CRASH COURSE ON HOMOLOGICAL ALGEBRA AND HEREDITARY ALGEBRAS

JAN ŠŤOVÍČEK

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Contents

1. Homological dimensions

From a course of homological algebra (see also [CE56] or [Rot09], or [ASS06, §A4] for a quick overview), we know that for any ring A and $n \geq 0$, there are functors

$$\mathsf{Ext}^n_A\colon (\mathsf{Mod}\, A)^{\mathrm{op}} \times \mathsf{Mod}\, A \longrightarrow \mathsf{Ab}$$

such that:

- Ext^0_A is the usual Hom-functor (which sends a pair of modules (X, M) to the abelian group $\mathsf{Hom}_A(X, M)$ and is contravariant in the first variable and covariant in the second one), and
- given any short exact sequence of right modules

  $$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

  and a right A-module X, there are long exact sequences of abelian groups of the form

  $$\begin{tikzcd} 0 \arrow[r] & \mathsf{Hom}_A(X,L) \arrow[r] & \mathsf{Hom}_A(X,M) \arrow[r] & \mathsf{Hom}_A(X,N) \arrow[dll, rounded corners] & \\ & \mathsf{Ext}^1_A(X,L) \arrow[r] & \mathsf{Ext}^1_A(X,M) \arrow[r] & \mathsf{Ext}^1_A(X,N) \arrow[dll, rounded corners] & \\ & \mathsf{Ext}^2_A(X,L) \arrow[r] & \mathsf{Ext}^2_A(X,M) \arrow[r] & \mathsf{Ext}^2_A(X,N) \arrow[r] & \cdots \end{tikzcd}$$

  and

  $$\begin{tikzcd} 0 \arrow[r] & \mathsf{Hom}_A(N,X) \arrow[r] & \mathsf{Hom}_A(M,X) \arrow[r] & \mathsf{Hom}_A(L,X) \arrow[dll, rounded corners] & \\ & \mathsf{Ext}^1_A(N,X) \arrow[r] & \mathsf{Ext}^1_A(M,X) \arrow[r] & \mathsf{Ext}^1_A(L,X) \arrow[dll, rounded corners] & \\ & \mathsf{Ext}^2_A(N,X) \arrow[r] & \mathsf{Ext}^2_A(M,X) \arrow[r] & \mathsf{Ext}^2_A(L,X) \arrow[r] & \cdots \end{tikzcd}$$

Recall also that the groups $\mathsf{Ext}^n_A(X, M)$ can be computed in two (nontrivially!) equivalent ways:

- Starting from any projective resolution

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow X \longrightarrow 0 \tag{1}$$

of X, we can compute $\mathsf{Ext}^n_A(X,M)$ as the n-th cohomology $\mathsf{Ker}\, d^*_{n+1}/\,\mathsf{Im}\, d^*_n$ of the complex of Hom-groups

$$\cdots \xleftarrow{d_3^*} \mathsf{Hom}_A(P_2,M) \xleftarrow{d_2^*} \mathsf{Hom}_A(P_1,M) \xleftarrow{d_1^*} \mathsf{Hom}_A(P_0,M) \xleftarrow{d_0^*} 0.$$

Note that we have removed the module X from the resolution and the map d_0^* is the zero map by convention.

- Dually, we can start with any injective resolution of M,

$$0 \longrightarrow M \longrightarrow E^0 \xrightarrow{d^1} E^1 \xrightarrow{d^2} E^2 \xrightarrow{d^3} \cdots \tag{2}$$

and compute $\mathsf{Ext}^n_A(X,M)$ is the n-th cohomology $\mathsf{Ker}\, d_*^{n+1}/\,\mathsf{Im}\, d_*^n$ of the complex

$$0 \xrightarrow{d_*^0} \mathsf{Hom}_A(X,E^0) \xrightarrow{d_*^1} \mathsf{Hom}_A(X,E^1) \xrightarrow{d_*^2} \mathsf{Hom}_A(X,E^2) \xrightarrow{d_*^3} \cdots.$$

Remark 1.1. The group $\mathsf{Ext}^1_A(X,M)$ has a nice representation that explains the notation for the Ext-functors—it classifies extensions of the A-module X by M. More precisely, there is a natural bijection between the elements of $\mathsf{Ext}^1_A(X,M)$ and equivalence classes $[\ 0 \to M \to E \to X \to 0\ ]_\sim$ of short exact sequences in $\mathsf{Mod}\, A$. Here two short exact sequences $0 \to M \to E_1 \to X \to 0$ and $0 \to M \to E_2 \to X \to 0$ are said to be equivalent if there exists a homomorphism $f\colon E_1 \to E_2$ that fits into a commutative diagram of the form

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d, dotted, "f"'] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & E_2 \arrow[r] & X \arrow[r] & 0.
\end{tikzcd}$$

Such an f is necessarily an isomorphism. The zero element of $\mathsf{Ext}^1_A(X,M)$ corresponds to the equivalence class of split short exact sequences $[\ 0 \to M \to M \oplus X \to X \to 0\ ]_\sim$ and the group operation is realized through so-called Baer sums (we will not discuss details here; see e.g. [Rot09, §7.2.1] or a short summary in [ASS06, §A.5]).

