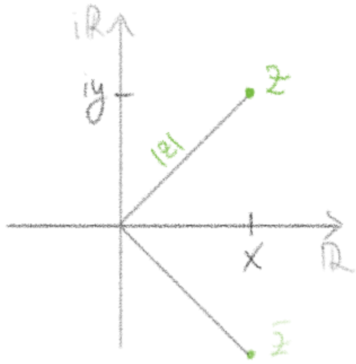
Cvičení 1

Téma: Komplexní čísla

Opakování: $i^2=-1,\ i^3=-i,\ i^4=1, \dots$ zbytek po dělení 4 $(i^{n \bmod 4})$



$z=x+iy$ … algebraický tvar komplexního čísla

$\operatorname{Re} z = x \in \mathbb{R}$ … reálná část komplexního čísla

$\operatorname{Im} z = y \in \mathbb{R}$ … imaginární část komplexního čísla

$|z|=\sqrt{x^2+y^2}$ … velikost komplexního čísla

$\bar{z}=x-iy$ … číslo komplexně sdružené k z

$z\cdot\bar{z}=x^2+y^2=|z|^2$

1) Nalezněte Re z a Im z.

$$z=(1+i)^2+\frac{1+i^{11}}{1+i}=2i-i=i=0+1\cdot i \qquad \operatorname{Re} z=0,\ \operatorname{Im} z=1$$

$$(1+i)^2=1+2i-1=2i$$

$$\frac{1+i^{11}}{1+i}=\frac{1+i^3}{1+i}=\frac{1-i}{1+i}\cdot\frac{1-i}{1-i}=\frac{1-2i-1}{1^2+1^2}=\frac{-2i}{2}=-i$$

$$z=\frac{2+3i}{1+2i}-\left(\frac{2-i}{1+2i}\right)^2=\frac{8}{5}-\frac{i}{5}-(-1)=\frac{13}{5}-\frac{i}{5} \qquad \operatorname{Re} z=\frac{13}{5},\ \operatorname{Im} z=-\frac{1}{5}$$

$$\frac{2+3i}{1+2i}\cdot\frac{1-2i}{1-2i}=\frac{2-4i+3i+6}{1+4}=\frac{8-i}{5}=\frac{8}{5}-\frac{i}{5}$$

$$\left(\frac{2-i}{1+2i}\right)^2=\frac{4-4i-1}{1+4i-4}=\frac{3-4i}{-3+4i}=-1$$

2) Dokažte $\overline{z_1z_2}=\overline{z_1}\cdot\overline{z_2}$ a $|z_1z_2|=|z_1|\cdot|z_2|$.

Označme $z_1=x_1+iy_1,\ z_2=x_2+iy_2$.

- $$\overline{z_1z_2}=\overline{(x_1+iy_1)(x_2+iy_2)}=\overline{x_1x_2+ix_1y_2+ix_2y_1-y_1y_2}=\overline{x_1x_2-y_1y_2+i(x_1y_2+x_2y_1)}$$
$$=x_1x_2-y_1y_2-i(x_1y_2+x_2y_1)$$

$$\overline{z_1}\cdot\overline{z_2}=\overline{x_1+iy_1}\cdot\overline{x_2+iy_2}=(x_1-iy_1)(x_2-iy_2)=x_1x_2-ix_1y_2-ix_2y_1-y_1y_2=$$
$$=x_1x_2-y_1y_2-i(x_1y_2+x_2y_1)$$

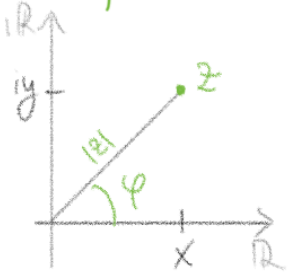
- $$|z_1z_2|=|x_1x_2-y_1y_2+i(x_1y_2+x_2y_1)|=\sqrt{(x_1x_2-y_1y_2)^2+(x_1y_2+x_2y_1)^2}=$$
$$=\sqrt{x_1^2x_2^2-2x_1x_2y_1y_2+y_1^2y_2^2+x_1^2y_2^2+2x_1x_2y_1y_2+x_2^2y_1^2}$$

$$= \sqrt{x_1^2 x_2^2 + y_1^2 y_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2}$$

$$|z_1||z_2| = \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} = \sqrt{x_1^2 x_2^2 + x_1^2 y_2^2 + y_1^2 x_2^2 + y_1^2 y_2^2}$$

