Cvičení 8.

Důležité limity

- $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- $\lim_{x\to 0} \frac{e^x-1}{x} = 1.$

Spočtěte limity

1. i) $\lim_{x\to 1} \frac{x^2-1}{2x^2-x-1}$

   ii) $\lim_{x\to \infty} \left(\frac{x^3}{x^2+1} - x\right)$

   iii) $\lim_{x\to 8} \frac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2}$

   iv) $\lim_{x\to 2} \left(\frac{1}{x^2-2x} - \frac{x}{x^2-4}\right)$

   v) $\lim_{x\to 1} \frac{x^{100}-2x+1}{x^{50}-2x+1}$

   vi) $\lim_{x\to 1} \frac{x+x^2+x^3+\ldots+x^n-n}{x-1}.$

2. i) $\lim_{x\to 0} \frac{(1+x)(1+2x)\ldots(1+nx)-1}{x}$

   ii) $\lim_{x\to 0} \frac{(1+mx)^n-(1+nx)^m}{x^2}$

   iii) $\lim_{x\to 2} \frac{(x^2-x-2)^{20}}{(x^3-12x+16)^{10}}.$

3. i) $\lim_{x\to \infty} x^{\frac{4}{3}} \left(\sqrt[3]{x^2+1} - \sqrt[3]{x^2-1}\right)$

   ii) $\lim_{x\to 16} \frac{\sqrt[4]{x}-2}{\sqrt{x}-4}$

   iii) $\lim_{x\to \infty} \sqrt{x^3} \left(\sqrt{x+1} + \sqrt{x-1} - 2\sqrt{x}\right)$

   iv) $\lim_{x\to a_+} \frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}.$

4. i) $\lim_{x\to 0} \frac{\cos x}{x}$

   ii) $\lim_{x\to 0} \frac{\mathrm{tg} x}{x}$

   iii) $\lim_{x\to a} \frac{\mathrm{tg} x-\mathrm{tg} a}{x-a}.$

Řešení:

1. i) $\lim_{x\to 1} \frac{x^2-1}{2x^2-x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(2x+1)} = \lim_{x\to 1} \frac{x+1}{2x+1} = \frac{2}{3}$.

ii) $\lim_{x\to\infty} \left(\frac{x^3}{x^2+1} - x\right) = \lim_{x\to\infty} \frac{x^3-x(x^2+1)}{x^2+1} = \lim_{x\to\infty} \frac{-1}{x+\frac{1}{x}} = 0$.

iii)

$$\lim_{x\to 8} \frac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2} = \lim_{x\to 8} \frac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2} \cdot \frac{\sqrt{9+2x}+5}{\sqrt{9+2x}+5} \cdot \frac{\sqrt[3]{x}^2+2\sqrt[3]{x}+4}{\sqrt[3]{x}^2+2\sqrt[3]{x}+4} = \lim_{x\to 8} \frac{9+2x-25}{x-8} \cdot \frac{\sqrt[3]{x}^2+2\sqrt[3]{x}+4}{\sqrt{9+2x}+5}$$
$$= \lim_{x\to 8} 2 \cdot \frac{\sqrt[3]{x}^2+2\sqrt[3]{x}+4}{\sqrt{9+2x}+5} = \frac{12}{5}.$$

iv) $\lim_{x\to 2} \left(\frac{1}{x^2-2x} - \frac{x}{x^2-4}\right) = \lim_{x\to 2} \frac{(x+2)-x^2}{x(x-2)(x+2)} = \lim_{x\to 2} \frac{-(x-2)(x+1)}{x(x-2)(x+2)} = \lim_{x\to 2} \frac{-(x+1)}{x(x+2)} = \frac{-3}{8}$.

v) $\lim_{x\to 1} \frac{x^{100}-2x+1}{x^{50}-2x+1} = \lim_{x\to 1} \frac{(x-1)(x^{99}+x^{98}+x^{97}+\ldots+x-1)}{x^{49}+x^{48}+\ldots+x-1} = \frac{98}{48} = \frac{49}{24}$.

vi)

$$\lim_{x\to 1} \frac{x+x^2+x^3+\ldots+x^n-n}{x-1} = \lim_{x\to 1} \frac{x-1+x^2-1+x^3-1+\ldots+x^n-1}{x-1}$$
$$= \lim_{x\to 1} \frac{(x-1)+(x-1)(x+1)+(x-1)(x^2+x+1)+..+(x-1)(x^{n-1}+x^{n-2}+\ldots+1)}{x-1}$$
$$= \lim_{x\to 1} \left(1+(x+1)+(x^2+x+1)+\ldots+(x^{n-1}+x^{n-2}+\ldots+1)\right)$$
$$= \lim_{x\to 1} \sum_{i=0}^{n-1} (n-1)x^i = \sum_{i=0}^{n-1} (n-1) = \frac{n(n+1)}{2}.$$

2. i) $\lim_{x\to 0} \frac{(1+x)(1+2x)\ldots(1+nx)-1}{x} = \lim_{x\to 0} \frac{1+(x+2x+\ldots+nx)+o(x)-1}{x} = 1+2+\ldots+n = \frac{n(n+1)}{2}$.

ii)

$$\lim_{x\to 0} \frac{(1+mx)^n-(1+nx)^m}{x^2} = \lim_{x\to 0} \frac{1+\binom{n}{1}mx+\binom{n}{2}m^2x^2+\ldots+m^nx^n-\left(1+\binom{m}{1}nx+\binom{m}{2}n^2x^2+\ldots+n^mx^n\right)}{x^2}$$
$$= \lim_{x\to 0} \frac{\binom{n}{2}m^2x^2-\binom{m}{2}n^2x^2+o(x^2)}{x^2} = \binom{n}{2}m^2-\binom{m}{2}n^2 = \frac{mn}{2}(n-m).$$

iii)

$$\lim_{x\to 2} \frac{(x^2-x-2)^{20}}{(x^3-12x+16)^{10}} = \lim_{x\to 2} \frac{((x-2)(x+1))^{20}}{((x-2)(x^2+2x-8))^{10}} = \lim_{x\to 2} \frac{((x-2)(x+1))^{20}}{((x-2)(x-2)(x+4))^{10}} = \lim_{x\to 2} \frac{(x+1)^{20}}{(x+4)^{10}}$$
$$= \frac{3^{20}}{6^{10}} = \frac{3^{10}}{2^{10}}.$$

3. i) $\lim_{x\to\infty} x^{\frac{4}{3}} \left(\sqrt[3]{x^2+1} - \sqrt[3]{x^2-1}\right) = \lim_{x\to\infty} x^{\frac{4}{3}} \left(\sqrt[3]{x^2+1} - \sqrt[3]{x^2-1}\right) \cdot \frac{\sqrt[3]{x^2+1}^2+\sqrt[3]{x^2+1}\cdot\sqrt[3]{x^2-1}+\sqrt[3]{x^2-1}^2}{\sqrt[3]{x^2+1}^2+\sqrt[3]{x^2+1}\cdot\sqrt[3]{x^2-1}+\sqrt[3]{x^2-1}^2}$

ii) $\lim_{x\to 16} \frac{\sqrt[4]{x}-2}{\sqrt{x}-4}$

iii) $\lim_{x\to\infty} \sqrt{x^3}\left(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x}\right)$

iv) $\lim_{x\to a_+} \frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$.

4. i) $\lim_{x\to 0} \frac{\cos x}{x}$

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iii) $\lim_{x\to a} \frac{\mathrm{tg} x-\mathrm{tg} a}{x-a}$.