If A happens to be a K-algebra over a field K, then all Hom-groups of modules have a natural structure of K-vector spaces, and (by the construction of the Ext-groups above), also $\mathsf{Ext}^n_A(M,X)$ is naturally a K-vector space for each $X, M \in \mathsf{Mod}\, A$ and $n \geq 0$. In fact, we have functors

$$\mathsf{Ext}^n_A\colon (\mathsf{Mod}\, A)^{\mathrm{op}} \times \mathsf{Mod}\, A \longrightarrow \mathsf{Mod}\, K \tag{3}$$

in this case.

If, moreover, $\dim_K A < \infty$ and $M, X \in \mathsf{mod}\, A$, then one can construct a projective resolution as in (1) with all the projective modules finitely generated, so also finite dimensional. For example, the minimal projective resolution has this property. Likewise, an injective resolution as in (2) can be taken in $\mathsf{mod}\, A$ (e.g., take a minimal projective resolution of DM in $\mathsf{mod}\, A^{\mathrm{op}}$ and apply the duality $D\colon (\mathsf{mod}\, A^{\mathrm{op}})^{\mathrm{op}} \to \mathsf{mod}\, A$, see §2 below). All in all, since $\dim_K \mathsf{Hom}_A(L,N) \leq (\dim_K L)\cdot(\dim_K N) < \infty$ for all $L, N \in \mathsf{mod}\, A$, the above computations of $\mathsf{Ext}^n(X,M)$ reveal that also $\dim_K \mathsf{Ext}^n(X,M) < \infty$ for all $n \geq 0$, and that (3) restricts to

$$\mathsf{Ext}^n_A\colon (\mathsf{mod}\, A)^{\mathrm{op}} \times \mathsf{mod}\, A \longrightarrow \mathsf{mod}\, K.$$

Here we will be interested in measuring how complicated algebras are from the homological point of view. This leads to the notions of projective, injective and global dimension. Typically, these concepts are defined a priori in categories of all (possibly infinitely generated) modules. We will follow this path and explain how to adapt the theory to categories of finitely generated modules over finite-dimensional algebras later in §2.

Definition 1.2. The *projective dimension* of a module $M \in \mathsf{Mod}\, A$ is defined as the smallest integer n such that M has a projective resolution

$$0 \to P_n \to \cdots \to P_0 \to M \to 0.$$

If no such finite projective resolution exists, we say that the projective dimension of M is infinite.

Dually, the *injective dimension* of $N \in \mathsf{Mod}\, A$ is the smallest n such that

$$0 \to N \to E^0 \to \cdots \to E^n \to 0,$$

or is infinite if there is no finite injective resolution.

These dimensions are closely related to the Ext groups. It is well known that a module $P \in \mathsf{Mod}\, A$ is projective if and only if each exact sequence of the form $0 \to M \to E \to P \to 0$ splits if and only if $\mathsf{Ext}^1(M, -) \equiv 0$ in $\mathsf{Mod}\, A$ if and only if $\mathsf{Ext}^i(M, -) \equiv 0$ for all $i > 0$. The long exact sequences of Ext groups allow us to show that

Proposition 1.3. *The following are equivalent for any module M over any ring A:*

1. *The projective dimension of M is at most n;*
2. $\mathsf{Ext}^i_A(M, -) \equiv 0$ *on* $\mathsf{Mod}\, A$ *for all* $i > n$*;*
3. $\mathsf{Ext}^{n+1}_A(M, -) \equiv 0$ *on* $\mathsf{Mod}\, A$*;*
4. *Given any exact sequence* $0 \to Y \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ *with* $P_0, \ldots, P_{n-1}$ *projective, then Y is projective as well.*

Injective modules and modules of finite injective dimension allow for a completely formally dual characterization.

