}}{i}-\frac{e^{-2i\pi}}{4i}+\frac{\pi}{2}\right)=$$

$$= \frac{i}{4}\left(-\cancel{\frac{2}{i}}+\cancel{\frac{1}{4i}}+\pi+\cancel{\frac{2}{i}}-\cancel{\frac{1}{4i}}\right) = \frac{\pi i}{4}$$

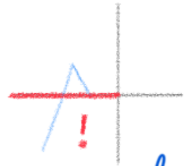
Věta: Nechť $F'(z) = f(z)$ na Ω, $C\subset\Omega$ s počátečním bodem z_1 a koncovým z_2. Pak $\int_C f(z)\,dz = F(z_2)-F(z_1)$.

6) Vypočtěte $\int_C \frac{1}{z}\,dz$, kde C je spojení 2 úseček $seg(-1,-2+i)$ a $seg(-2+i,-\sqrt{8}-i\sqrt{8})$.

Řešení – pomocí primitivní funkce



$(\ln(z))' = \frac{1}{z}$ na $\mathbb{C}\setminus\{z\in\mathbb{C},\ \operatorname{Re} z\le 0,\ \operatorname{Im} z=0\}$

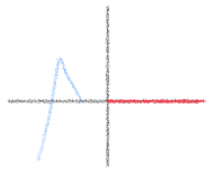


$\Rightarrow \ln(z)$ není vyhovující primitivní funkce pro $\int_C$

Zvolíme jinou větev logaritmu

$\ln_{2\pi}(z) := \ln|z| + i\varphi$, kde $\varphi = [0,2\pi) \cap \operatorname{Arg}(z)$

Pak $(\ln_{2\pi}(z))' = \frac{1}{z}$ na $\mathbb{C}\setminus\{z\in\mathbb{C},\ \operatorname{Re} z\ge 0,\ \operatorname{Im} z=0\}$



$\Rightarrow$ vhodná primitivní funkce pro $\int_C$

$$\int_C \frac{1}{z}\,dz = \ln_{2\pi}(-\sqrt{8}-i\sqrt{8}) - \ln_{2\pi}(-1) = \ln|-\sqrt{8}-i\sqrt{8}| + i\overbrace{\frac{5\pi}{4}}^{\operatorname{Arg}(-\sqrt{8}-i\sqrt{8})\cap[0,2\pi)} - \ln|-1| - i\pi =$$

$$= \ln\sqrt{8+8} + i\frac{5\pi}{4} - 0 - i\pi = \ln 4 + i\frac{\pi}{4}$$

7) Vypočtěte $\int_C \frac{1}{z^2+4}\,dz$, kde C je kladně orientovaná $k(2i,1)$.

Řešení – pomocí parciálních zlomků

$\varphi(t) = 2i + e^{it}$

$$\frac{1}{z^2+4} = \frac{1}{(z-2i)(z+2i)} = \frac{A}{z-2i} + \frac{B}{z+2i} \qquad A = \frac{1}{4i},\ B = \frac{-1}{4i}$$

$\int_C \frac{1}{z+2i}\,dz = 0$ z Cauchyovy věty, neboť $\frac{1}{z+2i}$ je holomorfní na Ω

$$\int_C \frac{1}{z-2i}\,dz = \int_{-\pi}^{\pi} \frac{1}{\cancel{2i}+e^{it}-\cancel{2i}}\, ie^{it} = \int_{-\pi}^{\pi} i\,dt = 2i\pi \quad \text{(z přednášky)}$$

$$\int_C \frac{1}{z^2+4}\,dz = \int_C \frac{\frac{1}{4i}}{z-2i}\,dz + \int_C \frac{-\frac{1}{4i}}{z+2i}\,dz = \frac{1}{4i}\cdot 2i\pi + 0 = \frac{\pi}{2}$$