Opakování: $z = |z| \cdot (\cos\varphi + i \sin\varphi)$ ... goniometrický tvar komplexního čísla

$= : |z| \cdot e^{i\varphi}$

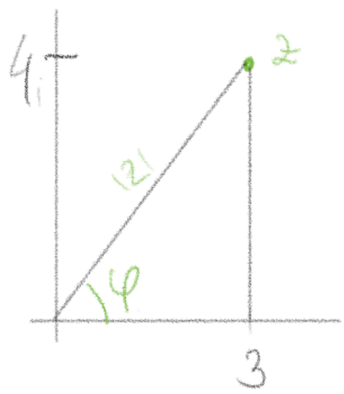


$\mathrm{Arg}\, z = \{ \varphi \in \mathbb{R} \mid |z| e^{i\varphi} = z \}$ ... množina argumentů čísla z

$\arg z = \mathrm{Arg}\, z \cap (-\pi, \pi]$ ... hlavní hodnota argumentu čísla z

3) Nalezněte goniometrický tvar komplexního čísla z

$z = 3 + 4i$



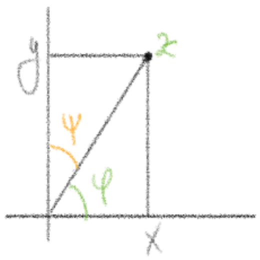
$$|z| = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$\mathrm{tg}\, \varphi = \frac{\text{protilehlá}}{\text{přilehlá}} = \frac{4}{3}$$

$$\varphi = \mathrm{arctg}\, \frac{4}{3}$$

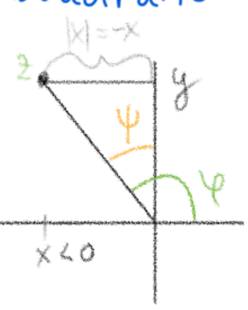
4) Ať $z = x + iy \neq 0$. Dokažte, že
$$\arg z = \begin{cases} \mathrm{arctg}\, \frac{y}{x} & , x > 0 \\ \frac{\pi}{2} - \mathrm{arctg}\, \frac{x}{y} & , y > 0 \\ -\frac{\pi}{2} - \mathrm{arctg}\, \frac{x}{y} & , y < 0 \\ \pi , & y = 0, x < 0 \end{cases}$$

1. kvadrant



- $\mathrm{tg}\, \varphi = \frac{y}{x} \Rightarrow \varphi = \mathrm{arctg}\, \frac{y}{x}$
- $\mathrm{tg}\, \psi = \frac{x}{y} \Rightarrow \psi = \mathrm{arctg}\, \frac{x}{y} \Rightarrow \varphi = \frac{\pi}{2} - \psi = \frac{\pi}{2} - \mathrm{arctg}\, \frac{x}{y}$

2. kvadrant



kladný úhel v △

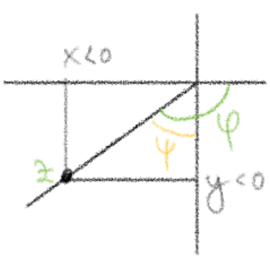
délka strany △

arctg je lichá

- $\mathrm{tg}\, \psi = \frac{-x}{y} \Rightarrow \psi = \mathrm{arctg}\, \frac{-x}{y} = -\mathrm{arctg}\, \frac{x}{y}$

$$\varphi = \frac{\pi}{2} + \psi = \frac{\pi}{2} + \left(-\mathrm{arctg}\, \frac{x}{y}\right) = \frac{\pi}{2} - \mathrm{arctg}\, \frac{x}{y}$$

