# APPROXIMATIONS OF MODULES
(LECTURE NOTES FOR NMAG531)

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Contents



## 1. $\mathcal{C}$-filtrations

The notion of an *extension* of a module $A$ by a module $B$, that is, of a module $C$ that fits in the short exact sequence $0 \to A \to C \to B \to 0$, generalizes the notion of a direct sum of $A$ and $B$ (which occurs in the particular case when the sequence above splits). Similarly as direct sums of pairs of modules are extended to finite, and arbitrary, direct sums of modules, one can iterate the extensions and obtain finite, and transfinite, extensions of modules from a given class $\mathcal{C}$. In more detail, we have

**Definition 1.1.** Let $R$ be a ring, $M$ a module, and $\mathcal{C}$ a class of modules.

A chain of submodules, $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$, of $M$ is called *continuous*, provided that $M_0 = 0$, $M_\alpha \subseteq M_{\alpha+1}$ for each $\alpha < \sigma$, and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$.

A continuous chain $\mathcal{M}$ is a *$\mathcal{C}$-filtration* of $M$, provided that $M = M_\sigma$, and each of the modules $M_{\alpha+1}/M_\alpha$ ($\alpha < \sigma$) is isomorphic to some element of $\mathcal{C}$.

If $M$ is a $\mathcal{C}$-filtered module, then $M$ is also called a *transfinite extension* of the modules in $\mathcal{C}$. A class $\mathcal{A}$ is said to be *closed under transfinite extensions* provided that $\mathcal{A}$ contains all $\mathcal{A}$-filtered modules. Clearly, this implies that $\mathcal{A}$ is closed under extensions and arbitrary direct sums.

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$M$ is called *$\mathcal{C}$-filtered*, provided that $M$ possesses at least one $\mathcal{C}$-filtration $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$. If $\sigma$ can be taken finite, then $M$ is called *finitely $\mathcal{C}$-filtered*.

We will use the notation $\mathrm{Filt}(\mathcal{C})$ for the class of all $\mathcal{C}$-filtered modules.

**Example 1.2.** 1. Let $R$ be any ring and $\mathcal{C}$ be a representative set of all countably generated projective modules. Since each extension by a projective module splits, $\mathcal{C}$-filtered modules coincide with the modules isomorphic to direct sums of modules from $\mathcal{C}$. Hence $\mathrm{Filt}(\mathcal{C}) = \mathcal{P}_0$ is the class of all projective modules, by a classic theorem of Kaplansky [2, 26.2].

2. Let $R$ be any ring and $\mathcal{C} = \operatorname{simp} R$ be a representative set of all simple modules. Then the $\mathcal{C}$-filtered modules coincide with the semiartinian modules, while the finitely $\mathcal{C}$-filtered modules are exactly the modules of finite length. The latter modules are the subject of the classic Jordan-Hölder theory. In a sense, the Hill Lemma and its applications presented below are extensions of this theory to the infinite setting where no dimensions and indecomposable direct sum decompositions are available in general.

3. Let $R = \mathbb{Z}$ (the integers) and $\mathcal{C} = \{\mathbb{Z}_p\}$ (the finite group of a prime order $p$). Then $\mathrm{Filt}(\mathcal{C})$ is the class of all abelian $p$-groups. This might look like an easy variation of 2., but despite the fact that abelian $p$-groups are just subgroups of direct sums of copies of the Prüfer group $\mathbb{Z}_{p^\infty}$, so their socle-sequences have length $\leq \omega$, and the countable ones are characterized by their Ulm-Kaplansky invariants, the full classification of all abelian $p$-groups is considered to be hopeless.

The easy fact that the class $\mathcal{P}_0$ of all projective modules from Example 1.2.1 is closed under transfinite extensions is a particular instance of a more important general phenomenon:

For a class of modules $\mathcal{C}$, we denote by ${}^\perp\mathcal{C}$ the common kernel of all the contravariant $\mathrm{Ext}^1$ functors induced by the elements of $\mathcal{C}$, that is,

$$ {}^\perp\mathcal{C} = \operatorname{Ker}\operatorname{Ext}^1_R(-,\mathcal{C}) = \{A \in \text{Mod–}R \mid \operatorname{Ext}^1_R(A,C) = 0 \text{ for all } C \in \mathcal{C}\}. $$

Similarly, for each $n \geq 1$, we define

$$ {}^{\perp_n}\mathcal{C} = \operatorname{Ker}\operatorname{Ext}^n_R(-,\mathcal{C}) = \{A \in \text{Mod–}R \mid \operatorname{Ext}^n_R(A,C) = 0 \text{ for all } C \in \mathcal{C}\}, $$

and

$$ {}^{\perp_\infty}\mathcal{C} = \bigcap_{n \geq 1} {}^{\perp_n}\mathcal{C} $$

For example, $\mathcal{P}_0 = {}^\perp(\text{Mod–}R)$.

**Lemma 1.3.** (The Eklof Lemma, [19]) *Let $\mathcal{C}$ be a class of modules. Then the class ${}^\perp\mathcal{C}$ is closed under transfinite extensions. That is, if $M$ is a ${}^\perp\mathcal{C}$-filtered module, then $M \in {}^\perp\mathcal{C}$.*

*Proof.* It suffices to prove the claim for the case when $\mathcal{C} = \{N\}$ for a single module $N$.

Let $(M_\alpha \mid \alpha \leq \kappa)$ be a ${}^\perp N$-filtration of $M$. So $\operatorname{Ext}^1_R(M_0, N) = 0$ and, for each $\alpha < \kappa$, $\operatorname{Ext}^1_R(M_{\alpha+1}/M_\alpha, N) = 0$. We will prove $\operatorname{Ext}^1_R(M, N) = 0$.

By induction on $\alpha \leq \kappa$ we will prove that $\operatorname{Ext}^1_R(M_\alpha, N) = 0$. This is clear for $\alpha = 0$.

The exact sequence

$$ 0 = \operatorname{Ext}^1_R(M_{\alpha+1}/M_\alpha, N) \to \operatorname{Ext}^1_R(M_{\alpha+1}, N) \to \operatorname{Ext}^1_R(M_\alpha, N) = 0 $$

proves the induction step.

Assume $\alpha \leq \kappa$ is a limit ordinal. Let $0 \to N \to I \xrightarrow{\pi} I/N \to 0$ be an exact sequence with $I$ an injective module. In order to prove that $\operatorname{Ext}^1_R(M_\alpha, N) = 0$, we show that the abelian group homomorphism $\mathrm{Hom}_R(M_\alpha, \pi) : \mathrm{Hom}_R(M_\alpha, I) \to \mathrm{Hom}_R(M_\alpha, I/N)$ is surjective.

Let $\varphi \in \mathrm{Hom}_R(M_\alpha, I/N)$. We now define by induction homomorphisms $\psi_\beta \in \mathrm{Hom}_R(M_\beta, I)$, $\beta < \alpha$, so that $\varphi \restriction M_\beta = \pi\psi_\beta$ and $\psi_\beta \restriction M_\gamma = \psi_\gamma$ for all $\gamma < \beta < \alpha$.

First define $M_{-1} = 0$ and $\psi_{-1} = 0$. If $\psi_\beta$ is already defined, the injectivity of $I$ yields the existence of $\eta \in \mathrm{Hom}_R(M_{\beta+1}, I)$, such that $\eta \restriction M_\beta = \psi_\beta$. Put $\delta = \varphi \restriction M_{\beta+1} - \pi\eta \in \mathrm{Hom}_R(M_{\beta+1}, I/N)$. Then $\delta \restriction M_\beta = 0$. Since $\mathrm{Ext}^1_R(M_{\beta+1}/M_\beta, N) = 0$, there is $\epsilon \in \mathrm{Hom}_R(M_{\beta+1}, I)$, such that $\epsilon \restriction M_\beta = 0$ and $\pi\epsilon = \delta$. Put $\psi_{\beta+1} = \eta + \epsilon$. Then $\psi_{\beta+1} \restriction M_\beta = \psi_\beta$ and $\pi\psi_{\beta+1} = \pi\eta + \delta = \varphi \restriction M_{\beta+1}$. For a limit ordinal $\beta < \alpha$, put $\psi_\beta = \bigcup_{\gamma<\beta} \psi_\gamma$.

Finally, put $\psi_\alpha = \bigcup_{\beta<\alpha} \psi_\beta$. By the construction, $\pi\psi_\alpha = \varphi$.
The claim is just the case of $\alpha = \kappa$. □

**Remark 1.4.** Let $\mathcal{C}$ be a class of modules.

(i) Since for each module $M$ and each $1 \leq n < \omega$, ${}^{\perp_{n+1}}M = {}^{\perp}\Omega^{-n}(M)$, where $\Omega^{-n}(M)$ denotes the $n$th cosyzygy in an injective coresolution of $M$, the Eklof Lemma also holds for the classes ${}^{\perp_n}\mathcal{C}$ $(1 < n < \omega)$ and ${}^{\perp_\infty}\mathcal{C}$. That is, all these classes are closed under transfinite extensions.

(ii) Also the class $\mathcal{T} = {}^{\perp_0}\mathcal{C} = \mathrm{Ker}\,\mathrm{Hom}_R(-, \mathcal{C})$ is closed under transfinite extensions. In fact, more is true: $\mathcal{T}$ is closed under extensions, direct sums, and homomorphic images, that is, $\mathcal{T}$ is a *torsion class*. Conversely, any torsion class of modules is of the form ${}^{\perp_0}\mathcal{D}$ for a class of modules $\mathcal{D}$.

(iii) The Eklof Lemma can be dualized as follows: Let $\mathcal{C}^\perp = \mathrm{Ker}\,\mathrm{Ext}^1_R(\mathcal{C}, -)$, and let $M$ be a $\mathcal{C}^\perp$-cofiltered module. Then $M \in \mathcal{C}^\perp$. For more details (and terminology), we refer to [27, 6.7].

One can also take the opposite point of view: given a class of modules closed under transfinite extensions, $\mathcal{D}$, try to find find its subset $\mathcal{C}$ such that $\mathcal{D} = \mathrm{Filt}(\mathcal{C})$. This is called the *deconstruction* of the class $\mathcal{D}$. If $\mathcal{D} = \mathrm{Filt}(\mathcal{C})$ where $\mathcal{C} = \mathcal{D}^{<\kappa}$ (the latter symbol denotes the class of all $< \kappa$-presented modules from $\mathcal{D}$), then the class $\mathcal{D}$ is called *$< \kappa$-deconstructible*.

Notice that here again, we generalize the classic setting where extensions are restricted to the split ones, whence $< \kappa$-deconstructibility amounts to $< \kappa$-decomposabilty of modules from $\mathcal{D}$ into direct sums of $< \kappa$-presented modules from $\mathcal{D}$.

Classes of modules occurring naturally in homological algebra are rarely decomposable (essentially, decomposability is available only for projective modules, and injective modules over right noetherian rings). In contrast, deconstructible classes are abundant. Deconstructibility yields approximations of modules, and the approximations make it possible to do relative homological algebra, where projective and injective modules are replaced by other classes of modules better fitting the particular settings of interest. Before turning to these facts, we will study $\mathcal{C}$-filtrations of modules in greater detail.

## 2. The Hill Lemma

When studying a particular $\mathcal{C}$-filtered module, we often need to replace the original $\mathcal{C}$-filtration by another one that better fits the study in case. A remarkable construction serving this purpose was discovered by Hill [28]. It expands a given $\mathcal{C}$-filtration, $\mathcal{M}$, of a module $M$ into a large family, $\mathcal{H}$, consisting of $\mathcal{C}$-filtered submodules of $M$. Moreover, $\mathcal{H}$ inherits the key property of $\mathcal{M}$: it forms a complete distributive sublattice of the modular lattice of all submodules of $M$:

**Theorem 2.1.** (Hill Lemma) *Let $R$ be a ring, $\kappa$ an infinite regular cardinal, and $\mathcal{C}$ a set of $< \kappa$-presented modules. Let $M$ be a module with a $\mathcal{C}$-filtration $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$. Then there is a family $\mathcal{H}$ consisting of submodules of $M$ such that*

(H1) $\mathcal{M} \subseteq \mathcal{H}$.
(H2) *$\mathcal{H}$ is closed under arbitrary sums and intersections. $\mathcal{H}$ is a complete distributive sublattice of the modular lattice of all submodules of $M$.*
(H3) *Let $N, P \in \mathcal{H}$ be, such that $N \subseteq P$. Then the module $P/N$ is $\mathcal{C}$-filtered.*
(H4) *Let $N \in \mathcal{H}$ and $X$ be a subset of $M$ of cardinality $< \kappa$. Then there is a $P \in \mathcal{H}$, such that $N \cup X \subseteq P$ and $P/N$ is $< \kappa$-presented.*

The key notion of the proof of Theorem 2.1 is that of a closed subset of the length, $\sigma$, of a $\mathcal{C}$-filtration:

**Definition 2.2.** Let $R$ be a ring and $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$ be a continuous chain of modules. Consider a family of modules $(A_\alpha \mid \alpha < \sigma)$, such that $M_{\alpha+1} = M_\alpha + A_\alpha$ for each $\alpha < \sigma$.
A subset $S$ of the ordinal $\sigma$ is *closed*, if every $\alpha \in S$ satisfies

$$M_\alpha \cap A_\alpha \subseteq \sum_{\beta \in S, \beta < \alpha} A_\beta.$$

The *height* $\mathrm{hgt}(x)$ of an element $x \in M_\sigma$ is defined as the least ordinal $\alpha < \sigma$, such that $x \in M_{\alpha+1}$. For any subset $S$ of $\sigma$, we define

$$M(S) = \sum_{\alpha \in S} A_\alpha.$$

For each ordinal $\alpha \leq \sigma$, we have $\alpha = \{\beta < \sigma \mid \beta < \alpha\}$, and $M_\alpha = \sum_{\beta<\alpha} A_\beta = M(\alpha)$. So $\alpha$ is a closed subset of $\sigma$.

Before embarking on the proof, we will need to collect properties of closed subsets:

**Lemma 2.3.** *Let $S$ be a closed subset of $\sigma$ and $x \in M(S)$. Let $S' = \{\alpha \in S \mid \alpha \leq hgt(x)\}$. Then $x \in M(S')$.*

*Proof.* Let $x \in M(S)$. Then $x = x_1 + \dots + x_k$, where $x_i \in A_{\alpha_i}$ for some $\alpha_i \in S$, $1 \leq i \leq k$. W.l.o.g., $\alpha_1 < \dots < \alpha_k$, and $\alpha_k$ is minimal.
If $\alpha_k > \mathrm{hgt}(x)$, then $x_k = x - x_1 - \dots - x_{k-1} \in M_{\alpha_k} \cap A_{\alpha_k} \subseteq \sum_{\alpha \in S, \alpha < \alpha_k} A_\alpha$, since $S$ is closed, in contradiction with the minimality of $\alpha_k$. $\square$

As an immediate corollary, we have

**Corollary 2.4.** *Let $S$ be a closed subset of $\sigma$ and $x \in M(S)$. Then $hgt(x) \in S$.*

*Proof.* If $\mathrm{hgt}(x) \notin S$ then by Lemma 2.3, $x \in M(\bar{S}) \subseteq M_{\mathrm{hgt}(x)}$ where $\bar{S} = \{\alpha \in S \mid \alpha < \mathrm{hgt}(x)\}$. Hence $x \in M_{\beta+1}$ for some $\beta < \mathrm{hgt}(x)$, a contradiction. $\square$

**Lemma 2.5.** *Let $(S_i \mid i \in I)$ be a family of closed subsets of $\sigma$. Then*

$$M(\bigcap_{i \in I} S_i) = \bigcap_{i \in I} M(S_i) \quad \text{and} \quad M(\bigcup_{i \in I} S_i) = \sum_{i \in I} M(S_i).$$

*Proof.* For the first equality, let $T = \bigcap_{i \in I} S_i$. Clearly, $M(T) \subseteq \bigcap_{i \in I} M(S_i)$. Suppose there is an $x \in \bigcap_{i \in I} M(S_i)$, such that $x \notin M(T)$, and choose such an $x$ of minimal height. Then $x = y + z$ for some $y \in A_{\mathrm{hgt}(x)}$ and $z \in M_{\mathrm{hgt}(x)}$. By Corollary 2.4, $\mathrm{hgt}(x) \in S_i$ for all $i \in I$, so $\mathrm{hgt}(x) \in T$ and $y \in M(T)$. Then $z \in \bigcap_{i \in I} M(S_i)$, $z \notin M(T)$ and $\mathrm{hgt}(z) < \mathrm{hgt}(x)$, in contradiction to minimality.

The second equality is immediate from Definition 2.2. □

The following remark explains the role of the closed subsets of $\sigma$:

**Remark 2.6.** Let $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$ be a continuous chain of modules. If $N$ is a submodule of $M = M_\sigma$, then $\mathcal{M}$ induces the continuous chain of submodules $\mathcal{N} = (N \cap M_\alpha \mid \alpha \leq \sigma)$ of $N$.

If, moreover, $N = M(S)$ for a subset $S \subseteq \sigma$, then another continuous chain of submodules of $N$ is given by $\mathcal{N}' = (M(S \cap \alpha) \mid \alpha \leq \sigma)$.

Notice that the set $S$ is closed in $\sigma$, if and only if the chains $\mathcal{N}$ and $\mathcal{N}'$ coincide. The only-if part holds, because $M(S) \cap M_\alpha = M(S) \cap M(\alpha) = M(S \cap \alpha)$ for each $\alpha \leq \sigma$ by Lemma 2.5. Conversely, if $\alpha \in S$, then $M_\alpha \cap A_\alpha \subseteq M(\alpha) \cap M(S) = M(S \cap \alpha) = \sum_{\beta \in S, \beta < \alpha} A_\beta$.

Next we prove that intersections and unions of closed subsets are again closed:

**Proposition 2.7.** *Let $(S_i \mid i \in I)$ be a family of closed subsets of $\sigma$. Then both the union and the intersection of this family are again closed in $\sigma$.*

*Proof.* As for the union, if $\beta \in S = \bigcup_{i \in I} S_i$, then $\beta \in S_i$ for some $i \in I$ and $M_\beta \cap A_\beta \subseteq \sum_{\alpha \in S_i, \alpha < \beta} A_\alpha \subseteq \sum_{\alpha \in S, \alpha < \beta} A_\alpha$.

For the intersection, let $\beta \in T = \bigcap_{i \in I} S_i$. Then $M_\beta \cap A_\beta \subseteq M(S_i \cap \beta)$ for each $i \in I$. Therefore Lemma 2.5 implies that

$$M_\beta \cap A_\beta \subseteq \bigcap_{i \in I} M(S_i \cap \beta) = M(T \cap \beta),$$

which exactly says that $T$ is closed. □

By Proposition 2.7, closed subsets form a complete distributive sublattice, $C(\sigma)$, of the complete Boolean lattice of all subsets of $\sigma$.

Assume that the chain $\mathcal{M}$ is strictly increasing, and $S, S' \in C(\sigma)$. Then $S \subseteq S'$, if and only if $M(S) \subseteq M(S')$. The only-if-part is trivial; to prove the if-part, assume that $M(S) \subseteq M(S')$ and there is an ordinal $\alpha$ in $S \setminus S'$. Then $A_\alpha \subseteq M(S \cap (\alpha+1)) = M(S) \cap M(\alpha+1) \subseteq M(S') \cap M(\alpha+1) = M(S' \cap (\alpha+1)) = M(S' \cap \alpha) \subseteq M_\alpha$, whence $M_{\alpha+1} = M_\alpha$, a contradiction.

Let $M = M_\sigma$, and let $L(M)$ denote the lattice of all submodules of $M$. We can summarize the above as

**Corollary 2.8.** *Assume that the chain $\mathcal{M}$ is strictly increasing. Then the map $\theta : S \mapsto M(S)$ is a complete lattice isomorphism of the complete distributive lattice $C(\sigma)$ onto a sublattice of the complete modular lattice $L(M)$.*

Even if $\mathcal{M}$ is not strictly increasing, being a homomorphic image of a distributive lattice, the image of $\theta$ is still a distributive sublattice of $L(M)$. This image yields the desired family of submodules $\mathcal{H}$, extending the given continuous chain $\mathcal{M}$:

*Proof of the Hill Lemma.* We start by fixing a family of $< \kappa$-generated modules $(A_\alpha \mid \alpha < \sigma)$, such that for each $\alpha < \sigma$:

$$M_{\alpha+1} = M_\alpha + A_\alpha,$$

as in Definition 2.2. Such family exists because $M_{\alpha+1}/M_\alpha$ is $< \kappa$-generated for each $\alpha < \sigma$. We claim that

$$\mathcal{H} = \{M(S) \mid S \text{ a closed subset of } \sigma\}$$

has properties (H1)-(H4).

Property (H1) is clear, since each ordinal $\alpha \leq \sigma$ is a closed subset of $\sigma$.

The first claim in (H2) follows by Proposition 2.7 and Lemma 2.5, the second by Corollary 2.8, because $\mathcal{H}$ is the image of $\theta$.

In order to prove property (H3), we will show that in the given setting, there exist an ordinal $\tau \leq \sigma$ and a continuous chain $(F_\gamma \mid \gamma \leq \tau)$ of elements of $\mathcal{H}$, such that $\mathcal{Q} = (F_\gamma/N \mid \gamma \leq \tau)$ is a $\mathcal{C}$-filtration of $P/N$, and for each $\gamma < \tau$ there is a $\beta < \sigma$ with $F_{\gamma+1}/F_\gamma$ isomorphic to $M_{\beta+1}/M_\beta$.

First, we have $N = M(S)$ and $P = M(T)$ for some closed subsets $S, T$. Since $S \cup T$ is closed, we can w.l.o.g. assume that $S \subseteq T$. For each $\beta \leq \sigma$, put

$$F_\beta = N + \sum_{\alpha \in T \setminus S, \alpha < \beta} A_\alpha = M(S \cup (T \cap \beta)) \qquad \text{and} \qquad \bar{F}_\beta = F_\beta/N.$$

Clearly $(\bar{F}_\beta \mid \beta \leq \sigma)$ is a filtration of $P/N$, such that $\bar{F}_{\beta+1} = \bar{F}_\beta + (A_\beta + N)/N$ for $\beta \in T \setminus S$ and $\bar{F}_{\beta+1} = \bar{F}_\beta$ otherwise. Let $\beta \in T \setminus S$. Then

$$\bar{F}_{\beta+1}/\bar{F}_\beta \cong F_{\beta+1}/F_\beta \cong A_\beta/(F_\beta \cap A_\beta),$$

and

$$F_\beta \cap A_\beta \supseteq \Big( \sum_{\alpha \in T, \alpha < \beta} A_\alpha \Big) \cap A_\beta = M_\beta \cap A_\beta,$$

where the latter equality holds because $\beta \in T$ and $T$ is closed in $\sigma$.

However, if $x \in F_\beta \cap A_\beta$, then $\mathrm{hgt}(x) \leq \beta$, so $x \in M(T')$ by Lemma 2.3, where $T' = \{\alpha \in S \cup (T \cap \beta) \mid \alpha \leq \beta\}$. By Proposition 2.7, we get $x \in M_\beta$, because $\beta \notin S$. Hence $F_\beta \cap A_\beta = M_\beta \cap A_\beta$ and $\bar{F}_{\beta+1}/\bar{F}_\beta \cong A_\beta/(M_\beta \cap A_\beta) \cong M_{\beta+1}/M_\beta$. The desired $\mathcal{C}$-filtration $\mathcal{Q}$ of $P/N$ is obtained from $(\bar{F}_\beta \mid \beta \leq \sigma)$ by removing possible repetitions, and (H3) follows. Denote by $\tau'$ the ordinal type of the well-ordered set $(T \setminus S, <)$. Notice that the length $\tau$ of the filtration can be taken as $1 + \tau'$ (the ordinal sum, hence $\tau = \tau'$ for $\tau'$ infinite).