Proposition 1.4. *The following are equivalent for any module N over any ring A:*

1. *The injective dimension of N is at most n;*
2. $\mathsf{Ext}^i_A(-, N) \equiv 0$ *on* $\mathsf{Mod}\, A$ *for all* $i > n$*;*
3. $\mathsf{Ext}^{n+1}_A(-, N) \equiv 0$ *on* $\mathsf{Mod}\, A$*;*
4. *Given any exact sequence* $0 \to N \to E^0 \to \cdots \to E^{n-1} \to Z \to 0$ *with* $E^0, \ldots, E^{n-1}$ *injective, then Y is injective as well.*

We will not spell out the arguments here, but rather refer to [Rot09] or [CE56] for details. The version for injective dimension has the advantage that we do not have to test the vanishing of $\mathsf{Ext}^1_A(-, N)$ on all modules, but only on cyclic modules. If A is a finite-dimensional algebra (or, more generally, a right artinian ring), we can do even better. The reason is that we have the Baer Criterion.

Lemma 1.5 (Baer Criterion)**.** *Let A be a ring and $N \in \mathsf{Mod}\, A$. Then the following are equivalent:*

1. *N is injective;*
2. *For any right ideal $I_A \subseteq A_A$, each homomorphism $f\colon I \to N$ extends to a homomorphism $A \to N$,*

$$\begin{tikzcd} I_A \arrow[r, "\subseteq"] \arrow[dr, "\forall f"'] & A_A \arrow[d, dotted, "\exists"] \\ & N_A; \end{tikzcd}$$

(3) $\mathsf{Ext}^1_A(A/I, N) = 0$ *for each right ideal* $I_A \subseteq A_A$.

Proof. The equivalence (1) $\Leftrightarrow$ (2) is classical; see [AF92, Lemma 18.3], [CE56, Theorem I.3.2] or [Rot09, Theorem 3.30]. The fact that (2) $\Leftrightarrow$ (3) follows from the exact sequence of abelian groups

$$\mathsf{Hom}_A(A, N) \to \mathsf{Hom}_A(I, N) \to \mathsf{Ext}^1_A(A/I) \to \mathsf{Ext}^1_A(A, N) = 0$$

obtained by applying $\mathsf{Hom}(-, N)$ to the short exact sequence $0 \to I \to A \to A/I \to 0$. Here we also use that $\mathsf{Ext}^1_A(A, N) = 0$ since A_A is projective. □

This allows us to improve Proposition 1.4.

Proposition 1.6. *Let* A *be a ring and* $N \in \mathsf{Mod}\, A$. *Then the following are equivalent:*

(1) *The injective dimension of* N *is at most* n*;*
(2) $\mathsf{Ext}^{n+1}_A(A/I, N) = 0$ *for each right ideal* $I_A \subseteq A_A$.

If, moreover, A *is a finite-dimensional algebra over a field (or, more generally, a right artinian ring), these are further equivalent to*

(3) $\mathsf{Ext}^{n+1}_A(S, N) \equiv 0$ *for each simple module* S_A.

Proof. To prove (1) $\Leftrightarrow$ (2), choose an exact sequence $0 \to N \to E^0 \xrightarrow{d^1} E^1 \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} E^{n-1} \to Z \to 0$ with all E^i injective and a right ideal I_A. Since $\mathsf{Ext}^j_A(A/I, E^i) = 0$ for all $i = 0, \dots, n-1$ and $j > 0$, the standard dimension shifting trick tells us that

$$\mathsf{Ext}^{n+1}_A(A/I, N) \cong \mathsf{Ext}^{n}_A(A/I, \mathsf{Im}\, d^1) \cong \cdots \cong \mathsf{Ext}^{2}_A(A/I, \mathsf{Im}\, d^{n-1}) \cong \mathsf{Ext}^{1}_A(A/I, Z).$$

So $\mathsf{Ext}^{n+1}_A(A/I, N) = 0$ for each I if and only if Z is injective (by the Baer Criterion) if and only if the injective dimension of N is at most n (by Proposition 1.4).

The implication (2) $\Rightarrow$ (3) is clear, as simple modules are precisely those of the form A/I where $I_A \subseteq A_A$ is a maximal right ideal.

It remains to prove (3) $\Rightarrow$ (2). Let $I_A \subseteq A_A$ be an ideal. Since we assume that A is a finite-dimensional algebra, the cyclic module A/I is of finite length, so has a composition series

$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_\ell = A/I$$

with F_{i+1}/F_i simple for each $i = 0, \dots, \ell - 1$ (the same is true if A is only right artinian by the Hopkins Theorem [AF92, Theorem 15.20 and Corollary 15.21]). We will prove by induction that $\mathsf{Ext}^{n+1}_A(F_i, N) = 0$ for all $0 \leq i < \ell$. The case $i = 0$ is trivial and for the inductive step it remains to notice that the short exact sequence $0 \to F_i \to F_{i+1} \to F_{i+1}/F_i \to 0$ induces an exact sequence

$$\mathsf{Ext}^{n+1}(F_{i+1}/F_i, N) \to \mathsf{Ext}^{n+1}(F_{i+1}, N) \to \mathsf{Ext}^{n+1}(F_i, N).$$

Now the leftmost term vanishes by assumption (3) and the rightmost term by the inductive hypothesis. □

At this point, we can define the global dimension of a ring. Strictly speaking, for general rings we should distinguish between the left and right global dimensions, depending on whether we choose left or right modules to define it. However, as we will see in the next section, the two values are equal for finite-dimensional algebras.

