The Pol - Inv Galois connection

Fix a set A.

<u>def</u>: $\underbrace{f: A^n \to A}_{\text{n-ary operation (function)}}$, $\underbrace{R \subseteq A^m}_{\text{m-ary relation}}$ are <u>compatible</u> if $\boxed{R \leq (A, f)^m}$ (subuniverse)

i.e., $\begin{pmatrix} x_{11} \\ \vdots \\ x_{m1} \end{pmatrix}, \dots, \begin{pmatrix} x_{1n} \\ \vdots \\ x_{mn} \end{pmatrix} \in R \Rightarrow \begin{pmatrix} f(x_{11}, \dots, x_{1n}) \\ \vdots \\ f(x_{m1}, \dots, x_{mn}) \end{pmatrix} \in R$

We also say: R is <u>invariant</u> w.r.t. f
f is a <u>polymorphism</u> of R

<u>Ex.</u>: f unary ... $(x_1, \dots, x_m) \in R \Rightarrow (f(x_1), \dots, f(x_m)) \in R$
i.e., f is an <u>endomorphism</u> of (A, R)
(cf. graph homomorphisms)

$R = \leq$ ... $x_1 \leq y_1, \dots, x_n \leq y_n \Rightarrow f(x_1, \dots, x_n) \leq f(y_1, \dots, y_n)$
i.e., f is <u>monotone</u> w.r.t. $\leq$

$R = \{a\}$ (unary rel.) ... $a \in R, \dots, a \in R \Rightarrow f(a, \dots, a) \in R$
i.e., $f(a, \dots, a) = a$

$\left.\begin{array}{l} R = \emptyset \\ R = A^n \text{ (n-ary)} \end{array}\right\}$ ... all operations are compatible

$$\begin{array}{ccc} \text{algebras on } A & & \text{relational structures on } A \\ \updownarrow \ \ \underline{A}=(A,F) \leftrightarrow F \subseteq \bigcup A^{A^n} & & \updownarrow \ \ \mathbb{A}=(A,\mathcal{R}) \leftrightarrow \mathcal{R} \subseteq \bigcup \mathcal{P}(A^n) \\ \text{sets of operations on } A & \overset{\mathrm{Inv}}{\underset{\mathrm{Pol}}{\rightleftarrows}} & \text{sets of relations on } A \end{array}$$

given by the <u>compatibility</u> relation

i.e., $\mathrm{Inv}(F) = \{ R : R \leq (A,F)^m, m \in \mathbb{N} \} = SP_{fin}((A,F))$

$\mathrm{Pol}(\mathcal{R}) = \{ f : R \leq (A,f)^m \ \forall R \in \mathcal{R} \text{ } m\text{-ary} \}$

(we also write $\mathrm{Inv}(\underline{A})$, $\mathrm{Pol}(\mathbb{A})$ where $\underline{A}$ is an algebra, $\mathbb{A}$ is a rel. str.)

$\mathrm{Pol}(\mathcal{R})$ is a clone

Pf: - projections are compatible

- $\left.\begin{array}{l} g_1,\dots,g_n \text{ comp.} \\ f \text{ comp.} \end{array}\right\} \Rightarrow f(g_1,\dots,g_n) \text{ comp.}$

Ex.: $\mathrm{Pol}(\{\leq\})$ = monotone operations

$\mathrm{Pol}(\{\{a\} : a \in A\})$ = idempotent operations

$\mathrm{Pol}(\{B : B \subseteq A\})$ = conservative op's $\quad (f(a_1,\dots,a_n) \in \{a_1,\dots,a_n\})$

$\quad\quad\quad\quad\uparrow$ unary rel.

Pol(all relations) = projections [not so obvious!]

Inv(projections) = all relations

Inv(all operations) = ?? [not clear]

$\mathrm{Inv}(\{x \mapsto c\}) = \{ R : (c,\dots,c) \in R \}$

$\quad\quad\uparrow$ const.

$B \leq \underline{A} \iff B \in \mathrm{Inv}(\underline{A})$ (subuniverses = compatible unary rel's)

What are closed sets in the Pol–Inv connection?

i.e., describe $\mathrm{Pol}(\mathrm{Inv}(\underline{A}))$ in purely functional terms
$\mathrm{Inv}(\mathrm{Pol}(\mathbb{A}))$ relational

$\mathrm{Pol}(\mathbb{R})$ is always a clone

↳ and every clone is some $\mathrm{Pol}(\mathbb{R})$ (on finite sets):

<u>Theorem</u> (Geiger / Bodnarčuk, Kalužnin, Kotov, Romov 1969):

A finite $\Rightarrow$ $\mathrm{Pol}(\mathrm{Inv}(\underline{A})) = \mathrm{Clo}(\underline{A})$

<u>Proof:</u> $(\supseteq)$ $\mathrm{Clo}(\underline{A}) \subseteq \mathrm{Clo}(A, \mathrm{Pol}(\mathrm{Inv}(\underline{A}))) = \mathrm{Pol}(\mathrm{Inv}(\underline{A}))$

since $F \subseteq \mathrm{Pol}(\mathrm{Inv}(F))$

since $\mathrm{Pol}(\mathrm{Inv}(\underline{A}))$ is already a clone

$(\subseteq)$ let $\underbrace{f \in \mathrm{Pol}(\mathrm{Inv}(\underline{A}))}_{\text{i.e., } f \text{ is compatible with any } R \in \mathrm{Inv}(\underline{A})}$ n-ary, ? $f \in \mathrm{Clo}(\underline{A})$?