For property (H4) we first prove that every subset $X$ of $\sigma$ of cardinality $< \kappa$ is contained in a closed subset of $\sigma$ of cardinality $< \kappa$. We will prove this by induction on the least $\beta \leq \sigma$ such that $X \subseteq \beta$. Since $\kappa$ is an infinite regular cardinal, in the induction step, we can even assume that $X$ is a one-element subset of $\beta$ and apply Proposition 2.7. So we are left to prove that every $\beta < \sigma$ is contained in a closed subset of cardinality $< \kappa$. For $\beta < \kappa$, we just take $S = \beta + 1$. Otherwise, the short exact sequence

$$0 \to M_\beta \cap A_\beta \to A_\beta \to M_{\beta+1}/M_\beta \to 0$$

shows that $M_\beta \cap A_\beta$ is $< \kappa$-generated. Thus $M_\beta \cap A_\beta \subseteq \sum_{\alpha \in S_0} A_\alpha$ for a subset $S_0 \subseteq \beta$ of cardinality $< \kappa$. Moreover, we can assume that $S_0$ is closed in $\sigma$ by inductive premise, and put $S = S_0 \cup \{\beta\}$. To show that $S$ is closed, it suffices to check the definition only for $\beta$. But $M_\beta \cap A_\beta \subseteq M(S_0) = \sum_{\alpha \in S, \alpha < \beta} A_\alpha$.

Finally, let $N = M(S)$, where $S$ is closed in $\sigma$, and let $X$ be a subset of $M$ of cardinality $< \kappa$. Then $X \subseteq \sum_{\alpha \in T} A_\alpha$ for a subset $T$ of $\sigma$ of cardinality $< \kappa$. By the preceding paragraph, we can assume that $T$ is closed in $\sigma$. Let $P = M(S \cup T)$. Then $P/N$ is $\mathcal{C}$-filtered by property (H3), and the filtration can be chosen indexed by 1+ the ordinal type of $T \setminus S$, which is certainly less than $\kappa$. In particular, $P/N$ is $< \kappa$-presented. $\square$

**Remark 2.9.** If we assume the stronger assumption that each module in $\mathcal{C}$ possesses a projective resolution consisting of $< \kappa$-generated modules, then the same is true of the module $P/N$ in (H4) (see e.g. [27, Corollary 7.2]).

## 3. First consequences of the Hill Lemma

Here is our first application of the Hill Lemma. It makes it possible to replace the original filtration of a module by a filtration respecting the choice of a generating set of $M$:

**Corollary 3.1.** *Let $R$ be a ring, $\kappa$ an infinite regular cardinal, and $\mathcal{C}$ a set of $< \kappa$-presented modules. Let $\mathcal{D}$ be the class of all modules possessing a $\mathcal{C}$-filtration of length $< \kappa$ (so $\operatorname{Filt}(\mathcal{D}) = \operatorname{Filt}(\mathcal{C})$).*

*Let $M$ be a $\mathcal{D}$-filtered module with $\operatorname{gen}(M) = \lambda \geq \kappa$. Let $\{m_\beta \mid \beta < \lambda\}$ be a set of $R$-generators of $M$.*

*Then there exists a $\mathcal{D}$-filtration $\mathcal{M}' = (M'_\beta \mid \beta \leq \lambda)$ of $M$ such that $\operatorname{gen}(M'_\beta) < \lambda$ and $m_\beta \in M'_{\beta+1}$ for each $\beta < \lambda$.*

*Proof.* Since $\operatorname{Filt}(\mathcal{D}) = \operatorname{Filt}(\mathcal{C})$, w.l.o.g., $M$ has a $\mathcal{C}$-filtration $\mathcal{M}$. Consider the corresponding Hill family $\mathcal{H}$ as in Theorem 2.1.

The desired $\mathcal{D}$-filtration of $M$ will be selected from $\mathcal{H}$ by induction as follows: $M'_0 = 0$; if $M'_\beta \in \mathcal{H}$ is defined and $m_\beta \in M'_\beta$, then we put $M'_{\beta+1} = M'_\beta$. If $m_\beta \notin M'_\beta$, we use property (H4) to find a module $M'_{\beta+1} \in \mathcal{H}$ such that $M'_\beta \subseteq M'_{\beta+1}$, $m_\beta \in M'_{\beta+1}$ and $M'_{\beta+1}/M'_\beta$ is $< \kappa$-presented. By property (H3), $M'_{\beta+1}/M'_\beta$ has a $\mathcal{C}$-filtration of length $< \kappa$, so $M'_{\beta+1}/M'_\beta \in \mathcal{D}$.

If $\beta \leq \lambda$ is a limit ordinal, we let $M'_\beta = \bigcup_{\gamma<\beta} M'_\gamma$, which is a module from $\mathcal{F}$ by property (H2). Since $m_\beta \in M'_\lambda$ for each $\beta < \lambda$, $M'_\lambda = M$.

Finally, for each $\beta < \lambda$, $M'_\beta$ has a $\mathcal{D}$-filtration of length $\leq \beta$, so $\operatorname{gen}(M'_\beta) < \lambda$. □

Our second application concerns $\mathcal{C}$-socle sequences of modules.

It is well-known that if $R$ is a right semiartinian ring, then all modules are semiartinian, and each module has a socle sequence. There is obviously no bound on the lengths of transfinite composition series of modules, but the length of the socle sequence of any module $M$ (called the *Loewy length* of $M$) is bounded, the bound being the Loewy length of $R$.

Semiartinian modules coincide with the $\mathcal{C}$-filtered modules for $\mathcal{C} = \operatorname{simp} R$. Thus a question arises of whether these well-known facts can be extended to arbitrary $\mathcal{C}$-filtered modules. We are going to present a positive answer: if each module in $\mathcal{C}$ is $< \kappa$-presented, then each $\mathcal{C}$-filtered module has a $\mathcal{C}$-socle sequence of length $\leq \kappa$. The result, known under the slogan of 'shortening filtrations', goes back to Enochs [24]. The proof here, via the Hill Lemma, is due to Šťovíček [43].

We start with the definition of a $\mathcal{C}$-socle sequence of a module.

**Definition 3.2.** Let $R$ be a ring, $M$ be a module, and $\mathcal{C}$ be a class of modules. A continuous chain $\mathcal{N} = (N_\beta \mid \beta \leq \tau)$ of submodules of $M$ is called a *$\mathcal{C}$-socle sequence* of $M$, provided that $N_\tau = M$, and $N_{\beta+1}/N_\beta$ is isomorphic to a direct sum of elements of $\mathcal{C}$ for each $\beta < \tau$. The ordinal $\tau$ is called the *length* of the $\mathcal{C}$-socle sequence $\mathcal{N}$.

It is easy to see that a module $M$ possesses a $\mathcal{C}$-socle sequence, if and only if $M$ is $\mathcal{C}$-filtered. In fact, each $\mathcal{C}$-filtration of $M$ is also its $\mathcal{C}$-socle sequence. So unlike socle sequences, the $\mathcal{C}$-socle sequences are not unique in general. But the key property is the same: $\mathcal{C}$-socle sequences are shorter than $\mathcal{C}$-filtrations in general; moreover, if $\mathcal{C}$ consists of modules of bounded presentation, then each $\mathcal{C}$-filtered module possesses a $\mathcal{C}$-socle sequence of bounded length. Such $\mathcal{C}$-socle sequence can be extracted from the family $\mathcal{H}$ constructed in the Hill Lemma:

**Theorem 3.3.** *Let $R$ be a ring, $\kappa$ be an infinite regular cardinal, and $\mathcal{C}$ be a class of $< \kappa$-presented modules. Let $M$ be $\mathcal{C}$-filtered module. Then $M$ has a $\mathcal{C}$-socle sequence of length $\leq \kappa$.*

*Proof.* Let $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$ be a $\mathcal{C}$-filtration of $M$, and $(A_\alpha \mid \alpha < \sigma)$ a family of $< \kappa$-generated submodules of $M$, such that $M_{\alpha+1} = M_\alpha + A_\alpha$ for each $\alpha < \sigma$. Let $\mathcal{H} = \{M(S) \mid S \text{ a closed subset of } \sigma\}$ be the family of submodules of $M$ from Theorem 2.1.

Let $\alpha < \sigma$. By (the proof of) property (H4) in Theorem 2.1, the set $A = \{S \mid S \text{ is a closed subset of } \sigma \,\&\, \alpha \in S \,\&\, |S| < \kappa\}$ is not empty. Let $S_\alpha = (\alpha + 1) \cap \bigcap_{S \in A} S$. By Lemma 2.7, $S_\alpha$ is the least element of $A$, and $\alpha$ is the greatest element of $S_\alpha$.

Putting $\sup(\emptyset) = 0$, we define the *$\mathcal{C}$-socle level function* $\ell : \sigma \to \kappa$ by induction on $\alpha < \sigma$ by the formula $\ell(\alpha) = \sup\{\ell(\gamma)+1 \mid \alpha \neq \gamma \in S_\alpha\}$. Notice that $\ell(\gamma) < \ell(\alpha)$ whenever $\alpha \neq \gamma \in S_\alpha$. (Also, $\ell(\alpha) = 0$ is equivalent to $S_\alpha = \{\alpha\}$, which implies $M_{\alpha+1} = M_\alpha \oplus A_\alpha$.)

For each $\beta \leq \kappa$, let $T_\beta = \{\gamma < \sigma \mid \ell(\gamma) < \beta\}$ and $N_\beta = M(T_\beta)$. We will prove that $\mathcal{N} = (N_\beta \mid \beta \leq \kappa)$ is a $\mathcal{C}$-socle sequence of $M$.

First, we claim that $T_\beta$ is closed, hence $N_\beta \in \mathcal{H}$, for each $\beta \leq \kappa$. This will follow once we prove that $T_\beta = \bigcup_{\gamma<\sigma, \ell(\gamma)<\beta} S_\gamma$. However, if $\gamma \in T_\beta$ then $\ell(\gamma) < \beta$, and clearly $\gamma \in S_\gamma$. Conversely, assume that $\alpha \in S_\gamma$ for some $\gamma < \sigma, \ell(\gamma) < \beta$. If $\alpha = \gamma$, then $\alpha \in T_\beta$. Otherwise $\alpha < \gamma$, so $\ell(\alpha) + 1 \leq \ell(\gamma) < \beta$ and $\alpha \in T_\beta$, and the claim is proved.

Clearly $\mathcal{N}$ is a continuous chain of submodules of $M$, and $M = N_\kappa$. It remains to show that for each $\beta < \kappa$, $N_{\beta+1}/N_\beta = \bigoplus_{\gamma \in T_{\beta+1} \setminus T_\beta} \bar{M}_\gamma$, where $\bar{M}_\gamma$ is isomorphic to some element of $\mathcal{C}$ for each $\gamma \in T_{\beta+1} \setminus T_\beta$.

Let $\gamma \in T_{\beta+1} \setminus T_\beta$. Then $\ell(\gamma) = \beta$, and we define $\bar{M}_\gamma = (M(T_\beta) + A_\gamma)/M(T_\beta)$. Then

$$\bar{M}_\gamma = M(T_\beta \cup S_\gamma)/M(T_\beta) \cong M(S_\gamma)/(M(S_\gamma) \cap M(T_\beta)) = M(S_\gamma)/M(S_\gamma \cap T_\beta) =$$

$$= M(S_\gamma)/M(S_\gamma \cap \gamma) \cong A_\gamma/(A_\gamma \cap M(S_\gamma \cap \gamma)) = A_\gamma/(A_\gamma \cap M_\gamma) \cong M_{\gamma+1}/M_\gamma,$$

because $S_\gamma$ is closed, and $\gamma$ is the largest element of $S_\gamma$. However, $M_{\gamma+1}/M_\gamma$ is isomorphic to an element of $\mathcal{C}$ as $\mathcal{M}$ is a $\mathcal{C}$-filtration of $M$.

Clearly, $N_{\beta+1}/N_\beta = \sum_{\gamma \in T_{\beta+1} \setminus T_\beta} \bar{M}_\gamma$.

So it remains to prove that $\bar{M}_\gamma \cap \sum_{\gamma \neq \delta \in T_{\beta+1} \setminus T_\beta} \bar{M}_\delta = 0$, or equivalently,

$$(M(T_\beta) + A_\gamma) \cap (M(T_\beta) + \sum_{\gamma \neq \delta \in T_{\beta+1} \setminus T_\beta} A_\delta) = M(T_\beta).$$

We have $M(T_\beta) + A_\gamma = M(T_\beta \cup S_\gamma)$, and $M(T_\beta) + \sum_{\gamma \neq \delta \in T_{\beta+1} \setminus T_\beta} A_\delta = M(T_\beta \cup \bigcup_{\gamma \neq \delta \in T_{\beta+1}} S_\delta)$. Moreover,

$$M(T_\beta \cup S_\gamma) \cap M(T_\beta \cup \bigcup_{\gamma \neq \delta \in T_{\beta+1}} S_\delta) = M(T_\beta \cup (S_\gamma \cap \bigcup_{\gamma \neq \delta \in T_{\beta+1}} S_\delta)) = M(T_\beta),$$

because by the definition of the function $\ell$, $S_\gamma \cap S_\delta \subseteq T_\beta$ for all $\gamma \neq \delta \in T_{\beta+1}$. $\square$

Later on, we will present further applications of the Hill Lemma. But first we have to introduce the key concept of these notes, that of an approximation of a module.

## 4. Approximations

Throughout this section we assume that $R$ is a ring, $M$ is a (right $R$-) module and $\mathcal{C}$ a class of modules closed under isomorphic images and direct summands.

**Definition 4.1.** A map $f \in \mathrm{Hom}_R(M, C)$ with $C \in \mathcal{C}$ is a *$\mathcal{C}$-preenvelope* of $M$, provided that the map $\mathrm{Hom}_R(f, C') : \mathrm{Hom}_R(C, C') \to \mathrm{Hom}_R(M, C')$ is surjective for each $C' \in \mathcal{C}$. That is, for each homomorphism $f' : M \to C'$ there is a homomorphism $g : C \to C'$, such that $f' = gf$:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "f'"'] & C \arrow[d, dashed, "g"] \\ & C'. \end{tikzcd}$$

(Note that we require the existence, but not the uniqueness, of the map $g$.)
The $\mathcal{C}$-preenvelope $f$ is a *$\mathcal{C}$-envelope* of $M$. provided that $f$ is *left minimal*, that is, provided $f = gf$ implies $g$ is an automorphism for each $g \in \mathrm{End}_R(C)$.

**Example 4.2.** The embedding $M \hookrightarrow E(M)$ is easily seen to be the $\mathcal{I}_0$-envelope of a module $M$. Here, $\mathcal{I}_0$ denotes the class of all injective modules, and $E(M)$ the injective hull of $M$.

Clearly a $\mathcal{C}$-envelope of $M$ is unique in the following sense: if $f : M \to C$ and $f' : M \to C'$ are $\mathcal{C}$-envelopes of $M$, then there is an isomorphism $g : C \to C'$, such that $f' = gf$.

In general a module $M$ may have many non-isomorphic $\mathcal{C}$-preenvelopes, but no $\mathcal{C}$-envelope. Nevertheless, if the $\mathcal{C}$-envelope exists, its minimality implies that it is isomorphic to a direct summand in each $\mathcal{C}$-preenvelope of $M$:

**Lemma 4.3.** *Let $f : M \to C$ be a $\mathcal{C}$-envelope and $f' : M \to C'$ a $\mathcal{C}$-preenvelope of a module $M$. Then*

(a) *there exists a decomposition $C' = D \oplus D'$, where $\mathrm{Im}\, f' \subseteq D$ and $f' : M \to D$ is a $\mathcal{C}$-envelope of $M$;*
(b) *$f'$ is a $\mathcal{C}$-envelope of $M$, if and only if $C'$ has no proper direct summands containing $\mathrm{Im}\, f'$.*

*Proof.* (a) By definition there are homomorphisms $g : C \to C'$ and $g' : C' \to C$, such that $f' = gf$ and $g'g$ is an automorphism of $C$. So $g$ is a split monomorphism, $D = \mathrm{Im}\, g \cong C$ is a direct summand in $C'$ containing $\mathrm{Im}\, f'$. It follows that $f' : M \to D$ is a $\mathcal{C}$-envelope of $M$.
(b) by part (a). □

**Definition 4.4.** A class $\mathcal{C} \subseteq \text{Mod–}R$ is a *preenveloping class*, (*enveloping class*) provided that each module has a $\mathcal{C}$-preenvelope ($\mathcal{C}$-envelope).

For example, the class $\mathcal{I}_0$ of all injective modules from Example 4.2 is an enveloping class of modules.

Now we briefly discuss the dual concepts:

**Definition 4.5.** A map $f \in \mathrm{Hom}_R(C, M)$ with $C \in \mathcal{C}$ is a *$\mathcal{C}$-precover* of $M$, provided that the abelian group homomorphism $\mathrm{Hom}_R(C', f) : \mathrm{Hom}_R(C', C) \to \mathrm{Hom}_R(C', M)$ is surjective for each $C' \in \mathcal{C}$.
A $\mathcal{C}$-precover $f \in \mathrm{Hom}_R(C, M)$ of $M$ is called a *$\mathcal{C}$-cover* of $M$, provided that $f$ is *right minimal*, that is, provided $fg = f$ implies that $g$ is an automorphism for each $g \in \mathrm{End}_R(C)$.

$\mathcal{C} \subseteq \text{Mod–}R$ is a *precovering class*, (*covering class*) provided that each module has a $\mathcal{C}$-precover ($\mathcal{C}$-cover).

**Remark 4.6.** A $\mathcal{C}$-preenvelope $f : M \to C$ is also called a *left $\mathcal{C}$-approximation* of $M$, because all morphisms from $M$ to an element $C'$ of $\mathcal{C}$ are left multiples of $f$ by morphisms from $C$ to $C'$. A $\mathcal{C}$-envelope is then called a *minimal left $\mathcal{C}$-approximation.*

Similarly, $\mathcal{C}$-precovers and $\mathcal{C}$-precovers are sometimes referred to as *right $\mathcal{C}$-approximations* and *minimal right $\mathcal{C}$-approximations*, respectively.

If Mod–$R$ is replaced by its subcategory mod–$R$ in the definitions above, then preenveloping and precovering classes are called *covariantly finite* and *contravariantly finite*.

**Example 4.7.** Each module $M$ has a $\mathcal{P}_0$-precover (where $\mathcal{P}_0$ denotes the class of all projective modules), since each module is a homomorphic image of a projective module. Moreover, $M$ has a $\mathcal{P}_0$-cover, if and only if $M$ has a projective cover in the sense of Bass (that is, there is an epimorphism $f : P \to M$ with $P$ projective and $\mathrm{Ker}(f)$ a small submodule of $P$). So $\mathcal{P}_0$ is always a precovering class, and it is a covering class, if and only if $R$ is a right perfect ring.

$\mathcal{C}$-covers may not exist in general, but if they exist, they are unique up to isomorphism. As in Lemma 4.3, we get

**Lemma 4.8.** *Let $f : C \to M$ be the $\mathcal{C}$-cover of $M$. Let $f' : C' \to M$ be any $\mathcal{C}$-precover of $M$. Then*

(a) *there is a decomposition $C' = D \oplus D'$, where $D \subseteq \mathrm{Ker}\, f'$ and $f' \restriction D'$ is a $\mathcal{C}$-cover of $M$.*
(b) *$f'$ is a $\mathcal{C}$-cover of $M$, if and only if $C'$ has no non-zero direct summands contained in* $\mathrm{Ker}\, f'$.

*Proof.* Dual to the proof of Lemma 4.3. □

The following lemma is known as the Wakamatsu Lemma (see [46]). It shows that under rather weak assumptions on the class $\mathcal{C}$, $\mathcal{C}$-envelopes and $\mathcal{C}$-covers are special in the sense of the following definition:

**Definition 4.9.** Let $\mathcal{C} \subseteq \text{Mod–}R$. We define

$$\mathcal{C}^{\perp} = \mathrm{Ker}\,\mathrm{Ext}^1_R(\mathcal{C}, -) = \big\{N \in \text{Mod–}R \mid \mathrm{Ext}^1_R(C, N) = 0 \text{ for all } C \in \mathcal{C}\big\}.$$

Similarly,

$$\mathcal{C}^{\perp_\infty} = \bigcap_{i \geq 1} \mathrm{Ker}\,\mathrm{Ext}^i_R(\mathcal{C}, -) = \big\{N \in \text{Mod–}R \mid \mathrm{Ext}^i_R(C, N) = 0\ \forall\, C \in \mathcal{C}\ \forall\, i \geq 1\big\}.$$

For $\mathcal{C} = \{C\}$, we write for short $C^{\perp}$, $C^{\perp_\infty}$, ${}^{\perp}C$, and ${}^{\perp_\infty}C$ in place of $\{C\}^{\perp}$, $\{C\}^{\perp_\infty}$, ${}^{\perp}\{C\}$, and ${}^{\perp_\infty}\{C\}$, respectively.

Let $M \in \text{Mod–}R$. A $\mathcal{C}$-preenvelope $f : M \to C$ of $M$ is called *special*, provided that $f$ is injective and $\mathrm{Coker}\, f \in {}^{\perp}\mathcal{C}$. In other words, a special $\mathcal{C}$-preenvelope $f$ of $M$ is a morphism that fits into a short exact sequence

$$0 \to M \xrightarrow{f} C \to D \to 0$$

with $C \in \mathcal{C}$ and $D \in {}^{\perp}\mathcal{C}$. Indeed, such $f$ is always a $\mathcal{C}$-preenvelope, since for each $C' \in \mathcal{C}$, $\mathrm{Ext}^1_R(D, C') = 0$ implies that the abelian group homomorphism $\mathrm{Hom}_R(f, C') : \mathrm{Hom}_R(C, C') \to \mathrm{Hom}_R(M, C')$ is surjective.

Dually, a $\mathcal{C}$-precover $g : C \to M$ of $M$ is called *special*, if $g$ is surjective and $\operatorname{Ker} g \in \mathcal{C}^{\perp}$. Again, a special $\mathcal{C}$-precover $g$ of $M$ is just a map that fits into a short exact sequence

$$0 \to B \to C \xrightarrow{g} M \to 0$$

where $C \in \mathcal{C}$ and $B \in \mathcal{C}^{\perp}$.

If $\mathcal{C}$ is a class of modules such that each module $M$ has a special $\mathcal{C}$-preenvelope (special $\mathcal{C}$-precover), then $\mathcal{C}$ is called *special preenveloping* (*special precovering*).

**Lemma 4.10.** (Wakamatsu Lemma) *Let $M \in \text{Mod–}R$ and $\mathcal{C} \subseteq \text{Mod–}R$ be a class closed under extensions.*

(a) *Let $f : M \to C$ be a $\mathcal{C}$-envelope of $M$ and $D = \operatorname{Coker} f$. Then $D \in {}^{\perp}\mathcal{C}$.*
*In particular, each monic $\mathcal{C}$-envelope of $M$ is special.*

(b) *Let $g : C \to M$ be a $\mathcal{C}$-cover of $M$ and $E = \operatorname{Ker} g$. Then $E \in \mathcal{C}^{\perp}$.*
*In particular, each surjective $\mathcal{C}$-cover of $M$ is special.*

*Proof.* (a) Let $K = \operatorname{Ker} f$ and $\pi : M \to M/K$ be the canonical projection. Then $f = \bar{f}\pi$ for a unique monomorphism $\bar{f} \in \operatorname{Hom}_R(M/K, C)$, and there is an exact sequence

$$0 \to M/K \xrightarrow{\bar{f}} C \xrightarrow{g} D \to 0.$$

In order to prove that $D \in {}^{\perp}\mathcal{C}$, we take an arbitrary extension

$$0 \to C' \to X \xrightarrow{h} D \to 0$$

with $C' \in \mathcal{C}$. We will prove that $h$ splits. First consider the pullback of $g$ and $h$:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & C' \arrow[r, equal] \arrow[d] & C' \arrow[d] & \\
0 \arrow[r] & M/K \arrow[r, "\alpha"] \arrow[d, equal] & P \arrow[r, "\beta"] \arrow[d, "\gamma"'] & X \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & M/K \arrow[r, "\bar{f}"] & C \arrow[r, "g"] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Since $C, C' \in \mathcal{C}$, also $P \in \mathcal{C}$ by assumption. Since $f$ is a $\mathcal{C}$-preenvelope of $M$, there exists a homomorphism $\delta : C \to P$ such that $\alpha\pi = \delta f$. Then $f = \gamma\alpha\pi = \gamma\delta f$, so $\gamma\delta$ is an automorphism of $C$ because $f$ is left minimal.