Definition 1.7. Let A be a ring. The *(right) global dimension* of A is defined as the supremum of the projective dimensions of modules M, where M runs over all (not necessarily finitely generated) A-modules.

The following characterization of the global dimension is an immediate consequence of Propositions 1.3 and 1.4.

Proposition 1.8. *Let* A *be a ring and* $n \geq 0$. *Then the following are equivalent:*

(1) *The global dimension of A is at most n;*
(2) $\mathsf{Ext}^i(-,-) \equiv 0$ *on* $\mathsf{Mod}\, A$ *for all* $i > n$;
(3) $\mathsf{Ext}^{n+1}(-,-) \equiv 0$ *on* $\mathsf{Mod}\, A$;
(4) *Each* $N \in \mathsf{Mod}\, A$ *has injective dimension at most* n.

Finally, with the Baer Criterion and Proposition 1.6, we obtain the following important consequence. The second part is very convenient since we know that there are only finitely many isomorphism classes of simple modules over a finite-dimensional algebra over a field.

Corollary 1.9. (1) *The global dimension of a ring A is equal to the supremum of the projective dimensions of the cyclic right modules A/I (where $I_A \subseteq A_A$ runs over all right ideals).*
(2) *If A is a finite-dimensional algebra over a field (or just an right artinian ring), then the global dimension of A equals the supremum of the projective dimensions of simple right modules.*

Proof. (1) By Proposition 1.6, all right modules have injective dimension at most n if and only if $\mathsf{Ext}_A^{n+1}(A/I,-) \equiv 0$ for each right ideal I_A. It remains to apply Proposition 1.3 to $M = A/I$.

(2) This follows by the same argument using the second part of Proposition 1.6. □

2. Homological algebra and finite dimensional algebras

Throughout this section, let A be a finite-dimensional algebra over a field K. In this case, we notice that the projectivity and injectivity of finitely generated modules can be tested entirely within the category $\mathsf{mod}\, A$.

Lemma 2.1. (1) *The following are equivalent for $P \in \mathsf{mod}\, A$:*
(a) *P is projective;*
(b) *Given any epimorphism $p\colon M \twoheadrightarrow N$ in $\mathsf{mod}\, A$ and a homomorphism $f\colon P \to N$, there exists $g\colon P \to M$ such that $f = pg$.*
(c) $\mathsf{Ext}_A^1(P,-) \equiv 0$ *on* $\mathsf{mod}\, A$.

(2) *Dually, the following are equivalent for $E \in \mathsf{mod}\, A$:*
(a) *E is injective;*
(b) *Given any monomorphism $i\colon L \hookrightarrow M$ in $\mathsf{mod}\, A$ and a homomorphism $f\colon L \to E$, there exists $g\colon M \to E$ such that $f = gi$.*
(c) $\mathsf{Ext}_A^1(-,E) \equiv 0$ *on* $\mathsf{mod}\, A$.

Proof. (1) The implication (a)⇒(c) is obvious. To prove (c)⇒(b), we apply the functor $\mathsf{Hom}_A(P,-)$ to the short exact sequence $0 \to \mathsf{Ker}\, p \to M \xrightarrow{p} N \to 0$. Then we obtain an exact sequence

$$\mathsf{Hom}_A(P,M) \xrightarrow{p_*} \mathsf{Hom}_A(P,N) \longrightarrow \mathsf{Ext}^1(P,\mathsf{Ker}\, p) = 0.$$

It follows that p is surjective, which (upon unraveling) is precisely condition (b). Finally, assuming (b), consider a short exact sequence $0 \to L \to A^n \xrightarrow{p} P \to 0$. It exists for some n since P is finitely generated and, by (b) applied to the identity morphism $f = 1_P$, there exists $g\colon P \to A$ such that $1_P = pg$. Hence $A^n = \mathsf{Im}\, p \oplus \mathsf{Ker}\, g$ and P is projective.

(2) Again, (a)⇒(c) is obvious and (c)⇒(b) follows by a similar argument as in part (1) (apply $\mathsf{Hom}_A(-,E)$ to the short exact sequence $0 \to L \xrightarrow{i} M \to \mathsf{Coker}\, i \to 0$). Finally, (a)⇔(b) follows from the Baer criterion (Lemma 1.5). □

Remark 2.2. The complicated route in this course—first discussing homological algebra in $\mathsf{Mod}\, A$ and then specializing to finite-dimensional algebras and finitely generated modules over them—was taken to clarify the relation to other courses. In principle, one could alternatively take a shortcut and use Lemma 2.1(1)(b) and (2)(b) as the definition of projective and injective modules in $\mathsf{mod}\, A$.