<u>TRICK:</u> consider $\mathrm{Clo}_n(\underline{A})$ as $|A|^n$-ary relation on A

(since $\mathrm{Clo}_n(\underline{A}) \subseteq A^{A^n}$)

and observe that $\mathrm{Clo}_n(\underline{A}) \in \mathrm{Inv}(\underline{A})$

we already proved that $\mathrm{Clo}_n(\underline{A}) \leq \underline{A}^{A^n}$
(= free algebra)

$\Rightarrow$ in particular, f is compatible with $\mathrm{Clo}_n(\underline{A})$

$\Rightarrow$ $\Big[\ \pi_1^n, \dots, \pi_n^n \in \mathrm{Clo}_n(\underline{A}) \Rightarrow f(\pi_1^n, \dots, \pi_n^n) \in \mathrm{Clo}_n(\underline{A})\ \Big]$

$(x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n)$

i.e., $f \in \mathrm{Clo}_n(\underline{A})$ $\square$

<u>def</u>: <u>relational clone</u> on a set A $\equiv$

any set of relations closed w.r.t. <u>pp-definitions</u>
$\qquad \equiv$

<u>primitive positive</u>

<u>def</u>: R is <u>pp-definable</u> from $R_1, \dots, R_n$ $\equiv$

there is a formula Φ using $\exists, \wedge, =, R_1, \dots, R_n$

s.t. $\bar{x} \in R \iff \Phi(\bar{x})$

<u>def</u>: $\mathrm{RelClo}(A) :=$ the smallest relational clone containing all relations of A

<u>Ex.</u>: given R_1 unary, R_2 ~~binary~~ ternary $\rightsquigarrow$ R binary:

$$(x,y) \in R \iff (\exists u, v, w)\ R_1(u) \wedge (x = v) \wedge R_2(v, w, y)$$

<u>Ex.</u>: existence of a path of length n is pp-def. in graphs:

$$(x,y) \in R \iff (\exists u_1, \dots, u_{n-1})\ R(x, u_1) \wedge R(u_1, u_2) \wedge \dots \wedge R(u_{n-1}, y)$$

⊙ $\bigcap_{\text{finite}}$ is pp-def ... $R(\bar{x}) \iff R_1(\bar{x}) \wedge \dots \wedge R_n(\bar{x})$

⊙ change of variables is pp-def. ... $f : \{1, \dots, n\} \to \{1, \dots, n\}$, R n-ary

$$\rightsquigarrow R^f(\bar{x}) \iff R(x_{f(1)}, \dots, x_{f(n)})$$

⊙ projection onto a subset of variables is pp-def

$$\dots\ R'(\bar{x}) \iff (\exists x_n)\ R(\bar{x}, x_n) \qquad \text{(etc.)}$$

⊙ R is pp-def from $R_1, \dots, R_n$, each R_i is pp-def from $S_{i,1}, \dots, S_{i,m_i}$

$\Longrightarrow$ R is pp-def from $S_{1,1}, \dots, S_{n,m_n}$

⊙ WLOG: $\Phi(x_1, \dots, x_n) = (\exists x_{n+1}, \dots, x_{n+k}) \bigwedge^{u} R_{r_i}(x_{i_1}, \dots) \wedge \bigwedge^{v} x_{j_i} = x_{k_i}$

<u>Prop.</u>: $\mathrm{Inv}(\underline{A})$ is a relational clone

<u>Pf</u>: from the last $\overset{\text{""}}{\odot}$, we only need to prove the following:

(1) R comp. $\Rightarrow$ R^f comp. $\left(\dots\ R \rightsquigarrow R(x_{i_1}, \dots) \right.$ & glueing variables $\left. \right)$

(2) R_i comp. $\Rightarrow$ $\bigcap R_i$ comp. $(\dots \wedge R_{r_i})$

(3) R comp. $\Rightarrow$ $(\exists x_n)\, R(\bar{x}, x_n)$ comp.

All of that is fairly obvious from the definition. $\square$

<u>Theorem</u>: A finite $\Rightarrow$ $\mathrm{Inv}(\mathrm{Pol}(\mathbb{A})) = \mathrm{RelClo}(\mathbb{A})$

<u>Pf</u>: ($\supseteq$) $\mathrm{RelClo}(\mathbb{A}) \subseteq \mathrm{RelClo}(A, \mathrm{Inv}(\mathrm{Pol}(\mathbb{A}))) = \mathrm{Inv}(\mathrm{Pol}(\mathbb{A}))$

($\subseteq$) Idea: • consider $\mathrm{Pol}_m(\mathbb{A})$ as $|A|^m$-ary relation & prove that it is pp-def from $\mathbb{A}$

• take $R \in \mathrm{Inv}(\mathrm{Pol}(\mathbb{A}))$, $|R| = N$ & prove that R is pp-def from $\mathrm{Pol}_N(\mathbb{A})$ $\square$

<u>Applications</u>:

(1) How to prove that $f \notin \mathrm{Clo}(\underline{A})$?

... find R s.t. $R \in \mathrm{Inv}(\underline{A})$, f is not compatible with R

<u>Ex.</u>: $\underline{A} = (\{0,1\}, \wedge, \vee)$, $f(x,y) = x+y$, $R = \leq$

(2) the complexity of the CSP problem