Define $i \in \operatorname{Hom}_R(D, X)$ by $i(g(c)) = \beta\delta(\gamma\delta)^{-1}(c)$ for each $c \in C$. This is possible, since $\operatorname{Ker} g = \operatorname{Im} \bar{f}$ and $\beta\delta(\gamma\delta)^{-1} \restriction \operatorname{Im} \bar{f} = 0$, because for each $m \in M$,

$$\delta(\gamma\delta)^{-1}\bar{f}\pi(m) = \delta(\gamma\delta)^{-1}f(m) = \delta(\gamma\delta)^{-1}\gamma\delta f(m) = \alpha\pi(m) \in \operatorname{Ker}\beta.$$

Moreover, $hig = h\beta\delta(\gamma\delta)^{-1} = g\gamma\delta(\gamma\delta)^{-1} = g$, so $hi = \mathrm{id}_D$, and $h$ splits.

(b) is dual to (a). $\square$

**Remark 4.11.** The $\mathcal{C}$-envelope $f$ of a module $M$ must be monic, provided that $\mathcal{I}_0 \subseteq \mathcal{C}$. This is because $M \hookrightarrow E(M)$ factorizes through $f$. Similarly, $\mathcal{P}_0 \subseteq \mathcal{C}$ implies that any $\mathcal{C}$-cover of $M$ is surjective.

Also notice that the Wakamatsu Lemma holds when Mod–$R$ is replaced by the category of all finitely presented modules, mod–$R$. Indeed, then all the modules in the proof of Lemma 4.10, with the possible exception of $K$, are finitely presented.

We finish this section by listing basic closure properties of covering and enveloping classes:

**Lemma 4.12.** *Let $R$ be a ring and $\mathcal{C}$ be a class of modules closed under isomorphisms. Let $C \in \mathcal{C}$ and $D \oplus E = C$.*

(a) *Assume that $D$ has a $\mathcal{C}$-cover. Then $D \in \mathcal{C}$.*
*So if $\mathcal{C}$ is covering, then $\mathcal{C}$ is closed under direct summands.*

(b) *Assume that $D$ has a $\mathcal{C}$-envelope. Then $D \in \mathcal{C}$.*
*So if $\mathcal{C}$ is enveloping, then $\mathcal{C}$ is closed under direct summands.*

*Proof.* We will only prove the claim of (a), the proof of (b) is dual.

Assume there exists a $\mathcal{C}$-cover of $D$, $f' \in \operatorname{Hom}_R(C', D)$. Let $\nu \in \operatorname{Hom}_R(D, C)$ be the split inclusion of $D$ into $C$ and $\pi \in \operatorname{Hom}_R(C, D)$ the split projection onto $D$ corresponding to the decomposition $C = D \oplus E$. Then $\pi\nu = id_D$.

By the precovering property of $f'$, there exists $g \in \operatorname{Hom}_R(C, C')$ such that $\pi = f'g$. In particular, $f'$ is surjective.

Let $h = \nu f'$. Then $f'gh = \pi\nu f' = f'$. Since $f'$ is a $\mathcal{C}$-cover, $gh$ is an automorphism of $C'$. In particular, $h$ is a monomorphism, and so is $f'$.

Thus $f'$ is an isomorphism, and $D \in \mathcal{C}$. □

**Lemma 4.13.** *Let $R$ be a ring and $\mathcal{C}$ be a class of modules closed under isomorphisms and direct summands.*

(a) *Assume that $\mathcal{C}$ is precovering. Then $\mathcal{C}$ is closed under direct sums.*
(b) *Assume that $\mathcal{C}$ is preenveloping. Then $\mathcal{C}$ is closed under direct products.*

*Proof.* We will only prove the claim of (b), the proof of (a) is dual.

Assume $\mathcal{C}$ is preenveloping and let $(E_i \mid i \in I)$ be a family of modules in $\mathcal{C}$. Let $f : P \to C$ be a $\mathcal{C}$-preenvelope of the module $P = \prod_{i \in I} E_i$. Denote by $\pi_i : P \to E_i$ the canonical projection ($i \in I$).

Then there exist homomorphisms $g_i : C \to E_i$ such that $g_i f = \pi_i$ for each $i \in I$. Define a homomorphism $g : C \to P$ by $\pi_i g(c) = g_i(c)$ for all $c \in C$ and $i \in I$. Then $gf(x) = (g_i(f(x)) \mid i \in I) = x$ for all $x \in P$. Thus $P$ is isomorphic to a direct summand in $C$, and $P \in \mathcal{C}$ by our assumption on the class $\mathcal{C}$. □

**Corollary 4.14.** *Let $R$ be a ring and $\mathcal{C}$ be a class of modules closed under isomorphisms.*

(a) *Assume that $\mathcal{C}$ is covering. Then $\mathcal{C}$ is closed under direct summands and direct sums.*
(b) *Assume that $\mathcal{C}$ is enveloping. Then $\mathcal{C}$ is closed under direct summands and direct products.*

## 5. Cotorsion pairs

Besides the Wakamatsu Lemma, there is another reason for investigating special preenvelopes and precovers, namely the existence of an explicit duality between them arising from the notion of a cotorsion pair:

**Definition 5.1.** Let $\mathcal{A}, \mathcal{B} \subseteq$ Mod–$R$. The pair $(\mathcal{A}, \mathcal{B})$ is called a *cotorsion pair* if $\mathcal{A} = {}^\perp\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^\perp$.

Let $\mathcal{C}$ be a class of modules. Then $\mathcal{C} \subseteq {}^{\perp}(\mathcal{C}^{\perp})$ as well as $\mathcal{C} \subseteq ({}^{\perp}\mathcal{C})^{\perp}$. Moreover, $\mathfrak{G}_{\mathcal{C}} = ({}^{\perp}(\mathcal{C}^{\perp}), \mathcal{C}^{\perp})$ and $\mathfrak{C}_{\mathcal{C}} = ({}^{\perp}\mathcal{C}, ({}^{\perp}\mathcal{C})^{\perp})$ are easily seen to be cotorsion pairs, called the cotorsion pairs *generated* and *cogenerated*, respectively, by the class $\mathcal{C}$.

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *hereditary* in case $\mathrm{Ext}^i_R(A, B) = 0$ for all $i \geq 1$, $A \in \mathcal{A}$ and $B \in \mathcal{B}$ (see Lemma 5.7 below for a characterization of heredity in terms of the closure properties of the classes $\mathcal{A}$ and $\mathcal{B}$).

The class $\mathcal{A} \cap \mathcal{B}$ is called the *kernel* of the cotorsion pair $(\mathcal{A}, \mathcal{B})$.

For any ring $R$, the cotorsion pairs of right $R$-modules are partially ordered by inclusion of their first components. In fact, they form a complete lattice $L_{\mathrm{Ext}}$ (whose support, however, is not a set but a proper class, e.g. for any non-right perfect ring).

The largest element of $L_{\mathrm{Ext}}$ is $\mathfrak{G}_{\mathrm{Mod}-R} = (\mathrm{Mod}\text{–}R, \mathcal{I}_0)$, while the least is $\mathfrak{C}_{\mathrm{Mod}-R} = (\mathcal{P}_0, \mathrm{Mod}\text{–}R)$. These are the *trivial cotorsion pairs.*

Note that $(\bigcap_{\alpha<\kappa} \mathcal{A}_\alpha, (\bigcap_{\alpha<\kappa} \mathcal{A}_\alpha)^{\perp})$ is the infimum of a sequence of cotorsion pairs $\{(\mathcal{A}_\alpha, \mathcal{B}_\alpha) \mid \alpha < \kappa\}$ in $L_{\mathrm{Ext}}$, while $({}^{\perp}((\bigcup_{\alpha<\kappa} \mathcal{A}_\alpha)^{\perp}), \bigcap_{\alpha<\kappa} \mathcal{B}_\alpha)$ is its supremum.

Cotorsion pairs are analogues of the classical (non-hereditary) torsion pairs, where Hom ($= \mathrm{Ext}^0$ ) is replaced by $\mathrm{Ext}^1$. Similarly, one can define $F$-pairs for any additive bifunctor $F$ on Mod–$R$.

Now we present several important examples of cotorsion pairs:

**Example 5.2.** For any ring $R$ and any $n \geq 0$, there are cotorsion pairs $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$, $(\mathcal{F}_n, \mathcal{F}_n^{\perp})$, and $({}^{\perp}\mathcal{I}_n, \mathcal{I}_n)$ where $\mathcal{P}_n$, $\mathcal{F}_n$, and $\mathcal{I}_n$ denotes the class of all modules of projective (flat, injective) dimension $\leq n$, respectively.

If $R$ is an integral domain, then there is a cotorsion pair $(\mathcal{TF}, \mathcal{TF}^{\perp})$ where $\mathcal{TF}$ is the class of all torsion-free modules.

We now record an immediate corollary of Lemma 4.10:

**Corollary 5.3.** *Let $R$ be a ring and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. If $\mathcal{A}$ is covering, then $\mathcal{A}$ is special precovering, and if $\mathcal{B}$ is enveloping, then $\mathcal{B}$ is special preenveloping.*

The key property of cotorsion pairs is their relation to module approximations. This fact – discovered by Salce [38] – says that the mutually dual categorical notions of a special precover and a special preenvelope are tied up by the homological tie of a cotorsion pair. In a sense, this fact is a remedy for the non-existence of a duality involving the category of all modules over a ring.

**Lemma 5.4.** (Salce Lemma) *Let $R$ be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair of modules. Then the following are equivalent:*

(a) *Each module has a special $\mathcal{A}$-precover.*
(b) *Each module has a special $\mathcal{B}$-preenvelope.*

*In this case, the cotorsion pair $\mathfrak{C}$ is called* complete*.*

*Proof.* (a) implies (b): let $M \in \mathrm{Mod}\text{–}R$. There is an exact sequence

$$0 \to M \to I \xrightarrow{\pi} F \to 0,$$

where $I$ is injective. By assumption, there is a special $\mathcal{A}$-precover $\rho$ of $F$

$$0 \to B \to A \xrightarrow{\rho} F \to 0.$$

Consider the pullback of $\pi$ and $\rho$:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & M \arrow[r, equal] \arrow[d] & M \arrow[d] & \\
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d, "\gamma"'] & I \arrow[r] \arrow[d, "\pi"'] & 0 \\
0 \arrow[r] & B \arrow[r] & A \arrow[r, "\rho"] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Since $B, I \in \mathcal{B}$, also $P \in \mathcal{B}$. So the left-hand vertical exact sequence is a special $\mathcal{B}$-preenvelope of $M$.

(b) implies (a): by a dual argument. $\square$

We finish this section by considering the particular case of hereditary cotorsion pairs.

**Definition 5.5.** Let $R$ be a ring, $\mathcal{C}$ be a class of modules and $M \in \text{Mod}–R$.

(i) $M$ is called *$\mathcal{C}$-resolved*, if there is a *$\mathcal{C}$-resolution of $M$*, that is, a long exact sequence $\dots \to C_n \to \dots \to C_0 \to M \to 0$, such that $C_n \in \mathcal{C}$ for all $n < \omega$.

Assume $M$ is $\mathcal{C}$-resolved. If $M$ has a $\mathcal{C}$-resolution, such that $C_i = 0$ for all $i \geq n + 1$, then the least such $n$ (among all such $\mathcal{C}$-resolutions) is called the *$\mathcal{C}$-resolution dimension* of $M$. Otherwise $M$ is said to have $\mathcal{C}$-resolution dimension $\infty$.

(ii) Dually, $M$ is called *$\mathcal{C}$-coresolved*, if there is a *$\mathcal{C}$-coresolution of $M$*, that is, a long exact sequence $0 \to M \to C_0 \to \dots \to C_n \to \dots$, such that $C_n \in \mathcal{C}$ for all $n \leq \omega$.

Assume $M$ is $\mathcal{C}$-coresolved. If $M$ has a $\mathcal{C}$-coresolution, such that $C_i = 0$ for all $i \geq n + 1$, then the least such $n$ (among all such $\mathcal{C}$-coresolutions) is called the *$\mathcal{C}$-coresolution dimension* of $M$. Otherwise $M$ is said to have $\mathcal{C}$-coresolution dimension $\infty$.

Clearly, any module $M$ is $\mathcal{P}_0$-resolved, and the $\mathcal{P}_0$-resolution dimension of $M$ is exactly its projective dimension. Similarly, any module is $\mathcal{I}_0$-coresolved, the $\mathcal{I}_0$-coresolution dimension being the injective dimension.

**Definition 5.6.** Let $R$ be a ring and $\mathcal{C}$ be a class of modules.

(i) $\mathcal{C}$ is *resolving*, provided that $\mathcal{C}$ is closed under extensions, $\mathcal{P}_0 \subseteq \mathcal{C}$ and $A \in \mathcal{C}$, whenever $0 \to A \to B \to C \to 0$ is a short exact sequence, such that $B, C \in \mathcal{C}$.

(ii) $\mathcal{C}$ is *coresolving*, provided that $\mathcal{C}$ is closed under extensions, $\mathcal{I}_0 \subseteq \mathcal{C}$ and $C \in \mathcal{C}$, whenever $0 \to A \to B \to C \to 0$ is a short exact sequence, such that $A, B \in \mathcal{C}$.

Hereditary cotorsion pairs connect resolving and coresolving classes of modules:

**Lemma 5.7.** *Let $R$ be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then the following assertions are equivalent:*

(a) *$\mathcal{A}$ is resolving;*

(b) *$\mathcal{B}$ is coresolving;*
(c) *The cotorsion pair $\mathfrak{C}$ is hereditary.*

*Proof.* (a) implies (c) and (b): let $0 \to C \to P \to A \to 0$ be an exact sequence with $A \in \mathcal{A}$ and $P \in \mathcal{P}_0$. By the premise, $C \in \mathcal{A}$. Let $B \in \mathcal{B}$. Applying $\mathrm{Hom}_R(-, B)$, we get the exact sequence $0 = \mathrm{Ext}^1_R(C, B) \to \mathrm{Ext}^2_R(A, B) \to \mathrm{Ext}^2_R(P, B) = 0$. By induction we get (c).

In order to prove (b), we take an exact sequence $0 \to E \to F \to G \to 0$ with $E, F \in \mathcal{B}$. Consider $A \in \mathcal{A}$. Applying $\mathrm{Hom}_R(A, -)$, we get the exact sequence $0 = \mathrm{Ext}^1_R(A, F) \to \mathrm{Ext}^1_R(A, G) \to \mathrm{Ext}^2_R(A, E) = 0$. This proves that $G \in \mathcal{A}^\perp = \mathcal{B}$.

(b) implies (c) by a dual argument.

(c) implies (a): let $0 \to E \to D \to C \to 0$ be an exact sequence of modules, such that $C, D \in \mathcal{A}$. Take $B \in \mathcal{B}$ and apply $\mathrm{Hom}_R(-, B)$. Then the sequence $0 = \mathrm{Ext}^1_R(D, B) \to \mathrm{Ext}^1_R(E, B) \to \mathrm{Ext}^2_R(C, B) = 0$ is exact, whence $E \in \mathcal{A}$. □

Note that any resolving class is syzygy closed; any coresolving class is cosyzygy closed. Also, if $\mathcal{C}$ is any class of modules, then the class ${}^{\perp_\infty}\mathcal{C}$ is resolving, and $\mathcal{C}^{\perp_\infty}$ coresolving.

Assume that $\mathcal{C}$ is syzygy closed. Then $\mathcal{C}^\perp = \mathcal{C}^{\perp_\infty}$ is coresolving and ${}^\perp(\mathcal{C}^\perp) = {}^{\perp_\infty}(\mathcal{C}^{\perp_\infty})$ is resolving, and the cotorsion pair generated by $\mathcal{C}$ is hereditary. Dually, if $\mathcal{C}$ is cosyzygy closed, then ${}^\perp\mathcal{C} = {}^{\perp_\infty}\mathcal{C}$ is resolving and $({}^\perp\mathcal{C})^\perp = ({}^{\perp_\infty}\mathcal{C})^{\perp_\infty}$ is coresolving, and The cotorsion pair cogenerated by $\mathcal{C}$ is hereditary.

**Remark 5.8.** Complete cotorsion pairs $(\mathcal{A}, \mathcal{B})$ make it possible to develop relative homological algebra, with projective resolutions replaced by resolutions by modules from the class $\mathcal{A}$, and injective coresolutions by coresolutions by modules from $\mathcal{B}$. For more on relative homological algebra, we refer to [25, Chapter 8].

Further on, complete cotorsion pairs in the category Mod–$R$ are the main sources of Quillen model category structures on chain complexes of modules that make it possible to express the derived category $D(R)$ as the homotopy category associated to the particular model structure. In fact, this approach extends much further, e.g., to Grothendieck categories (such as categories of quasi-coherent sheaves on schemes), and even to particular exact categories. We refer to [29] and to the recent monograph [26] for more details.

## 6. The abundance of complete cotorsion pairs

The following theorem, showing that complete cotorsion pairs are abundant, was originally proved in [21]. Similar arguments have been used in homotopy theory since Quillen's fundamental work [37] under the name of *small object argument*. The proof presented here is a more categorical modification of the original one, coming from [1]:

**Theorem 6.1.** (Completeness of cotorsion pairs generated by sets) *Let $\mathcal{S}$ be a set of modules.*

(a) *Let $M$ be a module. Then there is a short exact sequence*

$$0 \to M \hookrightarrow P \to N \to 0,$$

*where $P \in \mathcal{S}^\perp$ and $N$ is $\mathcal{S}$-filtered.*

*In particular, $M \hookrightarrow P$ is a special $\mathcal{S}^\perp$-preenvelope of $M$.*

(b) *The cotorsion pair $({}^\perp(\mathcal{S}^\perp), \mathcal{S}^\perp)$ is complete.*

*Proof.* (a) Put $X = \bigoplus_{S \in \mathcal{S}} S$. Then $X^\perp = \mathcal{S}^\perp$. So w.l.o.g., we assume that $\mathcal{S}$ consists of a single module $S$.

Let $0 \to K \xrightarrow{\mu} F \to S \to 0$ be a short exact sequence with $F$ a free module. Let $\lambda$ be an infinite regular cardinal, such that $K$ is $< \lambda$-generated.

By induction we define an increasing chain $(P_\alpha \mid \alpha < \lambda)$ as follows:
First let $P_0 = M$. For $\alpha < \lambda$, choose the index set $I_\alpha = \mathrm{Hom}_R(K, P_\alpha)$. We define $\mu_\alpha$ as the direct sum of $|I_\alpha|$ copies of the homomorphism $\mu$, i.e.

$$\mu_\alpha := \mu^{(I_\alpha)} \in \mathrm{Hom}_R(K^{(I_\alpha)}, F^{(I_\alpha)}).$$

Then $\mu_\alpha$ is a monomorphism, and $\mathrm{Coker}\, \mu_\alpha$ is isomorphic to a direct sum of copies of $S$. Let $\varphi_\alpha \in \mathrm{Hom}_R(K^{(I_\alpha)}, P_\alpha)$ be the canonical morphism. Note that for each $\eta \in I_\alpha$ there are canonical embeddings $\nu_\eta \in \mathrm{Hom}_R(K, K^{(I_\alpha)})$ and $\nu'_\eta \in \mathrm{Hom}_R(F, F^{(I_\alpha)})$, such that $\eta = \varphi_\alpha \nu_\eta$ and $\nu'_\eta \mu = \mu_\alpha \nu_\eta$.

Now $P_{\alpha+1}$ is defined via the pushout of $\mu_\alpha$ and $\varphi_\alpha$:

$$\begin{tikzcd}
0 \arrow[r] & K^{(I_\alpha)} \arrow[r, "\mu_\alpha"] \arrow[d, "\varphi_\alpha"'] & F^{(I_\alpha)} \arrow[r] \arrow[d, "\psi_\alpha"'] & S^{(I_\alpha)} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_\alpha \arrow[r, "\subseteq"] & P_{\alpha+1} \arrow[r] & S^{(I_\alpha)} \arrow[r] & 0.
\end{tikzcd}$$

If $\alpha \leq \lambda$ is a limit ordinal, we put $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$, so the chain is continuous. Put $P = \bigcup_{\alpha<\lambda} P_\alpha$.

We will prove that $\nu : M \hookrightarrow P$ is a special $S^\perp$-preenvelope of $M$.
First we prove that $P \in S^\perp$. Since $F$ is projective, we are left to show that any $\varphi \in \mathrm{Hom}_R(K, P)$ factorizes through $\mu$:

Since $K$ is $< \lambda$-generated, there are an index $\alpha < \lambda$ and $\eta \in I_\alpha$, such that $\varphi(k) = \eta(k)$ for all $k \in K$. The pushout square gives $\psi_\alpha \mu_\alpha = \sigma_\alpha \varphi_\alpha$, where $\sigma_\alpha$ denotes the inclusion of $P_\alpha$ into $P_{\alpha+1}$. Altogether we have $\psi_\alpha \nu'_\eta \mu = \psi_\alpha \mu_\alpha \nu_\eta = \sigma_\alpha \varphi_\alpha \nu_\eta = \sigma_\alpha \eta$. It follows that $\varphi = \psi' \mu$, where $\psi' \in \mathrm{Hom}_R(F, P)$ is defined by $\psi'(f) = \psi_\alpha \nu'_\eta(f)$ for all $f \in F$. This proves that $P \in S^\perp$.

It remains to prove that $N = P/M \in {}^\perp(\mathcal{S}^\perp)$. By construction, $N$ is the union of the continuous chain $(N_\alpha \mid \alpha < \lambda)$, where $N_\alpha = P_\alpha/M$.

Since $P_{\alpha+1}/P_\alpha$ is isomorphic to a direct sum of copies of $S$ by the pushout construction, so is $N_{\alpha+1}/N_\alpha \cong P_{\alpha+1}/P_\alpha$. Since $S \in {}^\perp(\mathcal{S}^\perp)$, also $N_{\alpha+1}/N_\alpha \in {}^\perp(\mathcal{S}^\perp)$, whence $N \in {}^\perp(\mathcal{S}^\perp)$ by the Eklof Lemma.

(b) follows by part (a) (cf. Lemma 5.4). □

**Remark 6.2.** Theorem 6.1 will be our main tool for proving completeness of cotorsion pairs in ZFC. However, the question of whether a particular cotorsion pair is complete may depend on the extension of ZFC that we work in.

For example, consider the cotorsion pair $\mathfrak{C}$ in Mod–$\mathbb{Z}$ cogenerated by $\mathbb{Z}$. Its left hand class is the class of all *Whitehead groups.* So under Gödel's Axiom of Constructibility (V = L), $\mathfrak{C}$ is trivial (see e.g. [19, 4.1(i)]), and hence complete. However, Eklof and Shelah [20] have shown that it is also consistent with ZFC + GCH that the class of all Whitehed groups is not precovering, and whence $\mathfrak{C}$ is not complete.

In contrast, Cox [18] has recently proved that under the (large cardinal) hypothesis of Vopěnka's Principle (VP), all cotorsion pairs over any right hereditary ring (such as $\mathbb{Z}$) are cogenerated by a set. Moreover, the Weak Diamond Principle implies that each hereditary cotorsion pair cogenerated by a set over a ring of finite global dimension is also generated by a set (cf. [27, 11.5]). Since the Weak Diamond Principle is relatively consistent with VP, it follows that assuming VP, it is consistent that *each* cotorsion pair over a right hereditary ring is complete.