Recall the vector space duality $D = \mathsf{Hom}_K(-, K)$ from [ASS06, §I.2.9] and that it induces duality between the categories of finitely generated left and right modules, which we typically write as an equivalence

$$D\colon (\mathsf{mod}\, A)^{\mathrm{op}} \xrightarrow{\simeq} \mathsf{mod}\, A^{\mathrm{op}}. \tag{4}$$

One merit of having Lemma 2.1 is that it explains why the duality sends projective modules to injective modules and vice versa. This was left without proof in [ASS06, Theorem I.5.13(b)]. Using the lemma, we can also extend the duality to Ext functors.

Proposition 2.3. *Let A be a finite-dimensional algebra, and $X, M \in \mathsf{mod}\, A$. Then there is a natural isomorphism $\mathsf{Ext}^n_A(X, M) \cong \mathsf{Ext}^n_{A^{\mathrm{op}}}(DM, DX)$ for each $n \geq 0$.*

Proof. The equivalence (4) provides the isomorphims for $n = 0$ (i.e. $\mathsf{Hom}_A(M, N) \cong \mathsf{Hom}_{A^{\mathrm{op}}}(DN, DM)$). For general n, consider a projective resolution of X in $\mathsf{mod}\, A$ as in (1). Then

$$0 \longrightarrow D(X) \longrightarrow D(P_0) \xrightarrow{D(d_1)} D(P_1) \xrightarrow{D(d_2)} D(P_2) \xrightarrow{D(d_3)} \cdots$$

is an injective resolution of $D(X)$ by [ASS06, Theorem I.5.13] and we have the following commutative diagram with isomorphisms in columns by (4):

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{Hom}_A(P_0, M) \arrow[r] \arrow[d, "\cong"'] & \mathsf{Hom}_A(P_1, M) \arrow[r] \arrow[d, "\cong"'] & \mathsf{Hom}_A(P_2, M) \arrow[r] \arrow[d, "\cong"'] & \cdots \\
0 \arrow[r] & \mathsf{Hom}_{A^{\mathrm{op}}}(DM, D(P_0)) \arrow[r] & \mathsf{Hom}_{A^{\mathrm{op}}}(DM, D(P_1)) \arrow[r] & \mathsf{Hom}_{A^{\mathrm{op}}}(DM, D(P_2)) \arrow[r] & \cdots
\end{tikzcd}$$

This induces isomorphism of the cohomology of the rows, so that $\mathsf{Ext}^n_A(X, M) \cong \mathsf{Ext}^n_{A^{\mathrm{op}}}(DM, DX)$. The fact that the latter isomorphism is natural in both X and M follows by standard arguments, similar to those that prove that Ext^n_A is a functor in the first place. We refer to [Rot09, CE56]. $\square$

As a consequence, we obtain the equality between the left and right global dimensions.

Corollary 2.4. *Let A be a finite-dimensional algebra, and $n \geq 0$. Then the following are equivalent:*

(1) *The global dimension of A is at most n;*
(2) *$\mathsf{Ext}^i_A(-, -) \equiv 0$ on $\mathsf{mod}\, A$ for all $i > n$;*
(3) *$\mathsf{Ext}^{n+1}_A(-, -) \equiv 0$ on $\mathsf{mod}\, A$;*

Proof. (1) $\Rightarrow$ (2) follows from Proposition 1.8 and (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Let $S, T \in \mathsf{mod}\, A$ be simple modules. Then

$$\mathsf{Ext}^{n+1}_A(S, T) \cong \mathsf{Ext}^{n+1}_{A^{\mathrm{op}}}(D(T), D(S)) = 0,$$

so the module $D(S) \in \mathsf{mod}\, A^{\mathrm{op}}$ has injective dimension at most n by Proposition 1.6 and we can choose an injective resolution $0 \to D(S) \to E^0 \to \cdots \to E^m \to 0$ in $\mathsf{mod}\, A^{\mathrm{op}}$ with $m \leq n$ (note that injective envelopes of finitely generated modules are again finitely generated by [ASS06, Corollary I.5.14]). By duality, we obtain a projective resolution $0 \to D(E^m) \to \cdots D(E^0) \to S \to 0$ of S, so all simple modules in $\mathsf{mod}\, A$ have projective dimension at most n. Consequently, the global dimension of A is at most n by Corollary 1.9. $\square$

Corollary 2.5. *If A is a finite-dimensional algebra, the global dimensions of A and A^{op} are equal.*

Proof. By Proposition 2.3, we have $\mathsf{Ext}_A^{n+1}(-,-) \equiv 0$ on $\mathsf{mod}\, A$ if and only if $\mathsf{Ext}_{A^{\mathrm{op}}}^{n+1}(-,-) \equiv 0$ on $\mathsf{mod}\, A^{\mathrm{op}}$ □

Remark 2.6. The left and right global dimensions are in fact equal much more generally, in particular for each left and right notherian ring [Rot09, Corollary 8.28].