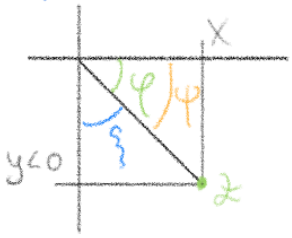
3. kvadrant



- $\mathrm{tg}\, \psi = \frac{-x}{-y} = \frac{x}{y} \Rightarrow \psi = \mathrm{arctg}\, \frac{x}{y}$

$$\varphi = -\frac{\pi}{2} - \psi = -\frac{\pi}{2} - \mathrm{arctg}\, \frac{x}{y}$$

4. kvadrant



• $tg\,\psi = \frac{-y}{x} \Rightarrow \psi = arctg\frac{-y}{x} = -arctg\frac{y}{x}$

$\varphi = -\psi = arctg\frac{y}{x}$

• $tg\,\xi = \frac{x}{-y} \Rightarrow \xi = arctg\frac{x}{-y} = -arctg\frac{x}{y}$

$\varphi = -\frac{\pi}{2} + \xi = -\frac{\pi}{2} - arctg\frac{x}{y}$

5) Ať $z \in \mathbb{C}\setminus\{0\}$, $\varphi \in Arg\, z$. Dokažte $Arg\, z = \{\varphi + 2k\pi \mid k \in \mathbb{Z}\}$.

$\supseteq$ Vezměme $k \in \mathbb{Z}$. Pak

z předpokladu $\varphi \in Arg\, z$

$$|z|e^{i(\varphi+2k\pi)} = |z|e^{i\varphi}\cdot \overbrace{e^{i2k\pi}}^{1} = |z|e^{i\varphi} = z$$

Z definice $Arg\, z$ pak $\varphi + 2k\pi \in Arg\, z$.

Nyní víme $Arg\, z \supseteq \{\varphi+2k\pi \mid k\in\mathbb{Z}\}$. Může nastat buď $\supsetneq$ nebo $=$.

Vyvrátíme $\supsetneq$

Ať $\psi \notin \{\varphi + 2k\pi \mid k \in \mathbb{Z}\}$. Pak $e^{i\psi} \neq e^{i\varphi}$

$$|z|e^{i\psi} \neq |z|e^{i\varphi} = z$$

Z definice $Arg\, z$ pak $\psi \notin Arg\, z$.

Tedy $Arg\, z = \{\varphi + 2k\pi \mid k \in \mathbb{Z}\}$.

6) Dokažte: Ať $z, w \in \mathbb{C}\setminus\{0\}$, $\varphi \in Arg\, z$, $\psi \in Arg\, w$. Pak

$$\boxed{z=w} \iff \boxed{|z|=|w| \;\&\; \exists k\in\mathbb{Z}: \varphi = \psi + 2k\pi}$$

$\Rightarrow$ Víme $|z|\cdot e^{i\varphi} = |w|\cdot e^{i\psi}$.

Pak $\begin{cases} |z| = |w| \\ e^{i\varphi} = e^{i\psi} \quad |\cdot e^{-i\psi} \end{cases}$

$e^{i(\varphi-\psi)} = 1 = e^{i2k\pi}$

$\varphi - \psi = 2k\pi \Rightarrow \varphi = \psi + 2k\pi.$

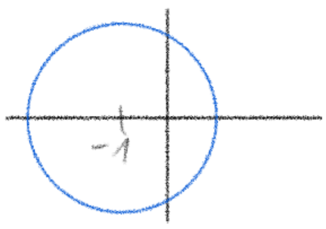
$\Leftarrow$ $z = |z|e^{i\varphi} = |w|\cdot e^{i(\psi+2k\pi)} = |w|\cdot e^{i\psi}\cdot \underbrace{e^{i2k\pi}}_{1} = |w|\cdot e^{i\psi} = w$