Any cotorsion pair generated by a set of modules $\mathcal{S}$ is also generated by the single module $M = \bigoplus_{S\in\mathcal{S}} S$. So the following corollary of Theorem 6.1 provides a characterization of the (complete) cotorsion pairs generated by sets of modules:

**Corollary 6.3.** *Let $M$ be a module. Denote by $\mathcal{C}_M$ the class of all modules $C$, such that there is an exact sequence $0 \to F \to C \to G \to 0$, where $F$ is free and $G$ is $\{M\}$-filtered. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. The following are equivalent*

(a) *$\mathfrak{C}$ is generated by $M$ (that is, $\mathcal{B} = M^\perp$).*

(b) *$\mathcal{A}$ consists of all direct summands of elements of $\mathcal{C}_M$ (and for each $A \in \mathcal{A}$, there are $C \in \mathcal{C}_M$ and $B \in \mathcal{K}_{\mathfrak{C}}$, such that $A \oplus B \cong C$).*

*Proof.* (a) implies (b): by assumption, $\mathcal{B} = M^\perp$. Take $A \in \mathcal{A}$, and let $0 \to N \xrightarrow{\mu} F \to A \to 0$ be a short exact sequence with $F$ free. By Theorem 6.1 (a), there is a special $\mathcal{B}$-preenvelope, $\nu : N \hookrightarrow B$ of $N$, such that $G = B/N$ is $\{M\}$-filtered. Let $(G_\alpha \mid \alpha \leq \lambda)$ be an $\{M\}$-filtration of $G$. Consider the pushout of $\mu$ and $\nu$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & N \arrow[r, "\mu"] \arrow[d, "\nu"'] & F \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "\pi"'] & A \arrow[r] & 0 \\
 & G \arrow[r, equal] \arrow[d] & G \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

The second column gives $C \in \mathcal{C}_M$. The second row splits since $B \in \mathcal{B}$ and $A \in \mathcal{A}$, so $A \oplus B \cong C$. Finally, since $F, G \in \mathcal{A}$, we have $C \in \mathcal{A}$, so $B \in \mathcal{K}_{\mathfrak{C}}$.

(b) implies (a): by the Eklof Lemma, $M^\perp = \mathcal{A}^\perp = \mathcal{B}$. □

**Corollary 6.4.** *Let $\mathcal{S}$ be a set of modules containing $R$. Then the class ${}^\perp(\mathcal{S}^\perp)$ consists of all direct summands of $\mathcal{S}$-filtered modules.*

*Proof.* By Corollary 6.3 and the Eklof Lemma. □

In general, we cannot omit the term “direct summands” in Corollary 6.4. For example, if $\mathcal{S} = \{R\}$, then ${}^\perp(\mathcal{S}^\perp) = \mathcal{P}_0$ is the class of all projective modules, while $\mathcal{S}$-filtered modules coincide with the free modules.

However, there is a way of getting rid of the direct summands on the account of enlarging the set $\mathcal{S}$. In the particular case of $\mathcal{S} = \{R\}$, this is the point of the celebrated Kaplansky theorem [2, 26.2], saying that each projective module is a direct sum of countably generated projective modules.

In general, let $\kappa$ be a regular uncountable cardinal and $\mathcal{S}$ a set of $< \kappa$-presented modules. Let $\mathcal{A} = {}^\perp(\mathcal{S}^\perp)$ (so $\mathcal{A}$ is a special precovering class by Theorem 6.1). Then $\mathcal{A} = \mathrm{Filt}(\mathcal{A}^{<\kappa})$, where $\mathcal{A}^{<\kappa}$ denotes the class of all $< \kappa$-presented modules from $\mathcal{A}$. This generalization of Kaplansky's theorem will follow will follow from our next lemma which is yet another consequence of the Hill Lemma:

**Lemma 6.5.** *Let $\kappa$ be an uncountable regular cardinal and $\mathcal{C}$ be a class of $\leq \kappa$-presented modules. Denote by $\mathcal{A}$ the class of all direct summands of $\mathcal{C}$-filtered modules. Then every module in $\mathcal{A}$ is $\mathcal{A}^{<\kappa}$-filtered.*

*Proof.* Let $K \in \mathcal{A}$, so there is a $\mathcal{C}$-filtered module $M$, such that $M = K \oplus L$ for some $L \subseteq M$. Denote by $\pi_K : M \to K$ and $\pi_L : M \to L$ the corresponding projections. Let $\mathcal{H}$ be the family of submodules of $M$ as in Theorem 2.1. We proceed in two steps:

Step I: By induction, we construct a continuous chain, $(N_\alpha \mid \alpha \leq \tau)$, of submodules of $M$, such that $N_\tau = M$, and for each $\alpha < \tau$,

(a) $N_\alpha \in \mathcal{H}$,
(b) $N_\alpha = \pi_K(N_\alpha) + \pi_L(N_\alpha)$, and
(c) the module $N_{\alpha+1}/N_\alpha$ is $< \kappa$-presented.

First $N_0 = 0$, and $N_\beta = \bigcup_{\alpha<\beta} N_\alpha$ for all limit ordinals $\beta \leq \tau$. Suppose we have $N_\alpha \subsetneqq M$. In order to construct $N_{\alpha+1}$, we take $x \in M \setminus N_\alpha$; by property (H4), there is $Q_0 \in \mathcal{H}$, such that $N_\alpha \cup \{x\} \subseteq Q_0$ and $Q_0/N_\alpha$ is $< \kappa$-presented. Let $X_0$ be a subset of $Q_0$ of cardinality $< \kappa$, such that the set $\{x + N_\alpha \mid x \in X_0\}$ generates $Q_0/N_\alpha$. Put $Z_0 = \pi_K(Q_0) \oplus \pi_L(Q_0)$. Clearly $Q_0/N_\alpha \subseteq Z_0/N_\alpha$. Since $\pi_K(N_\alpha), \pi_L(N_\alpha) \subseteq N_\alpha$, the module $Z_0/N_\alpha$ is generated by the set

$$\{x + N_\alpha \mid x \in \pi_K(X_0) \cup \pi_L(X_0)\}.$$

Thus we can find $Q_1 \in \mathcal{H}$, such that $Z_0 \subseteq Q_1$ and $Q_1/N_\alpha$ is $< \kappa$-presented. Similarly, we infer that $Z_1/N_\alpha$ is $< \kappa$-generated for $Z_1 = \pi_K(Q_1) \oplus \pi_L(Q_1)$, and we find $Q_2 \in \mathcal{H}$ such that $Z_1 \subseteq Q_2$ and $Q_2/N_\alpha$ is $< \kappa$-presented. In this way we obtain a chain $Q_0 \subseteq Q_1 \subseteq \dots$, such that for all $i < \omega$: $Q_i \in \mathcal{H}$, $Q_i/N_\alpha$ is $< \kappa$-presented, and $\pi_K(Q_i) + \pi_L(Q_i) \subseteq Q_{i+1}$. It is easy to see that $N_{\alpha+1} = \bigcup_{i<\omega} Q_i$ satisfies the properties (a)-(c).

Step II: by condition (b), we have

$$\pi_K(N_{\alpha+1}) + N_\alpha = \pi_K(N_{\alpha+1}) \oplus \pi_L(N_\alpha)$$

and similarly for $L$. Hence

$$\begin{aligned}
&(\pi_K(N_{\alpha+1}) + N_\alpha) \cap (\pi_L(N_{\alpha+1}) + N_\alpha) \\
&\quad = (\pi_K(N_{\alpha+1}) \oplus \pi_L(N_\alpha)) \cap (\pi_L(N_{\alpha+1}) \oplus \pi_K(N_\alpha)) \\
&\quad = \big(\pi_K(N_{\alpha+1}) \cap (\pi_L(N_{\alpha+1}) \oplus \pi_K(N_\alpha))\big) \oplus \pi_L(N_\alpha) \\
&\quad = \pi_K(N_\alpha) \oplus \pi_L(N_\alpha) = N_\alpha
\end{aligned}$$

and

$$N_{\alpha+1}/N_\alpha = (\pi_K(N_{\alpha+1}) + N_\alpha)/N_\alpha \oplus (\pi_L(N_{\alpha+1}) + N_\alpha)/N_\alpha.$$

By condition (a), $N_{\alpha+1}/N_\alpha$ is $\mathcal{C}$-filtered. Since

$$(\pi_K(N_{\alpha+1}) + N_\alpha)/N_\alpha \cong \pi_K(N_{\alpha+1})/\pi_K(N_\alpha),$$

$\pi_K(N_{\alpha+1})/\pi_K(N_\alpha)$ is isomorphic to a direct summand of a $\mathcal{C}$-filtered module, whence $\pi_K(N_{\alpha+1})/\pi_K(N_\alpha) \in \mathcal{A}$. Since the class of all $< \kappa$-presented modules is closed under direct summands, (c) yields that $\pi_K(N_{\alpha+1})/\pi_K(N_\alpha)$ is $< \kappa$-presented. So $(\pi_K(N_\alpha) \mid \alpha \leq \tau)$ is the desired $\mathcal{A}^{<\kappa}$-filtration of $K = \pi_K(N_\tau)$. □

**Theorem 6.6.** (Kaplansky theorem for cotorsion pairs) *Let $R$ be a ring, $\kappa$ a regular uncountable cardinal and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a cotorsion pair of modules. Then the following conditions are equivalent:*

(a) *$\mathfrak{C}$ is generated by a class $\mathcal{C}$ consisting of $< \kappa$-presented modules.*
(b) *Every module in $\mathcal{A}$ is $\mathcal{A}^{<\kappa}$-filtered.*

*Proof.* (a) $\implies$ (b). W.l.o.g., $\mathcal{C}$ is a set and $R \in \mathcal{C}$. By Corollary 6.4, $\mathcal{A}$ consists of all direct summands of $\mathcal{C}$-filtered modules. So statement (b) follows by Lemma 6.5.

(b) $\implies$ (a). By the Eklof Lemma 1.3, every $\mathcal{A}$-filtered module is contained in $\mathcal{A}$. Thus (b) implies that $\mathcal{A} = \mathrm{Filt}(\mathcal{A}^{<\kappa})$, whence $\mathfrak{C}$ is generated by the class $\mathcal{A}^{<\kappa}$. □

**Remark 6.7.** Even in the particular case of projective modules, it is not possible to extend Theorem 6.6 to the case of $\kappa = \aleph_0$. Namely, there exist rings $R$ which admit countably generated projective modules that are not direct sums of finitely generated projective ones.

For a concrete example, consider the commutative ring $R$ consisting of all continuous real-valued functions on $[0,1]$ and its ideal $P$ consisting of all functions $f \in R$ vanishing on some interval $[0, \epsilon(f)]$, where $\epsilon(f) \in (0,1)$. Then $P$ is countably generated and projective, but it is not a direct sum of finitely generated projective modules.

By the above, all cotorsion pairs $(\mathcal{A}, \mathcal{B})$ generated by sets of modules are complete; moreover, the special precovering class $\mathcal{A}$ is of the form $\mathcal{A} = \mathrm{Filt}(\mathcal{S})$ for a subset $\mathcal{S} \subseteq \mathcal{A}$. In fact all classes of the form $\mathrm{Filt}(\mathcal{S})$ for a set of modules $\mathcal{S}$ are special precovering in the following weaker sense (not requiring the precover to be surjective):

**Theorem 6.8.** *Let $\mathcal{S}$ be any set of modules. Then the class $\mathcal{C} = \mathrm{Filt}(\mathcal{S})$ is precovering. Moreover, for each module $M$ there exists a $\mathcal{C}$-precover $f \in \mathrm{Hom}_R(C, M)$ such that $\mathrm{Ker}\, f \in \mathcal{C}^\perp$.*

*Proof.* Let $M \in \text{Mod–}R$. Let $N = \sum_{C \in \mathcal{C}, g \in \mathrm{Hom}_R(C,M)} \mathrm{Im}\, g$ (this is the *trace* of $\mathcal{C}$ in $M$).

First, we will prove that the module $N$ has a special $\mathcal{C}$-precover in the sense of Definition 4.9: We have a short exact sequence $0 \to K \stackrel{\mu}{\to} L \to N \to 0$ where $L$ is a direct sum of copies of modules from $\mathcal{C}$, hence also $L \in \mathcal{C}$. By Theorem 6.1(i), there is also a short exact sequence $0 \to K \stackrel{\nu}{\to} P \to Q \to 0$ such that $P \in \mathcal{S}^\perp$ and $Q \in \mathcal{C}$. Consider the pushout of $\mu$ and $\nu$:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r, "\mu"] \arrow[d, "\nu"'] & L \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P \arrow[r] \arrow[d] & C \arrow[r, "f"] \arrow[d] & N \arrow[r] & 0 \\
 & Q \arrow[r, equal] \arrow[d] & Q \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}
$$

Since $L, Q \in \mathcal{C}$, also $C \in \mathcal{C}$. Moreover, $P = \mathrm{Ker}\, f \in \mathcal{S}^\perp = \mathcal{C}^\perp$ by the Eklof Lemma, so $f$ is a special $\mathcal{C}$-precover of $N$.

Consider the embedding $\tau : N \to M$. Each homomorphism $g$ from an element of $\mathcal{C}$ to $M$ actually maps into $N$, that is, it factorizes through $\tau$, and hence through $\tau f$, because $f$ is a $\mathcal{C}$-precover of $N$. Thus $\tau f$ is a $\mathcal{C}$-precover of $M$. Finally, $\tau$ is monic, so $\mathrm{Ker}\, \tau f = \mathrm{Ker}\, f \in \mathcal{C}^\perp$. □

That $\mathcal{C} = \mathrm{Filt}(\mathcal{S})$ is a precovering class for any set of modules $\mathcal{S}$ was first proved in [24] and [39]. The short proof above, providing for the additional special property of the precover, is due to Estrada, and appears, e.g., in [17].

Moreover, [17] contains a short proof of another result from [39], namely that the class $\mathcal{C}$ is also preenveloping provided that $\mathcal{C}$ is closed under direct products. The proof is yet another application of the Hill Lemma:

**Proposition 6.9.** *Let $\mathcal{S}$ be any set of modules. Assume that the class $\mathcal{C} = \mathrm{Filt}(\mathcal{S})$ is closed under direct products. Then $\mathcal{C}$ is preenveloping.*

*Proof.* Let $N$ be a module and $\kappa$ be a regular uncountable cardinal such that each module in $\mathcal{S}$ is $< \kappa$-presented and $\kappa > \mathrm{gen}\, N + |R|$. Let $\mathcal{T}$ be a representative set of the class of all $< \kappa$-presented modules in $\mathcal{C}$, and $\mathcal{F}$ be the set of all homomorphisms of the form $f \in \mathrm{Hom}_R(N, T_f)$ where $T_f \in \mathcal{T}$.

Let $C = \prod_{f \in \mathcal{F}} T_f$, and let $h \in \mathrm{Hom}_R(N, C)$ be defined by $f = \pi_f h$ where $\pi_f \in \mathrm{Hom}_R(C, T_f)$ is the canonical projection for each $f \in \mathcal{F}$. Notice that $C \in \mathcal{C}$ by our assumption on the class $\mathcal{C}$.

We will prove that $h$ is a $\mathcal{C}$-precover of $N$. Let $g \in \mathrm{Hom}_R(N, M)$ for some $M \in \mathcal{C}$. Let $\mathcal{M}$ be any $\mathcal{S}$-filtration of $M$, and consider the corresponding family $\mathcal{H}$ from Theorem 2.1. Since $\mathrm{gen}\, N < \kappa$, there is a subsetet $X \subseteq M$ of cardinality $< \kappa$ such that $\mathrm{Im}\, g \subseteq \langle X \rangle$. By conditions (H3) and (H4) of Theorem 2.1, $\mathrm{Im}\, g \subseteq P$ for a $< \kappa$-presented $\mathcal{S}$-filtered submodule $P$ of $M$. So there are an $f \in \mathcal{F}$ and an isomorphism $\tau \in \mathrm{Hom}_R(P, T_f)$ such that $f = \tau g$. Thus $g = \tau^{-1} f = \tau^{-1} \pi_f h$, proving that $g$ factorizes through $h$. $\square$

**Remark 6.10.** Let $\mathcal{S}$ be a set of modules, $\mathcal{C} = \mathrm{Filt}(\mathcal{S})$, and $\mathcal{D}$ be the class of all direct summands of modules from $\mathcal{C}$. Then $\mathcal{D}$ is precovering, and if $\mathcal{C}$ is closed under direct products, then $\mathcal{D}$ is also preenveloping.

This follows from Theorem 6.8 and Proposition 6.9, since the $\mathcal{C}$-precovers and $\mathcal{C}$-preenvelopes constructed there are easily seen to be also $\mathcal{D}$-precovers and $\mathcal{D}$-preenvelopes, respectively.

As the class $\mathcal{D}$ is closed under isomorphisms and direct summands, we infer from Lemma 4.13(b) that $\mathcal{D}$ is preenveloping, iff $\mathcal{D}$ is closed under direct products.

## 7. Modules of bounded homological dimensions

In this section, we will see that Theorem 6.1 applies to many cotorsion pairs arising in classical homological algebra. Hence it yields numerous generalizations of the notions of projective (pre-)covers and injective envelopes.

We start with the cotorsion pairs $({}^\perp\mathcal{I}_n, \mathcal{I}_n)$ for $n < \omega$, where $\mathcal{I}_n$ denotes the class of all modules of injective dimension $\leq n$. Their completeness is a simple consequence of Theorem 6.1 that follows from the Baer Criterion for injectivity [2, 18.3]:

**Theorem 7.1.** *Let $R$ be a ring and $n < \omega$. Then $({}^\perp\mathcal{I}_n, \mathcal{I}_n)$ is a complete hereditary cotorsion pair. In particular, every module has a special $\mathcal{I}_n$-preenvelope.*

*Proof.* Let $M$ be a module. Let

$$\mathcal{R}: \quad 0 \to M \to I_0 \to I_1 \to \dots \to I_{n-1} \to I_n \to \dots$$

be an injective coresolution of $M$. Let $C_n$ be the $n$-th cosyzygy of $M$ in $\mathcal{R}$. Then $M \in \mathcal{I}_n$, if and only if $C_n$ is injective.

By the Baer Criterion, the latter is equivalent to $\mathrm{Ext}^1_R(R/I, C_n) = 0$, and hence – by dimension shifting – to $\mathrm{Ext}^{n+1}_R(R/I, M) = 0$, for all right ideals $I$ of $R$. Denote by $S_I$ the $n$-th syzygy (in a projective resolution) of the cyclic module $R/I$. Then $\mathrm{Ext}_R^{n+1}(R/I, M) = 0$, if and only if $\mathrm{Ext}_R^1(S_I, M) = 0$. So $\mathcal{I}_n = (\bigoplus_{I \subseteq R} S_I)^\perp$, and the assertion follows by Theorem 6.1(b).

The cotorsion pair is hereditary since the class $\mathcal{I}_n$ is cosyzygy closed. $\square$

**Remark 7.2.** 1. Of course, there is more to say for $n = 0$: the class $\mathcal{I}_0$ is well known to be an enveloping class (as we will see in the next section, also this fact actually follows from a more general result concerning cotorsion pairs). Another case worth mentioning is for $n = 1$: since 1st syzygies of cyclic modules are just the right ideals of $R$, and the class $\mathcal{A}_1 = {}^\perp\mathcal{I}_1$ is closed under submodules, we infer that $\mathcal{A}_1$ is the class of all 'ideally filtered' modules, that is, $\mathcal{A}_1 = \mathrm{Filt}(\mathcal{J})$ where $\mathcal{J}$ is the class of all right ideals of the ring $R$.

2. The proof of Theorem 7.1 is based on the existence of a test module for injectivity, that is, on the Baer Criterion. In the dual case, the existence of test modules for projectivity depends on the structure of the base ring.

If $R$ is a right perfect ring, then a test module $M$ for projectivity does exist, for example, one can take $M =$ the direct product of all simple modules, cf. [27, 8.8]. Thus for each $n < \omega$, the cotorsion pair $(\mathcal{P}_n, \mathcal{P}_n^\perp)$ is cogenerated by a set.

However, if $R$ is not right perfect, then it is consistent with ZFC + GCH that there are no test modules for projectivity.

Nevertheless, a result dual to Theorem 7.1 is true for any ring. Before proving this fact, we will need some auxiliary facts.

The first one is known as Eilenberg's Trick:

**Lemma 7.3.** (Eilenberg's Trick) *Let $P$ be a projective module. Then there exists a free module $G$ such that $\mathrm{gen}(G) = \mathrm{gen}(P) + \aleph_0$, and $G \cong P \oplus G$.*

*If moreover $G$ is isomorphic to a direct summand of $P$, then $P \cong G$ is a free module.*

*Proof.* Since $P$ is projective, there exists a free module $F$ such that $\mathrm{gen}(F) = \mathrm{gen}(P)$, and a direct sum decomposition $F = A \oplus B$, such that $P \cong A$. Let $G = F^{(\omega)}$. Then

$$G = (A \oplus B) \oplus (A \oplus B) \oplus \cdots = A \oplus (B \oplus A) \oplus (B \oplus A) \oplus \cdots = A \oplus G \cong P \oplus G$$

and $\mathrm{gen}(G) = \mathrm{gen}(F) + \aleph_0 = \mathrm{gen}(P) + \aleph_0$.

If moreover $P \cong G \oplus Q$ for a module $Q$, then $G \cong P \oplus G \cong (G \oplus Q) \oplus G \cong (G \oplus G) \oplus Q \cong G \oplus Q \cong P$. $\square$

**Definition 7.4.** Let $R$ be a ring. Let $\kappa$ be a cardinal. Then $R$ is *right $\kappa$-noetherian*, provided that each right ideal $I$ of $R$ is $\leq \kappa$-generated.
The least infinite cardinal $\kappa$, such that $R$ is right $\kappa$-noetherian is the *right dimension* of $R$, denoted by $\dim(R)$.

**Lemma 7.5.** *Let $R$ be a ring and $\kappa$ be a cardinal, such that $\kappa \geq \dim(R)$. Then any submodule of a $\leq \kappa$-generated module is $\leq \kappa$-generated.*

*Proof.* First all submodules of cyclic modules are $\leq \dim(R)$-generated, since they are epimorphic images of right ideals. Further, any $\leq \kappa$-generated module $M$ is a union of a continuous chain, $(M_\alpha \mid \alpha \leq \kappa)$, of submodules, such that all the factors $M_{\alpha+1}/M_\alpha$ are cyclic. If $K \subseteq M$, then $(K \cap M_{\alpha+1})/(K \cap M_\alpha)$ embeds into $M_{\alpha+1}/M_\alpha$ for each $\alpha < \kappa$, and the assertion follows. $\square$

Now, we can prove Lemma 7.6:

**Lemma 7.6.** *Let $n < \omega$, $R$ be a ring, $\kappa = \dim(R)$, and $M \in \mathcal{P}_n$, where $\mathcal{P}_n$ denotes the class of all modules of projective dimension $\leq n$. Then $M$ is $\mathcal{P}_n^{\leq\kappa}$-filtered.*

*Proof.* Let $\lambda = \max\{\kappa, \rho\}$, where $\rho$ is the minimal number of generators of the module $M$. Since $\lambda \geq \dim(R)$ and $M$ is $\leq \lambda$-generated, applying Lemma 7.5 repeatedly, and then Lemma 7.3, we obtain a free resolution of $M$,

$$\mathcal{R} \quad : \quad 0 \to R^{(A_n)} \overset{f_n}{\to} R^{(A_{n-1})} \to \cdots \to R^{(A_1)} \overset{f_1}{\to} R^{(A_0)} \overset{f_0}{\to} M \to 0,$$

such that $|A_i| \leq \lambda$ for each $i \leq n$.