The last question which we will focus on in the section is how to compute the global dimension of a given algebra or, else (in view of Corollary 1.9), how to compute the projective dimension of simple modules. A finitely generated module may have many projective resolutions, and as the name suggests, the minimal projective resolution in the sense of [ASS06, Definition I.5.7] is also of minimal length. The conceptual reason for that is that the projective cover of a module $M \in \mathsf{mod}\, A$ is a summand of any epimorphism $Q \twoheadrightarrow M$ with Q projective (cf. [AF92, Lemma 17.17]), and by induction one can prove that each term of a given projective resolution of M has the corresponding term of the minimal resolution as a summand.

However, we will use another approach here to prove that will be technically easier to work out here and we will use it later in the proof of [ASS06, Lemma III.2.12].

Proposition 2.7. *Let A be a finite-dimensional algebra,*

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow M \longrightarrow 0$$

be a minimal projective resolution of $M \in \mathsf{mod}\, A$ and let $S \in \mathsf{mod}\, A$ be a simple module. Then $\mathsf{Ext}_A^n(M,S) \cong \mathsf{Hom}_A(P_n,S)$ for each $n \geq 0$.

Proof. Recall that the Ext groups are the cohomologies of the complex

$$\cdots \xleftarrow{d_3^*} \mathsf{Hom}_A(P_2,S) \xleftarrow{d_2^*} \mathsf{Hom}_A(P_1,S) \xleftarrow{d_1^*} \mathsf{Hom}_A(P_0,S) \xleftarrow{d_0^*} 0,$$

so it suffices to prove that $d_n^* = 0$ for all $n \geq 1$, where by definition $d_n^*(f) = f \circ d_n$ for each $f \in \mathsf{Hom}_A(P_{n-1},S)$. By the minimality of the projective resolution, the epimorphism $P_{n-1} \twoheadrightarrow \mathsf{Im}\, d_{n-1}$ (or $P_0 \twoheadrightarrow M$ if $n=0$) is a projective cover, so

$$\mathsf{Ker}(P_{n-1} \twoheadrightarrow \mathsf{Im}\, d_{n-1}) = \mathsf{Ker}\, d_{n-1} = \mathsf{Im}\, d_n \subseteq \mathsf{rad}\, P_{n-1}$$

by the construction of projective covers in [ASS06, Theorem I.5.8]. Consider now a homomorphism $f\colon P_{n-1} \to S$. Then

$$\mathsf{Im}(f \circ d_n) = f(\mathsf{Im}\, d_n) \subseteq f(\mathsf{rad}\, P_{n-1}) \subseteq \mathsf{rad}\, S = 0.$$

Here, the last inclusion follows from the functoriality of the radical [ASS06, Proposition I.3.7(c)]. So $d_n^*(f) = 0$ for each f, or in other words, $d_n^* = 0$. □

As a consequence, we obtain the following.

Corollary 2.8. *Let A be a finite-dimensional algebra and*

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow M \longrightarrow 0$$

be a minimal projective resolution of a non-zero module $M \in \mathsf{mod}\, A$. Then the projective dimension of M equals to minimal $n \geq 0$ such that $P_{n+1} = 0$, or is infinite if no such n exists.

Proof. By the Baer Criterion (Proposition 1.6) and the duality (Proposition 2.3), the projective dimension of M equals the minimal $n \geq 0$ such that $\mathsf{Ext}_A^{n+1}(P,S) = 0$ for all simple modules S. By Proposition 2.7, $\mathsf{Ext}_A^{n+1}(P,S)$ vanishes if and only if $\mathsf{Hom}_A(P_{n+1},S)$ does. However, we always have an epimorphism $P_{n+1} \twoheadrightarrow$

$P_{n+1}/\operatorname{rad} P_{n+1}$ and the latter module is semisimple (since it is naturally a module over the semisimple ring $A/\operatorname{rad} A$). Hence $\mathsf{Hom}_A(P_{n+1}, S) = 0$ for all simple modules S implies that $P_{n+1}/\operatorname{rad} P_{n+1} = 0$, which in turn implies that $P_{n+1} = 0$ by the Nakayama Lemma [ASS06, Lemma I.2.2]. □

3. Hereditary algebras

A natural question to ask is what we can say about rings of low global dimensions. The case global dimension zero conincides with semisimple rings.