7) Popište geometricky množinu komplexních čísel

• $|z+1| = 2$

$|z-(-1)| = 2$ … "vzdálenost z od -1 je 2"

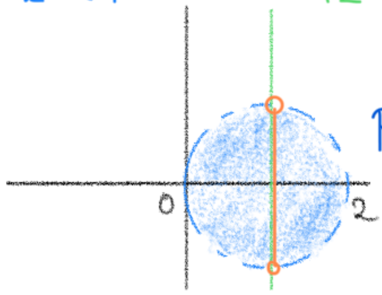
⇝ kružnice se středem -1 a poloměrem 2



• $|z-1| < 1$ & $|z| = |z-2|$

kruh

$|z-0| = |z-2|$ … "vzdálenost od 0 je stejná jako od 2"

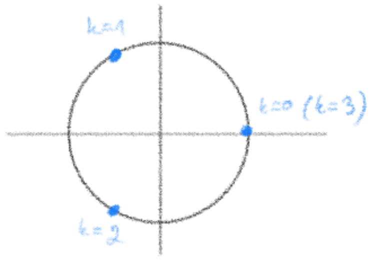


průnikem podmínek je úsečka mezi body $1+i$ a $1-i$ bez krajních bodů

Opakování: Binomická rovnice $z^n = a \in \mathbb{C} \setminus \{0\}$ má n řešení uspořádaných do pravidelného n-úhelníku

8) Řešte rovnici

• $z^3 = 1$ $\quad |z| \cdot e^{i\varphi} = z$

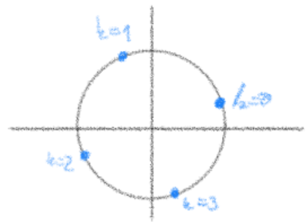


$|z|^3 e^{i3\varphi} = 1 = 1 \cdot e^{i0}$

Úloha 6

$\overset{\text{Úloha 6}}{\Longleftrightarrow} \begin{cases} |z|^3 = 1 & \Leftrightarrow |z| = 1 \\ 3\varphi = 0 + 2k\pi & \Leftrightarrow \varphi = \frac{2}{3}k\pi \end{cases}$ $z \in \{1 \cdot e^{i\frac{2}{3}k\pi}, k = 0,1,2\}$.

• $z^4 = 81i$ $\quad |z| \cdot e^{i\varphi} = z$



$|z|^4 \cdot e^{i4\varphi} = 81i = 81e^{i\frac{\pi}{2}}$

$\overset{\text{Úloha 6}}{\Longleftrightarrow} \begin{cases} |z|^4 = 81 & \Leftrightarrow |z| = 3 \\ 4\varphi = \frac{\pi}{2} + 2k\pi & \Leftrightarrow \varphi = \frac{\pi}{8} + \frac{k\pi}{2} \end{cases}$ $z \in \{3 \cdot e^{i(\frac{\pi}{8} + \frac{k\pi}{2})}, k = 0,1,2,3\}$

• $z^n = \bar{z}$, $n \in \mathbb{N}$. Toto není binomická rovnice

$|z|^n e^{in\varphi} = |z| e^{i\varphi}$

$\Leftrightarrow \begin{cases} |z|^n = |z| \; \begin{cases} |z|^{n-1} = 1, \; z \neq 0 \; \begin{cases} n = 1: |z| \in \mathbb{R}^+ \\ n \geq 2: |z| = 1 \end{cases} \\ z = 0 \end{cases} \\ n\varphi = -\varphi + 2k\pi \Leftrightarrow \varphi(n+1) = 2k\pi \Leftrightarrow \varphi = \frac{2k\pi}{n+1} \end{cases}$

$z \in \begin{cases} \{0; \; 1 \cdot e^{i\frac{2k\pi}{n+1}}; \; k = 0, \dots, n\}, \; n \in \mathbb{N} \setminus \{1\} \\ \mathbb{R}, \; n = 1 \end{cases}$