Let $(m_\alpha \mid \alpha < \lambda)$ be a set of $R$-generators of $M$. By induction on $\alpha$, we will construct a $\mathcal{P}_n^{<\kappa}$-filtration $(M_\alpha \mid \alpha < \lambda)$ of $M$ together with free resolutions $\mathcal{R}_\alpha$ of $M_\alpha$, which are restrictions of $\mathcal{R}$:

$$\mathcal{R}_\alpha \quad : \quad 0 \to F_{\alpha,n} \overset{f_n \restriction F_{\alpha,n}}{\to} F_{\alpha,n-1} \to \cdots \to F_{\alpha,1} \overset{f_1 \restriction F_{\alpha,1}}{\to} F_{\alpha 0} \overset{f_0 \restriction F_{\alpha,0}}{\to} M_\alpha \to 0,$$

so that $m_\alpha \in M_{\alpha+1}$, $F_{\alpha,i} = R^{(A_{\alpha,i})}$ for some $A_{\alpha,i} \subseteq A_i$, $A_{\alpha,i} \subseteq A_{\alpha+1,i}$, and $|A_{\alpha+1,i} \setminus A_{\alpha,i}| \leq \kappa$, for all $\alpha < \lambda$ and $i \leq n$.

First $M_0 = 0$ and $A_{0,i} = \emptyset$ for all $i \leq n$. Assume $M_\alpha$ and $\mathcal{R}_\alpha$ are defined. If $M_\alpha \neq M$, let $\gamma < \lambda$ be the least index, such that $m_\gamma \notin M_\alpha$. Clearly, there is a subset $B_0 \subseteq A_0$ of cardinality $\leq \kappa$ (in fact, a finite one), such that $m_\gamma \subseteq f_0(R^{(A_{\alpha,0} \cup B_0)})$.

Since

$$\operatorname{Ker}(f_0 \restriction R^{(A_{\alpha,0})}) = \operatorname{Ker}(f_0 \restriction R^{(A_{\alpha,0} \cup B_0)}) \cap R^{(A_{\alpha,0})},$$

we have

$$\begin{aligned} &\operatorname{Ker}(f_0 \restriction R^{(A_{\alpha,0} \cup B_0)}) / \operatorname{Ker}(f_0 \restriction R^{(A_{\alpha,0})}) \\ &\qquad \cong (R^{(A_{\alpha,0})} + \operatorname{Ker}(f_0 \restriction R^{(A_{\alpha,0} \cup B_0)}))/R^{(A_{\alpha,0})}. \end{aligned}$$

The latter module is a submodule in $R^{(A_{\alpha,0} \cup B_0)}/R^{(A_{\alpha,0})} \cong R^{(B_0)}$. So the exactness of $\mathcal{R}_\alpha$ at $F_{\alpha,0}$, of $\mathcal{R}$ at $R^{(A_0)}$ and Lemma 7.5 yield the existence of a subset $B_1 \subseteq A_1$ of cardinality $\leq \kappa$, such that $\operatorname{Ker}(f_0 \restriction R^{(A_{\alpha,0} \cup B_0)}) \subseteq f_1(R^{(A_{\alpha,1} \cup B_1)})$. Similarly, there is a subset $B_2 \subseteq A_2$ of cardinality $\leq \kappa$, such that $\operatorname{Ker}(f_1 \restriction R^{(A_{\alpha,1} \cup B_1)}) \subseteq f_2(R^{(A_{\alpha,2} \cup B_2)})$, etc. Finally, there is a subset $B_n \subseteq A_n$ of cardinality $\leq \kappa$, such that $\operatorname{Ker}(f_{n-1} \restriction R^{(A_{\alpha,n-1} \cup B_{n-1})}) \subseteq f_n(R^{(A_{\alpha,n} \cup B_n)})$.

Now there is a subset $B_{n-1} \subseteq B'_{n-1} \subseteq A_{n-1}$ of cardinality $\leq \kappa$, such that $f_n(R^{(A_{\alpha,n} \cup B_n)}) \subseteq R^{(A_{\alpha,n-1} \cup B'_{n-1})}$, etc. Finally, there is a subset $B_0 \subseteq B'_0 \subseteq A_0$ of cardinality $\leq \kappa$, such that $f_1(R^{(A_{\alpha,1} \cup B'_1)}) \subseteq R^{(A_{\alpha,0} \cup B'_0)}$.

Continuing this back and forth procedure in $\mathcal{R}$, we obtain, for each $i \leq n$, a countable chain $B_i \subseteq B'_i \subseteq B_i{}'' \subseteq \ldots$ consisting of subsets of $A_i$ of cardinality $\leq \kappa$. Let $C_i = B_i \cup B'_i \cup B_i{}'' \cup \ldots$. Then $C_i$ has cardinality $\leq \kappa$, and the sequence

$$\begin{aligned} \mathcal{R}_{\alpha+1} \ : \ 0 \to F_{\alpha+1,n} &\overset{f_n \restriction F_{\alpha+1,n}}{\to} F_{\alpha+1,n-1} \\ \ldots \to F_{\alpha+1,1} &\overset{f_1 \restriction F_{\alpha+1,1}}{\to} F_{\alpha+1,0} \overset{f_0 \restriction F_{\alpha+1,0}}{\to} M_{\alpha+1} \to 0, \end{aligned}$$

where $F_{\alpha+1,i} = R^{(A_{\alpha+1,i})}$ and $A_{\alpha+1,i} = A_{\alpha,i} \cup C_i$, is exact, and $\{m_\gamma\} \cup M_\alpha \subseteq N$. (The backward procedure takes care of kernels being inside images, while the forward one of the resulting sequence being a complex.)

Note that $\mathcal{R}_\alpha$ is an exact subcomplex of the exact complex $\mathcal{R}_{\alpha+1}$, so the factor complex $\mathcal{R}_{\alpha+1}/\mathcal{R}_\alpha$ is also exact, and hence it yields a free resolution of the module $M_{\alpha+1}/M_\alpha$. This shows that $M_{\alpha+1}/M_\alpha \in \mathcal{P}_n^{\leq\kappa}$.

For a limit ordinal $\alpha < \lambda$, we define $A_{\alpha,i} = \bigcup_{\beta<\alpha} A_{\beta,i}$ and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$. Then the corresponding restriction of $\mathcal{R}$ is a free resolution of $M_\alpha$. □

Now we easily derive

**Theorem 7.7.** *Let $R$ be a ring and $n < \omega$.*

(a) *Then* $\mathfrak{C}_n = (\mathcal{P}_n, \mathcal{P}_n^\perp)$ *is a complete hereditary cotorsion pair. In particular, every module has a special* $\mathcal{P}_n$*-precover.*
(b) *If* $R$ *is right* $\aleph_0$*-noetherian, then* $\mathfrak{C}_n$ *is generated by (a representative set of) the class* $\mathcal{P}_n^{\leq\omega}$*.*

*Proof.* Let $\kappa = \dim(R)$. By the Eklof Lemma $(\mathrm{Filt}(\mathcal{P}_n^{\leq\kappa}))^\perp = (\mathcal{P}_n^{\leq\kappa})^\perp$, so by Lemma 7.6, $\mathcal{P}_n^\perp = (\mathcal{P}_n^{\leq\kappa})^\perp$. Clearly, $\mathcal{P}_n^{\leq\kappa}$ has a representative set of elements. By Corollary 6.3(b) and the Eklof Lemma, we get ${}^\perp(\mathcal{P}_n^\perp) = \mathcal{P}_n$, so $\mathfrak{C}_n$ is a complete cotorsion pair. $\mathfrak{C}_n$ is hereditary because the class $\mathcal{P}_n$ is syzygy closed. □

In fact, Lemma 7.6 is a particular instance (for $\kappa = \dim(R)$, $\mathcal{C} = \mathcal{P}_0^{\leq\kappa}$, $\mathcal{D} = \mathcal{P}_0$, and $\mathcal{D}_n = \mathcal{P}_n$) of the following general result whose proof involves yet another application of the Hill Lemma:

**Theorem 7.8.** *Let* $R$ *be a ring,* $\kappa \geq \dim(R)$*, and* $n < \omega$*. Let* $\mathcal{C}$ *be any class of* $\leq \kappa$*-presented modules, and* $\mathcal{D} = \mathrm{Filt}(\mathcal{C})$*. Let* $\mathcal{D}_n$ *denote the class of all modules of* $\mathcal{D}$*-resolution dimension* $\leq n$ *(see Definition 5.5(i)). Then each module* $M \in \mathcal{D}_n$ *is* $\mathcal{D}_n^{\leq\kappa}$*-filtered.*

The proof of Theorem 7.8 is similar to the proof of Lemma 7.6 in the sense that it makes use of a back and forth procedure in a $\mathcal{D}$-resolution of length $\leq n$. However, while the classes of all projective and free modules are decomposable, the class $\mathcal{D}$ is only $\kappa$-deconstructible in general. This extra difficulty is overcome by employing the Hill Lemma to make the 'vertical' $\mathcal{C}$-filtrations of the terms of the resolution in various degrees compatible with the 'horizontal' maps forming the $\mathcal{D}$-resolution. For more details, we refer to [40, 3.4].

## 8. Purity, and the deconstruction of flat modules

First, we recall basics on purity in module categories (for more details, see [27, Section 2.1]):

**Definition 8.1.** A submodule $A$ of a module $B$ is a *pure submodule* ($A \subseteq_* B$ for short), if for each finitely presented module $F$, the functor $\mathrm{Hom}(F, -)$ preserves the exactness of the short exact sequence $0 \to A \to B \to B/A \to 0$. The embedding $A \subseteq_* B$ is then called a *pure embedding*, and the sequence $0 \to A \to B \to B/A \to 0$ a *pure-exact sequence*. An epimorphism $\pi : B \to C$ is a *pure epimorphism* provided that $\mathrm{Ker}\,\pi \subseteq_* B$.

Clearly, any split embedding is pure. Moreover, the following characterization holds true:

**Lemma 8.2.** *Let* $A \subseteq B$ *be modules. Denote by* $\mathcal{E}$ *the short exact sequence*

$$0 \to A \xrightarrow{\subseteq} B \xrightarrow{\pi} B/A \to 0$$

*in* Mod–$R$*. Then the following are equivalent:*
(a) $A \subseteq_* B$*;*
(b) *for all* $0 < m < \omega$*,* $0 < n < \omega$ *and all systems of* $R$*-linear equations* $(\mathcal{S})$ *in the variables* $x_j$ $(j < n)$ *with* $a_i \in A$ $(i < m)$*,* $r_{ji} \in R$ $(i < m,\ j < n)$

$$(\mathcal{S}) \qquad \sum_{j<n} x_j.r_{ji} = a_i \quad (i < m)$$

*the following holds:*
$(\mathcal{S})$ *has a solution in* $A$*, whenever* $(\mathcal{S})$ *has a solution in* $B$*;*
(c) $\mathcal{E}$ *is a direct limit of a direct system of split short exact sequences;*

(d) *the sequence* $0 \to A \otimes_R F \to B \otimes_R F \to B/A \otimes_R F \to 0$ *is exact for any left* $R$*-module* $F$.

*Proof.* We will only prove the equivalence of (a) and (b), and refer to [27, 2.19] for the remaining implications. Notice that each finitely presented module $F$ is isomorphic to $R^n/G$ for some $n < \omega$ and some $G \subseteq R^n$ generated by the elements $r_i = \sum_{j<n} 1_j.r_{ji}$ $(i < m,\ r_{ij} \in R)$, where $(1_j \mid j < n)$ is the canonical basis of $R^n$. Denote by $\rho$ the canonical projection of $R^n$ onto $R^n/G$. Then, for each $R$-homomorphism $f \in \mathrm{Hom}_R(R^n/G, B/A)$, we have $f\rho(1_j) = \pi b'_j$ for some $b'_j \in B$ $(j < n)$ with $\pi(\sum_{j<n} b'_j.r_{ji}) = 0$ for each $i < m$. The exactness of $\mathcal{E}$ then gives

$$\sum_{j<n} b'_j.r_{ji} = a'_i \text{ for some } a'_i \in A. \tag{1}$$

Assume (a). Consider a system $(\mathcal{S})$ as in (b) and let $F = R^n/G$ be as above. If $(b_j \mid j < n)$ are solutions of $(\mathcal{S})$ in $B$, we can define $f \in \mathrm{Hom}_R(F, B/A)$ by $f\rho(1_j) = \pi b_j$ (this is possible, because $\pi \restriction A = 0$). Then (a) yields $g \in \mathrm{Hom}_R(F, B)$, such that $\pi g = f$. Define $c_j = b_j - g\rho(1_j)$ $(j < n)$. Then $\pi(c_j) = 0$, so $c_j \in A$, and also $\sum_{j<n} c_j.r_{ji} = a_i$ for all $i < m$.

Assume (b). Let $F = R^n/G$ be a finitely presented module and consider $f \in \mathrm{Hom}_R(F, B/A)$. The equality (1) above and (b) yield the existence of $c_j \in A$ $(j < n)$ with $\sum_{j<n} c_j.r_{ji} = a'_i$ $(i < m)$. Hence we can define $g \in \mathrm{Hom}_R(F, B)$ by $g\rho(1_j) = b'_j - c_j$. Then $\pi g\rho(1_j) = \pi(b'_j) = f\rho(1_j)$, so $\pi g = f$, and (a) holds. □

We will also need the following elementary properties of purity in Mod–$R$:

**Lemma 8.3.** *Let* $\lambda \geq |R| + \aleph_0$.

(a) *Let* $M$ *be a module and* $X$ *a subset of* $M$ *with* $|X| \leq \lambda$. *Then there is a pure submodule* $N \subseteq_* M$, *such that* $X \subseteq N$ *and* $|N| \leq \lambda$.
(b) *Assume* $C \subseteq B \subseteq A$, $C \subseteq_* A$ *and* $B/C \subseteq_* A/C$. *Then* $B \subseteq_* A$.
(c) *If* $A \subseteq_* B$ *and* $B \subseteq_* C$, *then* $A \subseteq_* C$.
(d) *Assume* $A_0 \subseteq \cdots \subseteq A_\alpha \subseteq A_{\alpha+1} \subseteq \cdots$ *is a chain of pure submodules of* $M$. *Then* $\bigcup_\alpha A_\alpha$ *is a pure submodule of* $M$.

*Proof.* (a) We apply the characterization (b) from Lemma 8.2 to define $N = \bigcup_{i<\omega} N_i$, where $N_0$ is the submodule generated by $X$, and $N_{i+1}$ is the submodule generated by solutions in $M$ of all the $R$-linear equations with right-hand side in $N_i$. Since $\lambda \geq |R| + \aleph_0$ and $|X| \leq \lambda$, we can assume that $|N_{i+1}| \leq \lambda$, and (a) easily follows.

(b) follows directly from Definition 8.1, since the canonical projection $\pi_B : A \to A/B$ is a composition of two canonical projections which are pure epimorphisms by assumption: $\pi_B = \pi_{B/C}\pi_C$.

(c) and (d) follow by Lemma 8.2 (b). □

The submodule $N$ constructed in part (a) is called the *purification* of the subset $X$ in $M$.

Recall that a module $M$ is *flat*, iff $\mathrm{Tor}^1_R(M, F) = 0$ for each left $R$-module $F$. Let $\mathcal{F}_0$ denote the class of all flat modules. By the Ext-Tor relations, $\mathcal{F}_0 = {}^\perp\mathcal{D}$, where $\mathcal{D}$ is the class of all duals of all left $R$-modules (i.e., $\mathcal{D} = \{\mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z}) \mid F \in R\text{–Mod}\}$).

So there is a cotorsion pair $(\mathcal{F}_0, \mathcal{C})$ in Mod–$R$. The modules in the class $\mathcal{C}$ are called *cotorsion*.

We now recall a relation between purity and flatness:

**Lemma 8.4.** *Let $R$ be a ring. Let $\mathcal{E}$ be the short exact sequence*

$$0 \to A \overset{\subseteq}{\to} B \to C \to 0$$

*in* Mod–$R$. *Assume $B$ is a flat module.*

*Then $\mathcal{E}$ is pure, iff $C$ is a flat module. In this case, also $A$ is flat.*

*Proof.* Assume $\mathcal{E}$ is pure-exact. Let $F \in R$–Mod. Applying $- \otimes_R F$ to $\mathcal{E}$ we obtain a long exact sequence

$$\dots \to 0 = \mathrm{Tor}^1_R(B, F) \to \mathrm{Tor}^1_R(C, F) \to A \otimes_R F \to B \otimes_R F \to C \otimes_R F \to 0.$$

Condition (d) of Lemma 8.2 says that the map $A \otimes_R F \to B \otimes_R F$ is monic, so $\mathrm{Tor}^1_R(C, F) = 0$. Thus, $C$ is flat. Moreover, the long exact sequence for Tor gives $0 = \mathrm{Tor}^2_R(C, F) \to \mathrm{Tor}^1_R(A, F) \to \mathrm{Tor}^1_R(B, F) = 0$, hence also $A$ is a flat module.

Conversely, if $C$ is flat, then for each $F \in R$–Mod we have the short exact sequence $\mathrm{Tor}^1_R(C, F) = 0 \to A \otimes_R F \to B \otimes_R F \to C \otimes_R F \to 0$, whence $\mathcal{E}$ is pure-exact by condition (d) of Lemma 8.2. $\square$

Now, we can show that the class $\mathcal{F}_0$ of all flat modules is deconstructible:

**Lemma 8.5.** *Let $\lambda \geq |R| + \aleph_0$. Let $M$ be a flat module. Then $M$ is $\mathcal{F}_0^{\leq\lambda}$-filtered.*

*Proof.* By induction, we will construct a $\mathcal{F}_0^{\leq\lambda}$-filtration $(M_\alpha \mid \alpha \leq \sigma)$ of $M$ which will also be a chain of pure (and hence flat) submodules of $M$, as follows:

$M_0 = 0$. Assume $M_\alpha \subseteq_* M$ is defined and there exists $x \in M \setminus M_\alpha$. By Lemma 8.4, $M/M_\alpha$ is flat. By Lemma 8.3(a), there is a submodule $M_\alpha \subsetneq M_{\alpha+1} \subseteq M$ such that $x + M_\alpha \in M_{\alpha+1}/M_\alpha$, $|M_{\alpha+1}/M_\alpha| \leq \lambda$, and $M_{\alpha+1}/M_\alpha \subseteq_* M/M_\alpha$. By Lemma 8.3(b), $M_{\alpha+1} \subseteq_* M$.

If $\alpha$ is a limit ordinal, we let $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$. Then $M_\alpha \subseteq_* M$ by Lemma 8.3(d).

Since the chain constructed above is strictly increasing, there is a $\sigma$ such that $M_\sigma = M$, whence $(M_\alpha \mid \alpha \leq \sigma)$ is a $\mathcal{F}_0^{\leq\lambda}$-filtration of $M$. $\square$

**Theorem 8.6.** *For any ring $R$, $(\mathcal{F}_0, \mathcal{C})$ is a complete cotorsion pair in* Mod–$R$.

*Proof.* This follows by Lemma 8.5 and Theorem 6.1. $\square$

**Remark 8.7.** (i) For $n \geq 0$, denote by $\mathcal{F}_n$ the class of all modules of flat dimension $\leq n$. Then $(\mathcal{F}_n, \mathcal{F}_n^\perp)$ is a cotorsion pair, which is complete by Theorem 7.8.

(ii) The proof Theorem 8.6 gives as a by-product information about the structure of flat modules over arbitrary rings: each flat module is a transfinite extension of $\leq \kappa$-presented flat modules, where $\kappa = |R| + \aleph_0$. However, the proof gives no information on the structure of cotorsion modules in general. In fact, even for $R = \mathbb{Z}$, Theorem 8.6 implies that for each torsion group $T$ there exists a cotorsion group $C$ such that $T$ is the torsion part of $C$. Since a full classification of general (infinitely generated) torsion groups is considered hopeless, the same applies to cotorsion groups.

However, flat cotorsion modules over commutative noetherian rings are classified by their invariants (cf. [25, 5.3.28]). So one can use the existence of minimal flat resolutions (obtained by iterated flat covers from Theorem 9.5 below) to define and compute invariants of cotorsion modules in this setting dually to the classic Bass invariants of modules coming from their minimal injective coresolutions. For more details on this approach, we refer to [47, 5.2.2].

## 9. Minimal approximations, FCC, and the Enochs Problem

We have already seen in Section 7 that all cotorsion pairs generated by a set are complete. In particular, so are the cotorsion pairs $({}^{\perp}\mathcal{I}_n, \mathcal{I}_n)$, $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ and $(\mathcal{F}_n, \mathcal{F}_n^{\perp})$ for each $n \geq 0$.

In some cases, minimal approximations exist, that is, the cotorsion pairs are perfect in the sense of the following definition:

**Definition 9.1.** Let $R$ be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair.

(i) $\mathfrak{C}$ is called *perfect*, provided that $\mathcal{A}$ is a covering class and $\mathcal{B}$ is an enveloping class.
(ii) $\mathfrak{C}$ is called *closed*, provided that $\mathcal{A} = \varinjlim \mathcal{A}$, that is, the class $\mathcal{A}$ is closed under forming direct limits in Mod–$R$.

The term *perfect* comes from the classical result of Bass characterizing right perfect rings by the property that the cotorsion pair $(\mathcal{P}_0, \text{Mod–}R)$ is perfect.

By the Wakamatsu Lemma 4.10, any perfect cotorsion pair is complete. The converse fails in general: for example, the pair $(\mathcal{P}_0, \text{Mod–}R)$ is complete for any (not necessarily right perfect) ring.

In order to prove the existence of minimal approximations, we will use the following result by Enochs [23] (see also [47, §2.2]):

**Theorem 9.2.** *Let $R$ be a ring and $M$ be a module. Let $\mathcal{C}$ be a class of modules closed under direct limits.*

(a) *Assume moreover that $\mathcal{C}$ is closed under extensions, and that $M$ has a monic $\mathcal{C}^{\perp}$-preenvelope $\nu$ with* $\operatorname{Coker} \nu \in \mathcal{C}$*. Then $M$ has a $\mathcal{C}^{\perp}$-envelope.*
(b) *Assume that $M$ has a $\mathcal{C}$-precover. Then $M$ has a $\mathcal{C}$-cover.*

Before proving this theorem, we list some of its its immediate corollaries:

**Corollary 9.3.** *Let $R$ be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair. If $\mathfrak{C}$ is closed, then $\mathfrak{C}$ is perfect.*

*Proof.* This is immediate from Theorem 9.2 for $\mathcal{C} = \mathcal{A}$. □

**Remark 9.4.** Theorem 9.2 can be applied even to non-closed cotorsion pairs: Let $R$ be an integral domain, $Q$ be its quotient field, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be the (complete) cotorsion pair generated by $Q$. Let $\mathcal{C} = \text{Mod–}Q (\subseteq \text{Mod–}R)$. Then $\mathcal{B} = \mathcal{C}^{\perp}$, so $\mathcal{B}$ is an enveloping class of modules by Theorems 6.1(a) and 9.2(a).

However, the class $\mathcal{A}$ is covering only if $R$ is an *almost perfect* domain (i.e., all proper quotients of $R$ are perfect rings). This is true when $R$ is a Dedekind domain, but fails for any Prüfer domain which is not Dedekind. For more details, we refer to [6].

Corollary 9.3 and Theorem 8.6 yield the following celebrated result from [14]:

**Corollary 9.5** (The Flat Cover Conjecture (FCC))**.** *Let $R$ be a ring and $\mathfrak{C}_0 = (\mathcal{F}_0, \mathcal{C})$ be the cotorsion pair generated by the class of all flat modules. Then $\mathfrak{C}_0$ is perfect.*

Similarly, all the cotorsion pairs $\mathfrak{C}_n = (\mathcal{F}_n, (\mathcal{F}_n)^{\perp})$ $(n \geq 0)$ are perfect (see Remark 8.7).