Proposition 3.1. *Let A be a ring. Then the following are equivalent:*

(1) *The ring is semisimple.*
(2) *Each short exact sequence of A-modules $0 \to N \to E \to M \to 0$ splits.*
(3) $\mathsf{Ext}^1_A(-,-) \equiv 0$ *on* $\mathsf{Mod}\, A$.
(4) *Each A-module is projective.*
(5) *Each A-module is injective.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from [Rot09, Proposition 4.1], which says that a module is semisimple if and only if every submodule is a direct summand. We just apply this to the module A_A. The rest is left as an exercise—it is not so crucial in the next exposition and easily follows from the summary in §1. □

The more interesting case for us are the rings of global dimension (at most) one.

Definition 3.2. A ring A is right hereditary if the right global dimension of A is at most one.

Remark 3.3. As we know that the left and right global dimensions coincide for finite-dimensional algebras over a field, we will later speak only of hereditary algebras.

Right hereditary rings can be characterized in various ways.

Theorem 3.4. *The following are equivalent for a ring A:*

(1) *A is right hereditary;*
(2) *The projective dimension of A/I is at most one for each $I_A \subseteq A_A$,*
(3) *Each right ideal $I_A \leq A$ is projective,*
(4) *Each submodule of a projective right module is projective,*
(5) *Each factor of an injective right module is injective.*

If A is a finite-dimensional algebra, these are further equivalent to

(6) *The projective dimension of each simple module S_A is at mosat one.*

Proof. The equivalences (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (6) follow from the Baer Criterion Corollary 1.9, and (2) $\Leftrightarrow$ (3) follows from Proposition 1.3 applied to $M = A/I$.

(1) $\Rightarrow$ (4) Suppose that $L \subseteq P$ is a submodule of a projective module. Since the projective dimension of P/L is at most one, L is projective again by Proposition 1.3.

(4) $\Rightarrow$ (1) Any $M \in \mathsf{Mod}\, A$ is a factor of a projective (even free) module, so there is an exact sequence $0 \to L \to P \to M \to 0$ with P projective. Assuming (4), L is also projective, so the projective dimension of M is at most one.

(1) $\Rightarrow$ (5) This is proved dually using Proposition 1.4. □

Remark 3.5. Many important examples of hereditary rings exist outside the realm of finite-dimensional algebras; see, e.g., [Rot09, §4.3]. For example, the ring of integers $\mathbb{Z}$ (or to that end any principal ideal domain) is hereditary because of Theorem 3.4(3). Hereditary commutative domains are called *Dedekind domains* and there are two important sources of them:

(1) Rings of integral elements in finite field extensions of $\mathbb{Q}$, such as $\mathbb{Z}[i]$, $\mathbb{Z}[e^{\frac{2\pi i}{3}}]$ or $\mathbb{Z}[\sqrt{-5}]$;

(2) Coordinate rings of smooth affine algebraic curves, such as $\mathbb{C}[x]$ or $\frac{\mathbb{C}[x,y]}{(y^2-f(x))}$, where $f(x) \in \mathbb{C}[x]$ is a cubic polynomial with no multiple roots.

As this course focuses on path algebras, we introduce the main source of examples of hereditary algebras here.

Lemma 3.6 ([ASS06, Theorem VII.1.7(a)]). *If K is a field and Q is a finite acyclic quiver, then the path algebra KQ is hereditary.*

Proof. We will prove that each simple module has projective dimension at most one. Since each simple module is of the form $S \cong \varepsilon_i KQ/\varepsilon_i \,\mathsf{rad}(KQ) = \varepsilon_i A/\varepsilon_i R_Q$ for some $i \in Q_0$, it suffices to prove that $\varepsilon_i R_Q$ is a projective KQ-module (where $R_Q \subseteq KQ$ is the arrow ideal). As a vector space, $\varepsilon_i R_Q$ has a basis formed by the non-trivial parts starting at i, so

$$\varepsilon_i R_Q = \bigoplus_{(\alpha\colon i \to j)} \alpha KQ.$$

It remains to note that for any arrow $\alpha\colon i \to j$, the left multiplication $\alpha\cdot-$ induces an isomorphism of the right modules $\alpha KQ \cong \varepsilon_j KQ$, so $\varepsilon_i R_Q \cong \bigoplus_{(\alpha\colon i\to j)} \varepsilon_j KQ$. $\square$

Remark 3.7. The path algebra KQ is hereditary even if Q has oriented cycles. The proof is much harder, uses non-commutative Gröbner bases.