*Proof of Theorem 9.2.* In order to prove part (a) of Theorem 9.2, we introduce more notation:

We will call an exact sequence $0 \to M \to F \to C \to 0$ with $C \in \mathcal{C}$ an *Ext-generator*, provided that for each exact sequence $0 \to M \to F' \to C' \to 0$ with $C' \in \mathcal{C}$ there exist $f \in \mathrm{Hom}_R(F', F)$ and $g \in \mathrm{Hom}_R(C', C)$, such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \| & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

is commutative. By assumption, there exists an Ext-generator with the middle term $F \in \mathcal{C}^{\perp}$. The proof of part (a) is divided into three steps/auxiliary lemmas:

**Lemma 9.6.** *Assume $0 \to M \to F \to C \to 0$ is an Ext-generator. Then there exist an Ext-generator $0 \to M \to F' \to C' \to 0$ and a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0, \end{array}$$

*such that* $\mathrm{Ker}(f) = \mathrm{Ker}(f'f)$ *in any commutative diagram whose rows are Ext-generators:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \| & & {\scriptstyle f'}\downarrow & & {\scriptstyle g'}\downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & F'' & \longrightarrow & C'' & \longrightarrow & 0. \end{array}$$

*Proof.* Assume that the assertion does not hold. By induction, we will construct a direct system of Ext-generators indexed by ordinals as follows:
First let the second row be the same as the first one, that is, put $F' = F_0 = F$, $C' = C_0 = C$, $f = \mathrm{id}_F$ and $g = \mathrm{id}_C$. Then there exist $F_1 = F''$, $C_1 = C''$, $f_{10} = f'$ and $g_{10} = g'$, such that the diagram above commutes, its rows are Ext-generators and $\mathrm{Ker}\, f_{10} \supsetneq \mathrm{Ker}\, f = 0$.

Assume that the Ext-generator $0 \to M \to F_\alpha \to C_\alpha \to 0$ is defined together with $f_{\alpha\beta} \in \mathrm{Hom}_R(F_\beta, F_\alpha)$ and $g_{\alpha\beta} \in \mathrm{Hom}_R(C_\beta, C_\alpha)$ for all $\beta \leq \alpha$. Then there exist $F_{\alpha+1}$, $C_{\alpha+1} \in \mathcal{C}$, $f_{\alpha+1,\alpha}$ and $g_{\alpha+1,\alpha}$, such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F_\alpha & \longrightarrow & C_\alpha & \longrightarrow & 0 \\ & & \| & & {\scriptstyle f_{\alpha+1,\alpha}}\downarrow & & {\scriptstyle g_{\alpha+1,\alpha}}\downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & F_{\alpha+1} & \longrightarrow & C_{\alpha+1} & \longrightarrow & 0 \end{array}$$

commutes, its rows are Ext-generators and $\mathrm{Ker}\, f_{\alpha+1,0} \supsetneq \mathrm{Ker}\, f_{\alpha 0}$, where $f_{\alpha+1,\beta} = f_{\alpha+1,\alpha} f_{\alpha\beta}$ and $g_{\alpha+1,\beta} = g_{\alpha+1,\alpha} g_{\alpha\beta}$ for all $\beta \leq \alpha$.

If $\alpha$ is a limit ordinal, put $F_\alpha = \varinjlim_{\beta<\alpha} F_\beta$ and $C_\alpha = \varinjlim_{\beta<\alpha} C_\beta$. Consider the direct limit $0 \to M \to F_\alpha \to C_\alpha \to 0$ of the Ext-generators $0 \to M \to F_\beta \to C_\beta \to 0$, $(\beta < \alpha)$. Since $\mathcal{C}$ is closed under direct limits, we have $C_\alpha \in \mathcal{C}$. Since $0 \to M \to F_\beta \to C_\beta \to 0$ is an Ext-generator for (some) $\beta < \alpha$, also $0 \to M \to F_\alpha \to C_\alpha \to 0$ is an Ext-generator.

Put $f_{\alpha\beta} = \varinjlim_{\beta \leq \beta' < \alpha} f_{\beta'\beta}$ and $g_{\alpha\beta} = \varinjlim_{\beta \leq \beta' < \alpha} g_{\beta'\beta}$ for all $\beta < \alpha$. Then $\mathrm{Ker}(f_{\alpha 0}) \supseteq \mathrm{Ker}(f_{\beta 0})$, and hence $\mathrm{Ker}(f_{\alpha 0}) \supsetneq \mathrm{Ker}(f_{\beta 0})$, for each $\beta < \alpha$.

By induction, for each ordinal $\alpha$ we obtain a strictly increasing chain $(\mathrm{Ker}\, f_{\beta 0} \mid \beta < \alpha)$, consisting of submodules of $F$, a contradiction. $\square$

**Lemma 9.7.** *Assume $0 \to M \to F \to C \to 0$ is an Ext-generator. Then there exist an Ext-generator $0 \to M \to F' \to C' \to 0$ and a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & M \arrow[r] & F' \arrow[r] & C' \arrow[r] & 0,
\end{tikzcd}$$

*such that* $\operatorname{Ker}(f') = 0$ *in any commutative diagram whose rows are Ext-generators:*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F' \arrow[r] \arrow[d, "f'"] & C' \arrow[r] \arrow[d, "g'"] & 0 \\
0 \arrow[r] & M \arrow[r] & F'' \arrow[r] & C'' \arrow[r] & 0.
\end{tikzcd}$$

*Proof.* By induction on $n < \omega$, we infer from Lemma 9.6 that there is a countable direct system $\mathcal{D}$ of Ext-generators $0 \to M \to F_n \to C_n \to 0$ with homomorphisms $f_{n+1,n} \in \operatorname{Hom}_R(F_n, F_{n+1})$, $g_{n+1,n} \in \operatorname{Hom}_R(C_n, C_{n+1})$, such that the 0-th term of $\mathcal{D}$ is the given Ext-generator $0 \to M \to F \to C \to 0$,

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F_n \arrow[r] \arrow[d, "f_{n+1,n}"] & C_n \arrow[r] \arrow[d, "g_{n+1,n}"] & 0 \\
0 \arrow[r] & M \arrow[r] & F_{n+1} \arrow[r] & C_{n+1} \arrow[r] & 0
\end{tikzcd}$$

is commutative, and for each commutative diagram whose rows are Ext-generators

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F_{n+1} \arrow[r] \arrow[d, "\bar{f}"] & C_{n+1} \arrow[r] \arrow[d, "\bar{g}"] & 0 \\
0 \arrow[r] & M \arrow[r] & \bar{F} \arrow[r] & \bar{C} \arrow[r] & 0
\end{tikzcd}$$

we have $\operatorname{Ker}(f_{n+1,n}) = \operatorname{Ker}(\bar{f} f_{n+1,n})$.

Consider the direct limit $0 \to M \to F' \to C' \to 0$ of $\mathcal{D}$, so $F' = \varinjlim_{n<\omega} F_n$ and $C' = \varinjlim_{n<\omega} C_n$. Since $\mathcal{C}$ is closed under direct limits, we have $C' \in \mathcal{C}$, and $0 \to M \to F' \to C' \to 0$ is an Ext-generator.

This Ext-generator has the required injectivity property. Indeed, let $f_n : F_n \to F'$ and $g_n : C_n \to C'$ ($n < \omega$) denote the direct limit maps. Then for $f = f_0$ and $g = g_0$, the first diagram in 9.7 is commutative. If $f'(x) = 0$ for some $x \in F'$ in the second diagram, then there exist $n < \omega$ and $x_n \in F_n$ such that $x = f_n(x_n) = f_{n+1} f_{n+1,n}(x_n)$. Then $f' f_{n+1} f_{n+1,n}(x_n) = 0$, so by the inductive step of our construction, $x_n \in \operatorname{Ker}(f_{n+1,n})$. Hence $x = f_{n+1} f_{n+1,n}(x_n) = 0$. $\square$

**Lemma 9.8.** *Let $0 \to M \xrightarrow{\nu} F' \xrightarrow{\pi} C' \to 0$ be the Ext-generator constructed in Lemma 9.7. Then $\nu : M \to F'$ is a $\mathcal{C}^\perp$-envelope of $M$.*

*Proof.* First we prove that in each commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F' \arrow[r] \arrow[d, "f'"] & C' \arrow[r] \arrow[d, "g'"] & 0 \\
0 \arrow[r] & M \arrow[r] & F' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

$f'$ is an automorphism.

Assume this is not true. By induction we construct a direct system of Ext-generators, $0 \to M \to F_\alpha \to C_\alpha \to 0$, indexed by ordinals, with injective, but not surjective, homomorphisms $f_{\alpha\beta} \in \operatorname{Hom}_R(F_\beta, F_\alpha)$ ($\beta < \alpha$). In view of Lemma 9.7, we take

$$0 \to M \to F_\alpha \to C_\alpha \to 0 = 0 \to M \xrightarrow{\nu} F' \xrightarrow{\pi} C' \to 0$$

in case $\alpha = 0$ or $\alpha$ non-limit and $F_\alpha = \varinjlim F_\beta$ and $C_\alpha = \varinjlim C_\beta$, if $\alpha$ is a limit ordinal. Then for each non-limit ordinal $\alpha$ ($\operatorname{Im} f_{\alpha\beta} \mid \beta$non-limit, $\beta < \alpha$) is a strictly increasing sequence of submodules of $F'$, a contradiction.

It remains to prove that $F' \in \mathcal{C}^\perp$. Consider an exact sequence $0 \to F' \xrightarrow{\mu} X \to C \to 0$, where $C \in \mathcal{C}$. We will prove that this sequence splits.
Consider the pushout of $\pi$ and $\mu$:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "\nu"] \arrow[d, equal] & F' \arrow[r, "\pi"] \arrow[d, "\mu"'] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & X \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
 & & C \arrow[r, equal] \arrow[d] & C \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

Since $\mathcal{C}$ is closed under extensions, we have $P \in \mathcal{C}$. Since $0 \to M \xrightarrow{\nu} F' \xrightarrow{\pi} C' \to 0$ is an Ext-generator, we also have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\mu'"'] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "\nu"] & F' \arrow[r, "\pi"] & C' \arrow[r] & 0.
\end{tikzcd}$$

By the first part of the proof, $\mu'\mu$ is an automorphism of $F'$. It follows that $0 \to F' \xrightarrow{\mu} X \to C \to 0$ splits. □

This finishes the proof of part (a) of Theorem 9.2.

The proof of part (b) is dual, so we just state the analogues of the three lemmas above for the dual settings, and only sketch their proofs:

**Lemma 9.9.** *For any $\mathcal{C}$-precover of $M$, $u \in \operatorname{Hom}_R(E, M)$, there exist a $\mathcal{C}$-precover of $M$, $v \in \operatorname{Hom}_R(F, M)$, and $f \in \operatorname{Hom}_R(E, F)$ such that*

(i) $vf = u$.

(ii) *If $w \in \operatorname{Hom}_R(G, M)$ is any $\mathcal{C}$-precover of $M$, and $g \in \operatorname{Hom}_R(F, G)$ satisfies $wg = v$, then* $\operatorname{Ker}(gf) = \operatorname{Ker}(f)$.

$$\begin{tikzcd}
E \arrow[r, "u"] \arrow[d, "f"'] & M \arrow[d, equal] \\
F \arrow[r, "v"] \arrow[d, "g"'] & M \arrow[d, equal] \\
G \arrow[r, "w"] & M
\end{tikzcd}$$

*Proof.* If the claim does not hold for a $\mathcal{C}$-precover $u \in \operatorname{Hom}_R(E, M)$, we can proceed by transfinite induction so that the kernels of the compositions of the left vertical maps form a strictly increasing continuous chain of submodules of $E$ indexed by ordinals (in limit steps, we just take direct limits, which is possible because $\mathcal{C}$ is closed under direct limits). This is a contradiction, since each such chain must have cardinality $\leq |E|$. $\square$

**Lemma 9.10.** *There exists a $\mathcal{C}$-precover $u \in \operatorname{Hom}_R(E, M)$ such that, if $v \in \operatorname{Hom}_R(F, M)$ is any $\mathcal{C}$-precover of $M$ and $f \in \operatorname{Hom}_R(E, F)$ is such that $vf = u$, then $f$ is injective.*

$$\begin{array}{ccc} E & \xrightarrow{u} & M \\ {\scriptstyle f}\downarrow & & \| \\ F & \xrightarrow{v} & M \end{array}$$

*Proof.* Starting with any $\mathcal{C}$-precover of $M$, we can iteratively use Lemma 9.9 and build a chain of $\mathcal{C}$-precovers $u_n \in \operatorname{Hom}_R(E_n, M)$ and $g_n \in \operatorname{Hom}_R(E_n, E_{n+1})$ with $u_n = u_{n+1}g_n$ $(n < \omega)$. Then $u = \varinjlim_{n<\omega} u_n \in \operatorname{Hom}_R(\varinjlim_{n<\omega} E_n, M)$ will be the desired $\mathcal{C}$-precover of $M$. $\square$

**Lemma 9.11.** *The $\mathcal{C}$-precover $u$ of $M$ constructed in Lemma 9.10 is a $\mathcal{C}$-cover.*

$$\begin{array}{ccc} E & \xrightarrow{u} & M \\ {\scriptstyle f}\downarrow & & \| \\ E & \xrightarrow{u} & M \end{array}$$

*Proof.* Assume there exists $f \in \operatorname{End} E$ such that $fu = u$ and $f$ is not an isomorphism. Then $f$ is monic by Lemma 9.10. By transfinite induction, we can construct for each ordinal $\alpha$ a strictly increasing chain of length $\alpha$ whose non-limit terms are copies of $E$ (in successors of limit steps, we factorize the direct limit of the chain, which is a module in $\mathcal{C}$ by assumption, via $u$ using the fact that $u$ is a $\mathcal{C}$-precover). This leads to a contradiction when $|\alpha| > |E|$. $\square$

This finishes the proof of part (b) of Theorem 9.2. $\square$

Theorem 9.2(b) remains true in an arbitrary Grothendieck category. Moreover, the following result was proved in [22]:

**Theorem 9.12.** *$\mathcal{C}$ is a covering class, whenever $\mathcal{C}$ is a class of objects in a Grothendieck category, such that $\mathcal{C}$ is closed under coproducts and directed colimits and there is a set of objects $\mathcal{S} \subseteq \mathcal{C}$, such that each object of $\mathcal{C}$ is a directed colimit of objects from $\mathcal{S}$.*

*Assuming the large cardinal principle VP (Vopěnka's Principle), any class of objects closed under coproducts and directed colimits is covering.*

**Remark 9.13.** We are mainly interested in approximations by classes occurring in (not necessarily closed) cotorsion pairs, so we deal a priori with classes of modules closed under extensions. However, Theorems 9.2(b) and 9.12 apply also to non-extension closed classes of modules, such as the class of all $\Gamma$-separated modules over Dedekind-like rings studied by Klingler and Levy in [32]. For more details on this application of approximation theory, see [33].

Whether covering classes with additional properties have to take a particular form may depend on extra set-theoretic assumptions. For example, in [13], it was proved that the statement "each covering class of modules closed under homomorphic images coincides with the class of all modules generated by a single module" is equivalent to Vopěnka's Principle (VP).

The question whether the converse of Corollary 9.3 holds true is a major open problem of the approximation theory, due to Enochs:

**Problem 9.14.** *1. Is any perfect cotorsion pair closed?*

A positive answer is known for various particular cases, notably for the case of tilting cotorsion pairs, and more in general, of the cotorsion pairs $(\mathcal{A}, \mathcal{B})$ such that the class $\mathcal{B}$ is closed under direct limits, see [4, Corollary 5.5] (and also [10, Sect. 7]). A still more general result has recently been proven in [11], providing a positive answer for all cotorsion pairs generated by a class of $\leq \aleph_n$-presented modules for a fixed $n < \omega$.

*2.* (The Enochs Problem) *Is any covering class of modules $\mathcal{C}$ closed under direct limits?*

For $\mathcal{C} = \operatorname{Add} M$ (= the class of all direct summands of arbitrary direct sums of copies of a module $M$), a positive answer has been obtained in [42] for two cases: (1) when there exists an $n < \omega$ such that $M$ is a direct sum of $\leq \aleph_n$-generated modules, and (2) for an arbitrary module $M$, but under the extra set-theoretic assumption of the existence of a proper class of cardinals $\kappa$ such that each stationary set in $\kappa^+$ contains a non-reflecting stationary subset. The latter is known to hold, e.g., under Gödel's Axiom of Constructibility.

In the case when $\mathcal{C} = \operatorname{Filt}(\mathcal{S})$ for a set of modules $\mathcal{S}$, consistency of a positive answer has recently been proven in [11].

## 10. Tilting modules

Our next goal is to study relations between the approximation theory and tilting.

We start by introducing an infinitely generated version of the classic notion of a tilting module, and of the associated tilting classes, going back to [16] and [3]. We will also briefly sketch their role in generalizing the classical Morita theory of equivalences.

**Definition 10.1.** Let $R$ be a ring. A module $T$ is *tilting*, provided that

(T1) $T$ has finite projective dimension (i.e., $T \in \mathcal{P}$),
(T2) $\operatorname{Ext}^i_R(T, T^{(\kappa)}) = 0$ for all $1 \leq i < \omega$ and all cardinals $\kappa$, and
(T3) There are $r \geq 0$ and a long exact sequence $0 \to R \to T_0 \to \dots \to T_r \to 0$, where $T_i \in \operatorname{Add}(T)$ for all $i \leq r$.

If $n < \omega$ and $T$ is tilting of projective dimension $\leq n$, then $T$ is called *$n$-tilting.* The class $T^{\perp_\infty}$ is called the *(right) $n$-tilting class* induced by $T$. Clearly,

$$({}^\perp(T^{\perp_\infty}), T^{\perp_\infty})$$

is a hereditary cotorsion pair, called the *$n$-tilting cotorsion pair* induced by $T$. The class ${}^\perp(T^{\perp_\infty})$ is the *left $n$-tilting class* induced by $T$.

If $T$ and $T'$ are tilting modules, then $T$ is said to be *equivalent* to $T'$, provided that the induced tilting classes coincide, that is, $T^{\perp_\infty} = (T')^{\perp_\infty}$.

We will call a module $M$ is *strongly finitely presented* (*strongly countably presented*), in case $M$ has a projective resolution consisting of finitely (countably) generated modules. A strongly finitely presented tilting module $T$ is called *classical.* A tilting module $T$ is called *good*, if $T$ satisfies the stronger condition (T3$'$) obtained from (T3) by replacing the class $\operatorname{Add}(T)$ with $\operatorname{add}(T)$. This condition is not particularly restrictive, because each tilting module is equivalent to a good tilting module (see Remark 11.4 below).

It is easy to see that 0-tilting modules are nothing else than the (not necessarily finitely generated) projective generators. Hence classical 0-tilting modules coincide with the progenerators.

Progenerators play a key role in the Morita theory of equivalence. This theory extends to an arbitrary $n \geq 0$: for any classical tilting module $T$ and any $i \leq n = \operatorname{proj\,dim} T$ there is a *tilting category equivalence* [34] between the categories $\bigcap_{j\neq i} \operatorname{Ker} \operatorname{Ext}^j_R(T,-)$ and $\bigcap_{j\neq i} \operatorname{Ker} \operatorname{Tor}^S_j(-,T)$ (where $S = \operatorname{End}(T_R)$):

$$(2) \qquad \bigcap_{j\neq i} \operatorname{Ker} \operatorname{Ext}^j_R(T,-) \underset{\operatorname{Tor}^S_i(-,T)}{\overset{\operatorname{Ext}^i_R(T,-)}{\rightleftarrows}} \bigcap_{j\neq i} \operatorname{Ker} \operatorname{Tor}^S_j(-,T)\,.$$

Notice that for $n = 0$, the tilting category equivalence is just the Morita equivalence between Mod–$R$ and Mod–$S$.

**Remark 10.2.** One can proceed and view the tilting equivalence as an instance of an equivalence between the (triangulated) bounded derived categories $D^b(\text{mod–}R)$ and $D^b(\text{mod–}S)$ of the rings $R$ and $S$, respectively, [36]. Further, one can extend the equivalence (2) to the case when $T$ is a good tilting module; however, the right hand classes then have to be restricted to the class $\mathcal{F} = \{X \in \mathcal{D}(S) \mid \operatorname{Hom}_{\mathcal{D}(S)}(\mathcal{E}, X) = 0\}$ where $\mathcal{E}$ is the kernel of the total left derived functor $\mathbb{L}(- \otimes_S T)$. Again, this is just an instance of the equivalence of triangulated categories

$$\mathcal{D}(R) \underset{\mathbb{R}(\operatorname{Hom}_R(T,-))}{\overset{\mathbb{L}(-\otimes_S T)}{\leftrightarrows}} \mathcal{E}_\perp\,,$$

where $\mathcal{D}(R)$ is the (unbounded) derived category of the ring $R$ and $\mathbb{R}(\operatorname{Hom}_R(T,-))$ is the total right derived functor of $\operatorname{Hom}_R(T,-)$. We refer to [9] for more details, and to [35] and [30] for further recent generalizations involving contramodules.

For $n \geq 1$, there is a rich supply of non-projective finitely generated $n$-tilting modules over artin algebras (cf. [5, Chap. VI]), but the situation is completely different for commutative rings. The following was first observed in the particular case of 1-tilting modules in [15].

**Lemma 10.3.** *Let $R$ be a commutative ring and $T$ be a strongly finitely presented module.*

(a) *Assume that* $1 \leq n = \operatorname{proj\,dim} T < \infty$. *Then* $\operatorname{Ext}^n_R(T,T) \neq 0$.
(b) *If $T$ is classical tilting, then $T$ is projective.*

*Proof.* (a) By assumption, there exists a projective resolution $\mathcal{R}$ of $T$ consisting of finitely generated modules. Let $M = \Omega^{(n-1)}(T)$ be the $(n-1)$th syzygy of $T$ in $\mathcal{R}$. Then $M$ is a finitely presented module of projective dimension 1, so there is a maximal ideal $m$ of $R$, such that $\operatorname{proj\,dim} R_{(m)} M_{(m)} = 1$. Moreover, $M_{(m)}$ is the $(n-1)$th syzygy of $T_{(m)}$ in $\mathcal{R}_m$, where $\mathcal{R}_m$ is the free resolution of the $R_{(m)}$-module $T_{(m)}$ obtained by applying the localisation functor $- \otimes_R R_{(m)}$ to $\mathcal{R}$.

Assume $\operatorname{Ext}^n_R(T,T) = 0$. Then $\operatorname{Ext}^1_R(M,T) \cong \operatorname{Ext}^n_R(T,T) = 0$, so localising, we obtain $\operatorname{Ext}^1_{R_{(m)}}(M_{(m)}, T_{(m)}) = 0$. Since $0 \neq T_{(m)}$ is finitely generated, $T_{(m)}$ contains a maximal $R_{(m)}$-submodule, and because $M_{(m)}$ has projective dimension 1, also $\operatorname{Ext}^1_{R_{(m)}}(M_{(m)}, R_{(m)}/m_{(m)}) = 0$.

Since $R_{(m)}$ is a local ring, the finitely presented $R_{(m)}$-module $M_{(m)}$ has a projective (= free) cover $F$ whose kernel $K$ is finitely generated. As $F/K \cong M_{(m)}$ has projective dimension 1, certainly $K \neq 0$. Then $K$ contains a maximal submodule $L$, so $K/L \cong R_{(m)}/m_{(m)}$ since $R_{(m)}$ is local. By the above $\operatorname{Ext}^1_{R_{(m)}}(M_{(m)}, K/L) = 0$, whence the projection $\pi : K \to K/L$ can be extended to $\sigma \in \operatorname{Hom}_{R_{(m)}}(F, K/L)$.

Then Ker $\sigma$ is a maximal submodule of $F$, so $K \subseteq \mathrm{Rad}(F) \subseteq \mathrm{Ker}\ \sigma$, and $\pi = \sigma \restriction K = 0$, a contradiction.

(b) By part (a). $\square$

In order to introduce interesting examples of tilting modules and their applications, we will concentrate on the setting of Iwanaga-Gorenstein rings (which generalizes $n$-Gorenstein rings from classic commutative algebra).

**Example 10.4.** Let $R$ be an *Iwanaga-Gorenstein ring*, that is, a left and right noetherian ring with finite injective dimension on either side. Then the left and the right injective dimensions of $R$ coincide with some $n < \omega$, and $R$ is called *$n$-Iwanaga-Gorenstein*.

If $R$ is $n$-Iwanaga-Gorenstein, then all (left or right) $R$-modules of finite injective (projective, flat) dimension have injective (projective, flat) dimension $\leq n$, so in Mod–$R$, we have $\mathcal{P} = \mathcal{P}_n = \mathcal{I} = \mathcal{I}_n = \mathcal{F}_n$ (for more detail, see [25, §9.1]).

0-Iwanaga-Gorenstein rings are also called QF-rings, and they include all group algebras of finite groups over arbitrary fields. Also, each Dedekind domain is easily seen to be a 1-Iwanaga-Gorenstein ring.

We will stop at the 1-dimensional commutative case for a moment:

**Example 10.5** (Bass tilting modules)**.** 1-Gor Let $R$ be a commutative 1-Iwanaga-Gorenstein ring (that is, a commutative noetherian ring with $\operatorname{inj\,dim} R \leq 1$). Let $P_0$ and $P_1$ denote the sets of all prime ideals of height 0 and 1, respectively. By a classical result of Bass, the minimal injective coresolution of $R$ has the form

$$0 \to R \to G \xrightarrow{\pi} \bigoplus_{p \in P_1} E(R/p) \to 0,$$

where $G = \bigoplus_{q \in P_0} E(R/q)$.

Consider a subset $P \subseteq P_1$. Put $R_P = \pi^{-1}(\bigoplus_{p \in P} E(R/p))$ and $T_P = R_P \oplus \bigoplus_{p \in P} E(R/p)$. We will show that $T_P$ is a 1-tilting module (called the *Bass tilting module*).

First we have $R_P/R \cong \bigoplus_{p \in P} E(R/p)$ and $G/R_P \cong \bigoplus_{p \in P_1 \setminus P} E(R/p)$. Since both $R_P$ and $R_P/R$ have injective (equivalently, projective) dimension $\leq 1$, so does $T_P$. As $\mathrm{Hom}_R(E(R/p), G/R_P) = 0$, we see that $\mathrm{Ext}^1_R(E(R/p), R_P) = 0$ for all $p \in P$, and similarly $\mathrm{Ext}^1_R(E(R/p), R_P^{(\kappa)}) = 0$ for all $p \in P$ and $\kappa > 0$.

Finally, the exact sequence $0 \to R \to R_P \to \bigoplus_{p \in P} E(R/p) \to 0$ yields condition (T3) for $T_P$, and also the equality $\mathrm{Ext}^1_R(R_P, R_P^{(\kappa)}) = 0$ for each cardinal $\kappa$, and thus condition (T2) for $T_P$.

The 1-tilting class induced by $T_P$ is $\{M \mid \mathrm{Ext}^1_R(E(R/p), M) = 0 \text{ for all } p \in P\}$. This class equals $\{M \mid \mathrm{Ext}^1_R(R/p, M) = 0 \text{ for all } p \in P\}$, in case $R$ is hereditary (in particular, when $R$ is a Dedekind domain).

**Remark 10.6.** In fact, each tilting module over a commutative 1-Iwanaga-Gorenstein ring is equivalent to a Bass tilting module $T_P$ for a unique subset $P \subseteq P_1$, so tilting modules are parametrized by the subsets of $P_1$, cf. [45].

We will finish this section by a simple example of an infinitely generated $n$-tilting module over an $n$-Iwanaga-Gorenstein ring which will come in handy later on, for studying finitistic dimensions of Iwanaga-Gorenstein rings.

**Example 10.7.** Let $n \geq 0$ and $R$ be an $n$-Iwanaga-Gorenstein ring. Let $0 \to R \to I_0 \to \cdots \to I_n \to 0$ be the minimal injective coresolution of $R$.

Then $T = \bigoplus_{i \leq n} I_i$ is an $n$-tilting module: Indeed, since $T$ is injective, $T$ has projective dimension $\leq n$, so condition (T1) of Definition 10.1 is satisfied. Since $R$ is noetherian, $T^{(\kappa)}$ is also injective, so (T2) holds. The minimal injective coresolution above yields condition (T3).

## 11. Tilting and approximations

Now we can continue with the basic properties of tilting cotorsion pairs:

**Lemma 11.1.** *Let $R$ be a ring and $T$ be an $n$-tilting module. Denote by $\mathfrak{T} = (\mathcal{A}, \mathcal{B})$ the $n$-tilting cotorsion pair induced by $T$.*

(a) *Let $0 \to P_n \to \cdots \to P_0 \to T \to 0$ be a projective resolution of $T$ with the syzygy modules $S_0 = T, \dots, S_n = P_n$. Let $S = \bigoplus_{i \leq n} S_i$. Then $\mathfrak{T}$ is the cotorsion pair generated by $S$. In particular, $\mathfrak{T}$ is complete.*

(b) *$\mathcal{A} \subseteq \mathcal{P}_n$ and $\mathcal{B} \subseteq \mathrm{Gen}(T)$, where $\mathrm{Gen}(T)$ denotes the class of all homomorphic images of (possibly infinite) direct sums of copies of $T$.*

*Each of the short exact sequences forming the long exact sequence in (T3) is given by a special $\mathcal{B}$-preenvelope of an element of $\mathcal{A}$. The length $r$ in (T3) can be taken $\leq n$.*

(c) *The kernel of $\mathfrak{T}$ equals $\mathrm{Add}(T)$.*

(d) *Let $m \geq 0$ and $M \in \mathcal{B} \cap \mathcal{P}_m$. Then $M$ has $\mathrm{Add}(T)$-resolution dimension $\leq m$.*

*Proof.* (a) We have $T^{\perp_\infty} = \bigcap_{i \leq n} S_i^{\perp} = S^{\perp}$.

(b) By assumption, $S \in \mathcal{P}_n$, so $\mathcal{A} \subseteq \mathcal{P}_n$.

Let $M \in \mathcal{B}$. Consider the long exact sequence from (T3):

$$0 \to R \xrightarrow{\varphi} T_0 \xrightarrow{\varphi_0} T_1 \xrightarrow{\varphi_1} \dots \xrightarrow{\varphi_{r-1}} T_r \xrightarrow{\varphi_r} 0.$$

Since $T_i \in \mathcal{A}$ for all $i \leq r$, and $\mathcal{A}$ is resolving (because $\mathfrak{T}$ is hereditary, see Lemma 5.7), we have $K_i = \mathrm{Ker}(\varphi_i) \in \mathcal{A}$. In particular, $K_i \in \mathcal{P}_n$. Let $f : R^{(\lambda)} \to M$ be an epimorphism and put $g = \varphi^{(\lambda)}$. Consider the exact sequence $0 \to R^{(\lambda)} \xrightarrow{g} T_0^{(\lambda)} \to K_1^{(\lambda)} \to 0$, and form the pushout of $f$ and $g$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R^{(\lambda)} & \xrightarrow{g} & T_0^{(\lambda)} & \longrightarrow & K_1^{(\lambda)} & \longrightarrow & 0 \\
& & f\big\downarrow & & h\big\downarrow & & \big\| & & \\
0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & K_1^{(\lambda)} & \longrightarrow & 0 \\
& & \big\downarrow & & \big\downarrow & & & & \\
& & 0 & & 0. & & & &
\end{array}$$

Since $M \in \mathcal{B}$, the second row splits, so $M$ is a direct summand in $G$. Since $h$ is surjective, $G \in \mathrm{Gen}(T_0) \subseteq \mathrm{Gen}(T)$, and $M \in \mathrm{Gen}(T)$. This proves that $\mathcal{B} \subseteq \mathrm{Gen}(T)$.

By (T2), $T_i \in \mathrm{Add}(T) \subseteq \mathcal{B}$ for all $i \leq r$. So the embedding $K_i \hookrightarrow T_i$ is a special $\mathcal{B}$-preenvelope of $K_i \in \mathcal{A}$ and $\mathrm{proj\,dim}\, K_i \leq n$ for each $i \leq r$. If $n < r$, then the short exact sequence $0 \to K_n \to T_n \to K_{n+1} \to 0$ splits, since $\mathrm{Ext}^1_R(K_{n+1}, K_n) \cong \dots \cong \mathrm{Ext}^n_R(K_{n+1}, K_1) \cong \mathrm{Ext}^{n+1}_R(K_{n+1}, K_0) = 0$. So we can assume $r \leq n$ in (T3).

(c) By (T2), $\mathrm{Add}(T) \subseteq \mathcal{A} \cap \mathcal{B}$.

Conversely, let $M \in \mathcal{A} \cap \mathcal{B}$. By part (b), $M \in \mathrm{Gen}(T)$. So the canonical map $\varphi \in \mathrm{Hom}_R(T^{(\mathrm{Hom}_R(T,M))}, M)$ is surjective, and there is a short exact sequence

$$0 \to L \to T^{(\mathrm{Hom}_R(T,M))} \xrightarrow{\varphi} M \to 0. \tag{3}$$

Applying $\mathrm{Hom}_R(T,-)$ to (3), we obtain the long exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_R(T,L) \to \mathrm{Hom}_R(T,T^{(\mathrm{Hom}_R(T,M))}) \overset{\mathrm{Hom}_R(T,\varphi)}{\to} \mathrm{Hom}_R(T,M) \\ &\to \mathrm{Ext}^1_R(T,L) \to \mathrm{Ext}^1_R(T,T^{(\mathrm{Hom}_R(T,M))}) \to \mathrm{Ext}^1_R(T,M) \to \dots \\ &\dots \to \mathrm{Ext}^i_R(T,L) \to \mathrm{Ext}^i_R(T,T^{(\mathrm{Hom}_R(T,M))}) \to \mathrm{Ext}^i_R(T,M) \to \dots . \end{aligned}$$

Since $\mathrm{Hom}_R(T,\varphi)$ is surjective, $\mathrm{Ext}^1_R(T,L)=0$ by (T2). As $\mathrm{Ext}^i_R(T,M)=0$ for all $0<i<\omega$, condition (T2) also implies that $L\in T^{\perp_\infty}=\mathcal{B}$. Since $M\in\mathcal{A}$, (3) splits, and $M\in\mathrm{Add}(T)$.

(d) Let $M\in\mathcal{B}\cap\mathcal{P}_m$. An iteration of special $\mathcal{A}$-precovers (of $M$ etc.) gives rise to a long exact sequence

$$0\to K_m\to E_m \xrightarrow{\psi_n} E_{m-1} \xrightarrow{\psi_{m-1}} \dots \xrightarrow{\psi_1} E_0 \xrightarrow{\psi_0} M\to 0,$$

where $E_i\in\mathrm{Add}(T)$, $K_i=\mathrm{Ker}\,\psi_i\in\mathcal{B}$ and $\psi_i$ induces a special $\mathcal{A}$-precover of its image for all $i\le m$. By assumption, $M\in\mathcal{P}_m$, so $\mathrm{Ext}^1_R(K_{m-1},K_m)\cong\dots\cong\mathrm{Ext}^m_R(K_0,K_m)\cong\mathrm{Ext}^{m+1}_R(M,K_m)=0$. It follows that $K_{m-1}\in\mathrm{Add}(T)$, so we can take $E_m=K_{m-1}$ and $K_m=0$. □

If $T$ is a tilting module, then its projective dimension is the maximum of projective dimensions of the modules in $\mathcal{A}={}^{\perp}(T^{\perp_\infty})$. In particular, by Lemma 11.1(b), equivalent tilting modules have equal projective dimensions.

By Lemma 11.1(c), the kernel of $\mathfrak{T}$ equals $\mathrm{Add}(T)$. The classes $\mathcal{A}$ and $\mathcal{B}$ can be recovered from the kernel simply using the equalities $\mathcal{B}=(\mathrm{Add}(T))^{\perp_\infty}$ and $\mathcal{A}={}^{\perp}\mathcal{B}$.

There is another way of recovering $\mathcal{A}$ and $\mathcal{B}$ from the kernel, via $\mathrm{Add}(T)$-resolutions and $\mathrm{Add}(T)$-coresolutions in the sense of Definition 5.5:

**Proposition 11.2.** *Let $R$ be a ring, $T$ be an $n$-tilting module and $(\mathcal{A},\mathcal{B})$ the $n$-tilting cotorsion pair induced by $T$.*

(a) *$\mathcal{A}$ equals the class of all $\mathrm{Add}(T)$-coresolved modules of $\mathrm{Add}(T)$-coresolution dimension $\le n$.*
(b) *$\mathcal{B}$ equals the class of all $\mathrm{Add}(T)$-resolved modules. In particular, $\mathcal{B}$ is closed under direct sums.*

*Proof.* (a) Since $\mathcal{A}$ is resolving, $M\in\mathcal{A}$ for any module $M$ of finite $\mathrm{Add}(T)$-coresolution dimension.

Conversely, let $A\in\mathcal{A}$. An iteration of special $\mathcal{B}$-preenvelopes (of $A$ etc.) yields a long exact sequence

$$0\to A\to E_0 \xrightarrow{\psi_0} E_1 \xrightarrow{\psi_1} \dots \xrightarrow{\psi_{n-1}} E_n \xrightarrow{\psi_n} K_{n+1}\to 0,$$

where $E_i\in\mathrm{Add}(T)$ for all $i\le n$ and $K_{n+1}\in\mathcal{A}$. Let $K_i=\mathrm{Ker}\,\psi_i$ ($i\le n$). By Lemma 11.1(b), $K_{n+1}\in\mathcal{P}_n$, so $\mathrm{Ext}^1_R(K_{n+1},K_n)\cong\dots\cong\mathrm{Ext}^n_R(K_{n+1},K_1)\cong\mathrm{Ext}^{n+1}_R(K_{n+1},A)=0$. It follows that $K_{n+1}\in\mathrm{Add}(T)$, so we can take $E_n=K_n$ and $K_{n+1}=0$.

(b) If $M\in\mathcal{B}$, then an $\mathrm{Add}(T)$-resolution is obtained by an iteration of special $\mathcal{A}$-precovers (of $M$ etc.).

Conversely, assume there exists an $\mathrm{Add}(T)$-resolution $B$

$$\dots\to E_n\to\dots\to E_0\to B\to 0.$$

Denote by $K_0$ the kernel of the epimorphism $E_0\to B$, by $K_1$ the kernel of the epimorphism $E_1\to K_0$, etc. Let $A\in\mathcal{A}$. Then $\mathrm{Ext}^1_R(A,B)\cong\mathrm{Ext}^2_R(A,K_0)\cong\dots\cong\mathrm{Ext}^{n+1}_R(A,K_{n-1})=0$ by Lemma 11.1(b), so $B\in\mathcal{B}$. □

Now we are in a position to give a characterization of tilting classes of modules, going back to [3]:

**Theorem 11.3.** *Let $R$ be a ring, $n < \omega$ and $\mathcal{C}$ be a class of modules. Then the following assertions are equivalent:*

(a) *$\mathcal{C}$ is $n$-tilting.*

(b) *$\mathcal{C}$ is coresolving, special preenveloping, closed under direct sums and direct summands and ${}^{\perp}\mathcal{C} \subseteq \mathcal{P}_n$.*

*Proof.* (a) implies (b): this follows from parts (a) and (b) of Lemma 11.1, and from Proposition 11.2(b).

(b) implies (a): first, the special $\mathcal{C}$-preenvelope of any injective module splits. Since $\mathcal{C}$ is closed under direct summands and it is coresolving, we have $\mathcal{I}_0 \subseteq \mathcal{C}$ and $\mathcal{C}$ is cosyzygy closed. So ${}^{\perp_\infty}\mathcal{C} = {}^{\perp}\mathcal{C}$.

The special $\mathcal{C}$-preenvelope of $R$ gives rise to a short exact sequence $0 \to K_0 \to T_0 \to K_1 \to 0$, where $K_0 = R$, $T_0 \in \mathcal{C}$ and $K_1 \in {}^{\perp}\mathcal{C} \subseteq \mathcal{P}_n$. Since $R \in {}^{\perp}\mathcal{C}$, we have $T_0 \in \mathcal{C} \cap {}^{\perp}\mathcal{C}$. By induction we obtain short exact sequences $0 \to K_i \to T_i \to K_{i+1} \to 0$ with $T_i \in \mathcal{C} \cap {}^{\perp}\mathcal{C}$ and $K_{i+1} \in {}^{\perp}\mathcal{C} \subseteq \mathcal{P}_n$. Since $K_{n+1} \in \mathcal{P}_n$, the sequence $0 \to K_n \to T_n \to K_{n+1} \to 0$ splits by dimension shifting. So we can assume that $K_{n+1} = 0$, and form the long exact sequence (with $T_i \in \mathcal{C} \cap {}^{\perp}\mathcal{C}$ for all $i \leq n$)

$$0 \to R \overset{\varphi_0}{\to} T_0 \overset{\varphi_1}{\to} T_1 \overset{\varphi_2}{\to} \dots \overset{\varphi_{n-1}}{\to} T_{n-1} \overset{\varphi_n}{\to} T_n \to 0. \tag{4}$$

Put $T = \bigoplus_{i \leq n} T_i$. We will prove that $T$ is $n$-tilting. First, $T \in \mathcal{C} \cap {}^{\perp}\mathcal{C} \subseteq \mathcal{P}_n$, so (T1) holds. Since $\mathcal{C}$ is closed under direct sums, $T^{(\kappa)} \in \mathcal{C}$ for each cardinal $\kappa$, and (T2) holds. The long exact sequence above gives (T3).

Finally, we will prove that $T^{\perp_\infty} = \mathcal{C}$. Since $T \in {}^{\perp}\mathcal{C}$, clearly $T^{\perp_\infty} \supseteq \mathcal{C}$. Conversely, let $C \in T^{\perp_\infty}$. Consider a special $\mathcal{C}$-preenvelope $\psi_0$ of $C$, a special $\mathcal{C}$-preenvelope $\psi_1$ of $\operatorname{Coker} \varphi_0$ etc. Since $\operatorname{Coker} \psi_{n+1} \in \mathcal{P}_n$, dimension shifting shows that $\psi_{n+1}$ splits. So there is a long exact sequence

$$0 \to C \overset{\psi_0}{\to} D_0 \overset{\psi_1}{\to} D_1 \overset{\psi_2}{\to} \dots \overset{\psi_{n-1}}{\to} D_{n-1} \overset{\psi_n}{\to} D_n \to 0$$

with $D_i \in \mathcal{C} \subseteq T^{\perp_\infty}$ for all $i < n$, and $D_n \in \mathcal{C} \cap {}^{\perp}\mathcal{C}$. Since $C \in T^{\perp_\infty}$ and $T^{\perp_\infty}$ is coresolving, we get $\operatorname{Coker} \psi_i \in T^{\perp_\infty}$ for all $i \leq n$. It remains to prove that $\mathcal{C} \cap {}^{\perp}\mathcal{C} \subseteq {}^{\perp}(T^{\perp_\infty})$ – then $\psi_n$ splits and, by induction, $\psi_0$ splits, so $C \in \mathcal{C}$.

Let $M \in \mathcal{C} \cap {}^{\perp}\mathcal{C}$ $(\subseteq T^{\perp_\infty} \cap \mathcal{P}_n)$. By Lemma 11.1(d), there is a long exact sequence

$$0 \to E_n \to \dots \to E_0 \overset{\eta_0}{\to} M \to 0,$$

where $E_i \in \operatorname{Add}(T)$ for all $i \leq n$. By the closure properties of $\mathcal{C}$, $\operatorname{Add}(T) \subseteq \mathcal{C} \cap {}^{\perp}\mathcal{C}$, and $\operatorname{Ker} \eta_0 \in \mathcal{C}$. So $\eta_0$ splits, and $M \in \operatorname{Add}(T) \subseteq {}^{\perp}(T^{\perp_\infty})$. □

**Remark 11.4.** Note that the proof of (b) implies (a) above is constructive: the tilting module $T$ is obtained as $T = \bigoplus_{i \leq n} T_i$, where $T_i$ form the long exact sequence (4) obtained by an iteration of special $\mathcal{C}$-preenvelopes, starting from a special $\mathcal{C}$-preenvelope of $R$, $\varphi_0 : R \to T_0$, over a special $\mathcal{C}$-preenvelope of the cokernel of $\varphi_0$, etc. In view of (4), $T$ is a good tilting module.

Moreover, if $T'$ is an arbitrary tilting module, then applying the construction above to the tilting class $\mathcal{C} = (T')^{\perp_\infty}$, we obtain a good tilting module $T$ which is equivalent to $T'$.

Similarly, we can characterize tilting cotorsion pairs by the closure properties of their components:

**Corollary 11.5.** *Let $n < \omega$. Let $R$ be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then the following assertions are equivalent:*

(a) *$\mathfrak{C}$ is an $n$-tilting cotorsion pair.*

(b) $\mathfrak{C}$ *is a hereditary and complete cotorsion pair, such that* $\mathcal{A} \subseteq \mathcal{P}_n$ *and* $\mathcal{B}$ *is closed under direct sums.*

*Proof.* By Theorem 11.3, for $\mathcal{C} = \mathcal{B}$. □

**Remark 11.6.** Surprisingly, the completeness of $\mathfrak{C}$ is redundant in the statement of part (b) of Corollary 11.5. That is, if $\mathfrak{C}$ is hereditary, $\mathcal{A} \subseteq \mathcal{P}_n$, and $\mathcal{B}$ is closed under direct sums, then $\mathfrak{C}$ is complete (and hence tilting). For a proof, we refer e.g. to [27, 8.21].

*In the rest of these notes, for a class of modules* $\mathcal{C}$*, the notation* $\mathcal{C}^{<\omega}$ *and* $\mathcal{C}^{\leq\omega}$ *will stand for the class of all strongly finitely presented, and strongly countably presented, modules, respectively (cf. Definition 10.1).*

## 12. Classes of finite type

Our Definition 10.1 of a tilting module admits infinitely generated modules. Indeed, most examples of tilting modules presented so far were not finitely generated (for a very good reason in the commutative case - see Lemma 10.3).

There is, however, an implicit finiteness condition hidden in the notion of a tilting module: every tilting module $T$ is of finite type. This says that, though $T$ is large, when computing the corresponding tilting class $T^{\perp_\infty}$, we can replace $T$ by a set $\mathcal{S}$ consisting of strongly finitely presented modules of bounded projective dimension, such that $T^{\perp_\infty} = \mathcal{S}^{\perp_\infty}$.

**Definition 12.1.** Let $R$ be a ring.
Let $\mathcal{C}$ be a class of modules. Then $\mathcal{C}$ is of *finite type* provided there exist $n < \omega$ and $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$ ($\mathcal{S} \subseteq \mathcal{P}_n^{\leq\omega}$), such that $\mathcal{C} = \mathcal{S}^{\perp_\infty}$ .
Let $T$ be a module. The $T$ is of *finite type* provided that the class $T^{\perp_\infty}$ is of finite type.

Let $\mathcal{C}$ be a class of finite type and $\mathcal{A} = {}^\perp\mathcal{C}$ ($= {}^{\perp_\infty}\mathcal{C}$). Then $(\mathcal{A}, \mathcal{C})$ is a hereditary cotorsion pair generated by the class $\mathcal{A}^{<\omega}$, so $\mathcal{S} = \mathcal{A}^{<\omega}$ is the largest possible choice for $\mathcal{S}$ in Definition 12.1.