We have already seen [ASS06, Theorem II.3.7] which says that for every basic finite dimensional algebra A over an algebraically closed field K, there is an admissible ideal $I \subseteq KQ_A$ such that $A \cong KQ_A/I$. A natural question now is which of these algebras are hereditary, where a partial answer is given by Lemma 3.6. We will show that, in fact, these are the only hereditary algebras in this context. As a first step, we show that the quiver of a basic hereditary algebra over $K = \overline{K}$ is acyclic.

Lemma 3.8 ([ASS06, Corollary VII.1.5(a)]). *Let A be hereditary and $f\colon Q \to P$ be a homomorphism between indecomposable projectives. If f is non-zero, then it is a monomorphism.*

Proof. Since A is hereditary, $\mathsf{Im}\, f \subseteq P$ is projective. So $f\colon Q \twoheadrightarrow \mathsf{Im}\, f$ is a split epimorphism. As Q is indecomposable and f non-zero, it follows that $\mathsf{Ker}\, f = 0$. $\square$

Proposition 3.9. *Suppose that A is a hereditary finite-dimensional algebra over $K = \overline{K}$. Then Q_A is acyclic.*

Proof. Recall from [ASS06, Theorem II.3.7] that each arrow $\alpha\colon i \to j$ in Q_A corresponds under the isomorphism $KQ_A/I \cong A$ to a basis element $\alpha + e_i(\mathsf{rad}\,A)^2 e_j$ of the vector space $e_i(\mathsf{rad}\,A)e_j/e_i(\mathsf{rad}\,A)^2 e_j$. Here $e_1, \ldots, e_n$ is a chosen complete set of primitive orthogonal idempotents. In particular, if we identify KQ_A/I with A, an arrow $\alpha\colon i \to j$ becomes a non-zero element of $e_i(\mathsf{rad}\,A)e_j$. So the left multiplication by α induces a non-zero homomorphism $\alpha \cdot -\colon e_j A \to e_i A$ whose image is contained in $\mathsf{rad}(e_i A)$. Therefore, this homomorphism is not surjective, but is injective by Lemma 3.8.

Now suppose by way of contradiction that there is an oriented cycle in Q_A. The discussion in the previous paragraph tells us that there is a corresponding cycle of injective but not surjective homomorphisms between indecomposable projective modules:

$$\begin{tikzcd}[column sep=small]
 & 1 \arrow[r, "\alpha_1"] & 2 \arrow[dr, bend left, "\alpha_2"] & & & & e_1A \arrow[dl, bend right, "\alpha_{k-1}\cdot -"'] & & e_2A \arrow[ll, "\alpha_1\cdot -"'] & \\
k-1 \arrow[ur, bend left, "\alpha_{k-1}"] & & & 3 & \Longrightarrow & e_{k-1}A & & & & e_3A \arrow[ul, bend right, "\alpha_2\cdot -"'] \\
\vdots & & & \vdots & & \vdots & & & & \vdots
\end{tikzcd}$$

The composition of the cycle also yields an endomorphism of $e_1 A$ which is injective but not surjective, which is absurd as $\dim_K e_1 A < \infty$. $\square$

Now we are ready for the final result of this section which (together with lemma 3.6) characterizes basic hereditary algebras over an algebraically closed field.

Theorem 3.10 ([ASS06, Theorem VII.1.7(b)]). *Let A be a basic hereditary finite-dimensional algebra over an algebraically closed field K. Then $A \cong KQ_A$.*

Proof. Without loss of generality, assume that $A = KQ_A/I$ with Q_A acyclic and $I \subseteq KQ_A$ admissible. Now we will use the argument from [ARS97, Lemma III.1.11] to prove that $I = 0$. To this end, we have an exact sequence of A-modules

$$0 \longrightarrow \frac{I}{R_Q \cdot I} \longrightarrow \frac{R_Q}{R_Q \cdot I} \xrightarrow{\ p\ } \frac{R_Q}{I} \longrightarrow 0.$$

Since R_Q is projective as a right KQ-module, $R_Q/R_Q I$ is projective as a right A-module. Indeed, there is a vector space basis of R_Q formed by all non-trivial paths in Q, so $R_Q = \bigoplus_{\alpha \in Q_1} \alpha KQ \cong \bigoplus_{\alpha \in Q_1} t(\alpha) KQ$ and $R_Q/R_Q I \cong \bigoplus_{\alpha \in Q_1} t(\alpha) KQ/t(\alpha) I$. Furthermore, $I/R_Q I \subseteq R_Q^2/R_Q I = \mathsf{rad}(R_Q/R_Q I)$ since I is admissible. So p is a projective cover in $\mathsf{mod}\, A$. Since A is hereditary, $\mathsf{rad}\, A = R_Q/I$ is already projective as a right A-module, so p is an isomorphism, and $I = R_Q I$. Now the Nakayama Lemma applied to the *left* KQ-module I shows that $I = 0$. $\square$

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