Any class of finite type is a tilting class, so there is a actually rich supply of tilting classes and modules available:

**Theorem 12.2.** *Let* $R$ *be a ring and* $\mathcal{C}$ *be a class of finite type. Then* $\mathcal{C}$ *is tilting, and definable (i.e., closed under pure submodules, direct limits and direct products).*

*Proof.* By assumption, there are $n < \omega$ and $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$, such that $\mathcal{C} = \mathcal{S}^{\perp_\infty}$.
Clearly, $\mathcal{C}$ is closed under direct products. Since $\mathcal{S}$ consists of strongly finitely presented modules, for each $A \in \mathcal{S}$ and each $i < \omega$, the functor $\operatorname{Ext}^i_R(A, -)$ commutes with direct limits, cf. [27, 6.6]. So the class $\mathcal{S}^{\perp_\infty}$ is closed under direct limits. Since $F^\perp$ is closed under pure submodules for any finitely presented module $F$, $\mathcal{C}$ is closed under pure submodules. This proves that $\mathcal{C}$ is a definable class.

Let $\mathfrak{C} = (\mathcal{A}, \mathcal{C})$ be the cotorsion pair cogenerated by $\mathcal{C}$. By Theorem 6.1, $\mathfrak{C}$ is complete. By Theorem 7.7, $\mathcal{A} \subseteq \mathcal{P}_n$. By Corollary 11.5, $\mathfrak{C}$ is an $n$-tilting cotorsion pair, that is, $\mathcal{C}$ is an $n$-tilting class. □

The converse of Theorem 12.2 also holds: all tilting classes and modules are of finite type. We will now only state this result, postponing a sketch of its proof to an appendix below, in order to present some of its remarkable consequences first.

**Theorem 12.3** (Finite type of tilting modules)**.** *Let* $R$ *be a ring,* $T$ *a tilting module and* $(\mathcal{A}, \mathcal{B})$ *the cotorsion pair induced by* $T$*. Then*

(a) $T$ *and* $\mathcal{B}$ *are of finite type.*

(b) *$T$ is equivalent to a tilting module $T_{fin}$, such that $T_{fin}$ is $\mathcal{A}^{<\omega}$-filtered.*

*Proof.* (a) will be proved in Section 14.

(b) By part (a), $\mathcal{B} = (\mathcal{A}^{<\omega})^{\perp}$. By Corollary 6.3 and Lemma 11.1(c), there are a $\mathcal{A}^{<\omega}$-filtered module $T_{fin}$ and a module $Q \in \operatorname{Add}(T)$, such that $T_{fin} = Q \oplus T$. Then $T_{fin}$ is a tilting module with $T^{\perp_\infty} = T_{fin}^{\perp_\infty}$, so $T_{fin}$ is equivalent to $T$. $\square$

**Remark 12.4.** There is an explicit construction of the tilting module $T_{fin}$ available: as remarked in 11.4, the proof of the implication (b) implies (a) in Theorem 11.3 shows that *any* iteration of special $\mathcal{B}$-preenvelopes: $\varphi_0 : R \to T_0$ of the ring $R$, $\varphi_1$ of the cokernel of $\varphi_0$ etc., yields a long exact sequence

$$0 \to R \overset{\varphi_0}{\to} T_0 \to T_1 \to \cdots \to T_{n-1} \to T_n \to 0,$$

such that $T' = \bigoplus_{i \le n} T_i$ is a tilting module equivalent to $T$. By part (a), $\mathcal{B} = \mathcal{C}^{\perp_\infty}$, where $\mathcal{C} = \mathcal{A}^{<\omega}$. By Theorem 6.1(a), each of the special $\mathcal{B}$-preenvelopes $\varphi_i$ above can be taken so that its cokernel is $\mathcal{C}$-filtered. But then also each $T_i$ ($i \le n$) is $\mathcal{C}$-filtered, and so is $T'$.

The tilting module $T$ itself need not in general possess an $\mathcal{A}^{<\omega}$-filtration. For example, if $T = R \oplus P$, where $P$ is a countably generated projective module, which has no finitely generated direct summands, then $T$ is not $\mathcal{P}_0^{<\omega}$-filtered (i.e., $T$ is not a direct sum of finitely generated projective modules).

## 13. Finitistic dimension conjectures

Our goal in this section is to present an application of tilting theory to proving results on finitistic dimensions of rings. First, we recall the relevant notions and their basic properties.

We will denote by $\operatorname{gl\,dim} R$ the (right) *global dimension* of $R$, i.e., is the supremum of the projective dimensions of all (right $R$-) modules.

**Definition 13.1.** Let $R$ be a ring. Denote by $\operatorname{Fin\,dim} R$ the *big finitistic dimension* of $R$, that is, the supremum of the projective dimensions of arbitrary modules of finite projective dimension.
Similarly, $\operatorname{fin\,dim} R$ will denote the *little finitistic dimension*, that is, the supremum of the projective dimensions of all finitely generated modules of finite projective dimension.
Obviously, $\operatorname{fin\,dim} R \le \operatorname{Fin\,dim} R \le \operatorname{gl\,dim} R$ for any ring $R$.

We recall a couple of simple and well-known facts:

**Lemma 13.2.** *Let $R$ be a ring, such that* $\operatorname{gl\,dim} R < \infty$.

(a) $\operatorname{fin\,dim} R = \operatorname{Fin\,dim} R = \operatorname{gl\,dim} R = \max\{\operatorname{proj\,dim} R/I \mid I \subseteq R\}$.
(b) *If $R$ is right semiartinian, then these dimensions are also equal to* $\max\{\operatorname{proj\,dim} S \mid S \in \operatorname{simp} R\}$.

*Proof.* This is an easy consequence of the Eklof Lemma 1.3. $\square$

So the little and the big finitistic dimensions provide a refinement of the homological dimension theory in the case when $\operatorname{gl\,dim} R = \infty$. The following example shows that such refinement is needed even in very simple cases:

**Example 13.3.** Let $R$ be a 0-Iwanaga-Gorenstein ring (= QF-ring) which is not completely reducible. For example, let $p$ be a prime integer, $n > 1$ and $R = \mathbb{Z}_{p^n}$.

Since all projective modules are injective, there is no module of projective dimension 1, hence no module of projective dimension $m$ for any $m \ge 1$. By assumption,

there is a non-projective simple module $M$, so $\operatorname{proj\,dim} M = \infty$. It follows that $\operatorname{fin\,dim} R = \operatorname{Fin\,dim} R = 0$, while $\operatorname{gl\,dim} R = \infty$.
Since $R = \mathbb{Z}_{p^n}$ is of finite representation type, it is certainly the finitistic dimension rather than the global dimension that reflects better the simple structure of the module category Mod–$\mathbb{Z}_{p^n}$.

Notice that, if $R$ is a right $\aleph_0$-noetherian ring, then the possible difference between $\operatorname{Fin\,dim} R$ and fin dim$R$ comes from (a representative set of) *countably* infinitely generated modules of finite projective dimension:

**Lemma 13.4.** *Assume that each right ideal of $R$ is countably generated. Then*

$$\operatorname{Fin\,dim} R = \sup \{ \operatorname{proj\,dim} M \mid M \in \mathcal{P}^{\leq \omega} \}.$$

*Proof.* This follows by Lemma 7.5. $\square$

**Example 13.5.** Let $R$ be a commutative noetherian ring. Then the little and the big finitistic dimensions are known to be closely related to other dimensions of the ring. Bass, Gruson and Raynaud proved that $\operatorname{Fin\,dim} R$ coincides with the Krull dimension of $R$. Auslander and Buchsbaum proved that, if $R$ is moreover local, then $\operatorname{fin\,dim} R = \operatorname{depth} R$, where $\operatorname{depth} R$ denotes the length of a maximal regular sequence in $\operatorname{Rad} R$. So in the local case, both dimensions are finite, but they coincide, if and only if $R$ is Cohen-Macaulay. Examples of commutative noetherian rings with $\operatorname{Fin\,dim} R = \operatorname{fin\,dim} R = \infty$ were constructed by Nagata.

If $R$ is an arbitrary ring, then the statements

- (I) $\operatorname{Fin\,dim} R = \operatorname{fin\,dim} R$,
- (II) $\operatorname{fin\,dim} R < \infty$

are known as the *first* and the *second finitistic dimension conjectures* for $R$, respectively.

In the case of artin algebras, (I) was disproved by Huisgen-Zimmermann: for each $n \geq 2$, she constructed a finite-dimensional monomial algebra $R$, such that $\operatorname{fin\,dim} R = n$ and $\operatorname{Fin\,dim} R = n + 1$ (see [31]).

Examples with arbitrarily big differences between the two dimensions were later constructed by Smalø: for each $n \geq 1$ there is a finite-dimensional algebra $R$ over a field, such that $\operatorname{fin\,dim} R = 1$ and $\operatorname{Fin\,dim} R = n$, [41].

The second finitistic dimension conjecture has been proved for all finite-dimensional monomial algebras, all algebras with representation dimension $\leq 3$, et al., however, this conjecture remains a major open of the represention theory of finite dimensional algebras.

We will finish this section by employing tilting modules in proving both finitistic dimension conjectures for Iwanaga-Gorenstein rings:

**Theorem 13.6** (Finitistic dimension conjectures for Iwanaga-Gorenstein rings)**.** *Let $n \geq 0$ and $R$ be an $n$-Iwanaga-Gorenstein ring.*

(a) *Let $0 \to R \to I_0 \to I_1 \to \cdots \to I_n \to 0$ be the minimal injective coresolution of $R$, and $T = \bigoplus_{i \leq n} I_i$ be the $n$-tilting module from Example 10.7. Let $(\mathcal{A}, \mathcal{B})$ be the tilting cotorsion pair generated by $T$.*
*Then* $\operatorname{Add}(T) = \mathcal{I}_0$, *and* $\mathcal{A} = \mathcal{P}_n$.

(b) $\operatorname{fin\,dim} R = \operatorname{Fin\,dim} R = n$.

*Proof.* First note that by Example 10.7, $\mathcal{P} = \mathcal{P}_n = \mathcal{I}_n = \mathcal{I}$, so $\operatorname{Fin\,dim} R = n$. Since $T$ is injective and $R$ is right noetherian, $\operatorname{Add}(T) \subseteq \mathcal{I}_0$.

We will prove that $\operatorname{Add}(T) = \mathcal{I}_0$. Since $\operatorname{Add}(T) = \mathcal{A} \cap \mathcal{B}$ by Lemma 11.1(c), it suffices to prove that $\mathcal{I}_0 \subseteq \mathcal{A}$. By Proposition 11.2(b), each module $B \in \mathcal{B}$ is

Add($T$)-resolved. Denote by $B'$ the kernel of the $n$-th map in a fixed Add($T$)-resolution of $B$. Since $\mathrm{Add}(T) \subseteq \mathcal{I}_0$, dimension shifting gives for each $I \in \mathcal{I}_0$ that $\mathrm{Ext}^1_R(I,B) \cong \mathrm{Ext}^{n+1}_R(I,B')$ . However, $\mathcal{I}_0 \subseteq \mathcal{P}_n$, so the latter Ext-group is zero, proving that $I \in \mathcal{A}$.

Since $\mathrm{Add}(T) = \mathcal{I}_0$, Proposition 11.2 yields that $\mathcal{A} = \mathcal{I}_n = \mathcal{P}_n = \mathcal{P}$, and $\mathcal{B}$ is the class of all $\mathcal{I}_0$-resolved modules. .

Finally, since $T$ is a tilting module, $T$ is of finite type by Theorem 12.3(a), so $\mathcal{B} = (\mathcal{A}^{<\omega})^\perp$. However, $\mathcal{A}^{<\omega} = \mathcal{P}^{<\omega}$. By Corollary 6.3, each module $P \in \mathcal{P}$ is a direct summand of a $\mathcal{P}^{<\omega}$-filtered module, so $\mathrm{fin\,dim}\, R = \mathrm{Fin\,dim}\, R$. □

Note that the equality $\mathrm{Add}(T) = \mathcal{I}_0$ proved above implies (by the Krull-Remak-Schmidt-Azumaya Theorem) that each indecomposable injective module occurs as a direct summand in some term of the minimal injective coresolution of $R$. In particular, the tilting module $T$ from 13.6 is an injective cogenerator in Mod–$R$.

For further applications of tilting and approximation theory to the finitistic dimension conjectures, we refer to [27, Chap. 17].

## 14. Appendix

This final section presents a sketch of the proof of Theorem 12.3(a), that is, of the fact that each tilting module is of finite type.

The proof is in three steps, each of which uses quite a different set of techniques: Step I is based on deep results from set-theoretic homological algebra, Step II involves a study of Mittag-Leffler conditions for inverse systems of modules, while Step III uses also model theoretic techniques. The original proof was given gradually in a series of papers [7], [44], [8] and [12], here we follow the simplified version from [27].

First we recall a piece of auxiliary notation needed in Step I: for a ring $R$, $\mathcal{G}$ will denote the class of all modules that are $\mathfrak{F}$-products of injective modules, where $\mathfrak{F}$ runs over all filters of the following form: there is a cardinal $\lambda$ and a regular infinite cardinal $\kappa \leq \lambda$, such that $\mathfrak{F}$ is the filter of all sets $I \subseteq \lambda$ with $|\lambda \setminus I| < \kappa$.

Here, an $\mathfrak{F}$-*product* is the submodule of the product $\prod_{\alpha<\lambda} I_\alpha$ of injective modules $I_\alpha$ that consists of all $(x_\alpha \mid \alpha < \lambda)$ such that $\{\alpha < \lambda \mid x_\alpha = 0\} \in \mathfrak{F}$. For example, if $\kappa = \aleph_0$, then the $\mathfrak{F}$-product is just the direct sum $\bigoplus_{\alpha<\lambda} I_\alpha$.

**Step I.** Let $R$ be a ring and $\mathcal{B}$ be a class of modules such that $\mathcal{B}$ is closed under direct sums and $\mathcal{B}$ contains the class $\mathcal{G}$ defined above. Let $n < \aleph_0$. Then each $M \in {}^{\perp_\infty}\mathcal{B} \cap \mathcal{P}_n$ is $\mathcal{C}$-filtered, where $\mathcal{C} = {}^{\perp_\infty}\mathcal{B} \cap \mathcal{P}_n^{\leq \aleph_0}$

The proof is by induction on $n$: the case of $n = 0$ is just the Kaplansky structure theorem for projective modules.

The inductive step (= [27, Lemma 13.33]) is proved by induction on the cardinality, $\lambda$, of a minimal generating subset of $M$.

For $\lambda$ a regular uncountable cardinal, we use the following more general result (= [27, Theorem 8.17]): If a class of modules $\mathcal{B}$ is closed under direct sums and $M \in {}^\perp\mathcal{B}$ is equipped with a $\lambda$-filtration $\mathcal{M}$ consisting of modules from ${}^\perp\mathcal{B}$, then $M$ is ${}^\perp\mathcal{B}$-filtered by a subfiltration of $\mathcal{M}$.

If $\lambda$ is singular, we use a consequence of Shelah's singular compactness theorem for $\mathcal{C}$-filtered modules (= [27, Theorem 7.29]), which says that if for each regular infinite cardinal $\kappa < \lambda$, $M$ contains a dense system consisting of $< \kappa$-generated $\mathcal{C}$-filtered submodules, then $M$ is $\mathcal{C}$-filtered.

Since each right tilting class $\mathcal{B}$ contains the class $\mathcal{G}$ (= [27, Corollary 13.29]), it follows from Step I that each left tilting class $\mathcal{A} = {}^{\perp_\infty}\mathcal{B}$ is $\aleph_0$-deconstructible.

**Step II.** ([27, Theorem 13.41]) Let $R$ be a ring and $\mathcal{B}$ be a class of modules closed under direct sums. Let $A$ be a countably presented module such that $\mathrm{Ext}^1_R(A,B) = 0$ for all $B \in \mathcal{B}$. Let $(B_i \mid i \in I)$ be any sequence of elements of $\mathcal{B}$ and $B'$ be a pure submodule of $\prod_{i\in I} B_i$. Then also $\mathrm{Ext}^1_R(A,B') = 0$.

If $\mathcal{B}$ is a right tilting class, then by Step I and Step II, $\mathcal{B}$ is a definable class (i.e., a class closed under direct products, pure submodules, and direct limits).

**Step III.** ([27, Lemma 13.44]) Let $T$ be a tilting module, and $(\mathcal{A},\mathcal{B})$ be the corresponding tilting cotorsion pair. Then $T$ is of finite type, iff $\mathcal{A}^{\leq\aleph_0} \subseteq \varinjlim \mathcal{A}^{<\aleph_0}$. The key point of the proof of the if-part is the fact, that two definable classes of modules are equal, iff they contain the same pure-injective modules.

The finite type of $T$ is then proved by induction on the projective dimension $n$ of $T$, using in the inductive step the fact (= [27, Lemma 13.45]) that the hereditary cotorsion pair generated by the 1st syzygy $\Omega(T)$ is an $(n-1)$-tilting cotorsion pair, and the trivial fact that the unique 0-tilting cotorsion pair, $(\mathcal{P}_0, \mathrm{Mod}\text{–}R)$, is generated by $R$, hence it is of finite type.

## References


[1] S. T. Aldrich, E. Enochs, O. Jenda, L. Oyonarte, *Envelopes and covers by modules of finite injective and projective dimensions*, J. Algebra **242** (2001), 447 – 459.

[2] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, 2nd ed., GTM 13, Springer-Verlag, N.Y. 1992.

[3] L. Angeleri Hügel, F. Coelho, *Infinitely generated tilting modules of finite projective dimension*, Forum Math. **13** (2001), 239 – 250.

[4] L. Angeleri Hügel, J. Šaroch, J.Trlifaj, *Tilting approximations and Mittag-Leffler conditions - the applications*, Israel J. Math. **226** (2018), 757 – 780.

[5] I. Assem, D. Simson, A. Skowronski, *Elements of the Representation Theory of Associative Algebras*, London Math. Soc. Student Texts **65**, Camb. Univ. Press, Cambridge 2006.

[6] S. Bazzoni, *Divisible envelopes, $\mathcal{P}_1$-covers, and weak injective modules*, J. Algebra Appl. **9** (2010), 531 – 542.

[7] S. Bazzoni, P. C. Eklof, J. Trlifaj, *Tilting cotorsion pairs*, Bull. London Math. Soc. **37** (2005), 683 – 696.

[8] S. Bazzoni, D. Herbera, *One dimensional tilting modules are of finite type*, Algebras and Repres. Theory **11** (2008), 43 – 61.

[9] S. Bazzoni, F. Mantese, A.Tonolo, *Derived equivalence induced by infinitely generated $n$–tilting modules*, Proc. Amer. Math. Soc. **139** (2011), 4225 – 4234.

[10] S. Bazzoni, L. Positselski, J. Šťovíček, *Projective covers of flat contramodules*, Int. Math. Res. Not. 2021, https://doi.org/10.1093/imrn/rnab202.

[11] S. Bazzoni, J. Šaroch, *Enochs' conjecture for cotorsion pairs and more*, Forum Math. **36**(2024), Issue 6, 1517 – 1528.

[12] S. Bazzoni, J. Šťovíček, *All tilting modules are of finite type*, Proc. Amer. Math. Soc. **135** (2007), 3771 – 3781.

[13] A. Ben Yassine, J. Trlifaj, *Dualizations of approximations, $\aleph_1$-projectivity, and Vopěnka's Principles*, Appl. Cat. Structures **32** (2024), Issue 6, Article 36.

[14] L. Bican, R. El Bashir, E.E. Enochs, *All modules have flat covers*, Bull. London Math. Soc. **33** (2001), 385 – 390.

[15] R. Colpi, C. Menini, *On the structure of $*$–modules*, J. Algebra **158** (1993), 400 – 419.

[16] R. Colpi, J. Trlifaj, *Tilting modules and tilting torsion theories*, J. Algebra **178** (1995), 614 – 634.

[17] M. Cortés-Izurdiaga, *Tor-pairs: Products and approximations*, Contemp. Math. **751** (2020), 119 – 133.

[18] S. Cox, *Salce's problem on cotorsion pairs is undecidable*, Bull. London Math. Soc. **54**(2022), 1363 – 1374.

[19] P. C. Eklof, *Homological algebra and set theory*, Trans. Amer. Math. Soc. **227** (1977), 207 – 225.

[20] P. C. Eklof, S. Shelah, *On the existence of precovers*, Illinois J. Math. 47(2003), 173 – 188.

[21] P. C. Eklof, J. Trlifaj, *How to make Ext vanish*, Bull. London Math. Soc. **33** (2001), 41 – 51.
[22] R. El Bashir, *Covers and directed colimits*, Algebras and Repres. Theory. **9** (2006), 423 – 430.
[23] E. Enochs, *Injective and flat covers, envelopes and resolvents*, Israel J. Math. **39** (1981), 33 – 38.
[24] E. Enochs, *Shortening filtrations*, Sci. China Math. **55** (2012), 687 – 693.
[25] E. Enochs, O. Jenda, *Relative Homological Algebra*, Vol. 1, 2nd ed., W. de Gruyter, Berlin 2011.
[26] J. Gillespie, *Abelian Model Category Theory*, Cambridge Studies in Advanced Math. 215, CUP, Cambridge 2025.
[27] R. Göbel and J. Trlifaj, *Approximations and Endomorphism Algebras of Modules*, 2nd Ed., vols. 1 & 2 - Approximations, Expositions in Mathematics **41**, W. de Gruyter, Berlin 2012.
[28] P. Hill, *The third axiom of countability for abelian groups*, Proc. Amer. Math. Soc. **82** (1981), 347 – 350.
[29] M. Hovey, *Cotorsion pairs, model category structures, and representation theory*, Math. Zeitschrift **241** (2002), 553 -– 592.
[30] M. Hrbek, *Topological endomorphism rings of tilting complexes*, arXiv:2205.11105v1.
[31] B. Huisgen–Zimmermann, *Homological domino effects and the first finitistic dimension conjecture*, Invent. Math. **108** (1992), 369 – 383.
[32] L. Klingler, L. Levy, *Representation type of commutative noetherian rings III: Global wildness and tameness*, Mem. Amer. Math. Soc. **832**, Amer. Math. Soc., Providence 2005.
[33] L. Levy, J. Trlifaj, *Γ–separated covers*, in Abelian Groups, Rings, Modules and Homological Algebra, LNPAM 249, CRC Press, New York 2006, 203 – 216.
[34] Y. Miyashita, *Tilting modules of finite projective dimension*, Math. Z. **193** (1986), 113 – 146.
[35] L. Positselski, J. Šťovíček, *The tilting-cotilting correspondence*, Int. Math. Res. Not. 2021, 189 – 274.
[36] J. Rickard, *Morita theory for derived categories*, J. London Math. Soc. **39** (1989), 436 – 456.
[37] D. Quillen, *Homotopical Algebra*, Lect. Notes Math. **43**, Springer, Berlin 1967.
[38] L. Salce, *Cotorsion theories for abelian groups*, Symposia Math. **23** (1979), 11 – 32.
[39] M. Saorín, J. Šťovíček, *On exact categories and applications to triangulated adjoints and model structures*, Adv. Math. **228** (2011), 968-1007.
[40] A. Slávik, J. Trlifaj, *Approximations and locally free modules*, Bull. London Math. Soc. **46** (2014), 76 – 90.
[41] S. Smalø, *Homological differences between finite and infinite dimensional representations of algebras*, in Trends in Mathematics, Birkhäuser, Basel 2000, 425 – 439.
[42] J. Šaroch, *Enochs' conjecture for small precovering classes of modules*, Israel J. Math. **255** (2023), 401 — 415.
[43] J. Šťovíček, *Deconstructibility and Hill lemma in Grothendieck categories*, Forum Math. 25 (2013), 193 – 219.
[44] J. Šťovíček, J. Trlifaj, *All tilting modules are of countable type*, Bull. London Math. Soc. **39** (2007), 121 – 132.
[45] J. Trlifaj, D. Pospíšil, *Tilting and cotilting classes over Gorenstein rings*, in Rings, Modules and Representations, Contemp. Math. **480** (2009), 319 – 334.
[46] J. Wakamatsu, *Stable equivalence of self–injective algebras and a generalization of tilting modules*, J. Algebra **134** (1990), 298 – 325.
[47] J. Xu, *Flat Covers of Modules*, Lect. Notes Math. **1634**, Springer, New York 1996.