Zadání

1. Vypočtěte limity:
   (a) $\lim_{n\to\infty}\left(\frac{n-2n^2}{3n^2+2}\right)$
   (b) $\lim_{n\to\infty}\left(n^3-4n^2+2\right)$
   (c) $\lim_{n\to\infty}\left(\frac{2n^2-3n}{n^4+1}\right)$
   (d) V závislosti na parametru $a\in\mathbb{R}$ určete $\lim_{n\to\infty}(a^n)$
2. Vypočtěte limity:
   (a) $\lim_{n\to\infty}\frac{1000n-789}{0.003n^2+34n-43}$
   (b) $\lim_{n\to\infty}\frac{\sqrt[3]{n}}{n+1}$
   (c) $\lim_{n\to\infty}(n^3-n^2+n-1)$
   (d) $\lim_{n\to\infty}(n^3+4n^4-n^5+8n^2-12n)$
   (e) $\lim_{n\to\infty}\frac{n^3-3n}{2n^3+7n-5}$
   (f) $\lim_{n\to\infty}\frac{n^2-3n}{2n^3+7n-5}$
   (g) $\lim_{n\to\infty}\left(\frac{2n^3+3n-5}{n^3+4n^2-3}\right)^2$
   (h) $\lim_{n\to\infty}\frac{n^3-3n}{3n^2+7n-5}$
   (i) $\lim_{n\to\infty}\frac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}}$
3. Vypočtěte limity:
   (a) $\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+...+\frac{n-1}{n^2}\right)$
   (b) $\lim_{n\to\infty}\left(\frac{1+2+...+n}{n+2}-\frac{n}{2}\right)$
   (c) $\lim_{n\to\infty}\frac{1^2+2^2+...+n^2}{n^2}$
   (d) $\lim_{n\to\infty}\frac{1^3+2^3+...+n^3}{n^4}$
   (e) $\lim_{n\to\infty}\frac{(2n-3)^{20}(3n+2)^{30}}{(2n+1)^{50}}$
4. Vypočtěte limitu:
   (a) $\lim_{n\to\infty}(-1)^{n!}$

Pomocné vzorce:

- $1^2+2^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$
- $1^3+2^3+...+n^3=\left(\frac{n(n+1)}{2}\right)^2$

Řešení

1. (a) $\lim_{n\to\infty} \frac{n-2n^2}{3n^2+2} = \lim_{n\to\infty} \frac{n^2\left(\frac{1}{n}-2\right)}{n^2\left(3+\frac{2}{n^2}\right)} = \lim_{n\to\infty} \frac{\frac{1}{n}-2}{3+\frac{2}{n^2}} = \frac{-2}{3}$

   (b) $\lim_{n\to\infty} \left(n^3 - 4n^2 + 2\right) = \lim_{n\to\infty} n^3 \left(1 - \frac{4}{n} + \frac{2}{n^3}\right) = \infty$

   (c) $\lim_{n\to\infty} \frac{2n^2-3n}{n^4+1} = \lim_{n\to\infty} \frac{n^2\left(2-\frac{3}{n}\right)}{n^2\left(n^2+\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{2-\frac{3}{n}}{n^2+\frac{1}{n^2}} = \frac{2-0}{\infty+0} = 0$

   (d) I. $a > 1$: $\lim_{n\to\infty} (a^n) = \infty$
   II. $a = 1$: $\lim_{n\to\infty} (a^n) = 1$
   III. $1 > a > -1$: $\lim_{n\to\infty} (a^n) = 0$
   IV. $-1 \geq a$: $\lim_{n\to\infty} (a^n)$ neexistuje

2. (a) $\lim_{n\to\infty} \frac{1000n-789}{0.003n^2+34n-43} = \lim_{n\to\infty} \frac{n^2}{n^2} \cdot \frac{\frac{1000}{n}-\frac{789}{n^2}}{0.003+\frac{34}{n}-\frac{43}{n^2}} = \lim_{n\to\infty} \frac{\frac{1000}{n}-\frac{789}{n^2}}{0.003+\frac{34}{n}-\frac{43}{n^2}} = \frac{0+0}{0.003+0+0} = 0$

   (b) $\lim_{n\to\infty} \frac{\sqrt[3]{n}}{n+1} = \lim_{n\to\infty} \frac{n}{n} \cdot \frac{n^{-2/3}}{1+\frac{1}{n}} = \lim_{n\to\infty} \frac{n^{-2/3}}{1+\frac{1}{n}} = \frac{0}{1+0} = 0$

   (c) $\lim_{n\to\infty}(n^3 - n^2 + n - 1) = \lim_{n\to\infty} n^3 \cdot (1 - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3}) = \infty \cdot 1 = \infty$

   (d) $\lim_{n\to\infty}(n^3 + 4n^4 - n^5 + 8n^2 - 12n) = \lim_{n\to\infty} n^5 \cdot (\frac{1}{n^2} + \frac{4}{n} - 1 + \frac{8}{n^3} - \frac{12}{n^4}) = \infty \cdot (-1) = -\infty$

   (e) $\lim_{n\to\infty} \frac{n^3-3n}{2n^3+7n-5} = \lim_{n\to\infty} \frac{n^3}{n^3} \cdot \frac{1-\frac{3}{n^2}}{2+\frac{7}{n^2}-\frac{5}{n^3}} = 1 \cdot \frac{1+0}{2+0} = \frac{1}{2}$

   (f) $\lim_{n\to\infty} \frac{n^2-3n}{2n^3+7n-5} = \lim_{n\to\infty} \frac{2-\frac{3}{n}}{n+\frac{7}{n}-\frac{5}{n^2}} = \frac{2-0}{\infty+0-0} = 0$

   (g) $\lim_{n\to\infty} \left(\frac{2n^3+3n-5}{n^3+4n^2-3}\right)^2 = \lim_{n\to\infty} \left(\frac{2+\frac{3}{n^2}-\frac{5}{n^3}}{1+\frac{4}{n}-\frac{3}{n^3}}\right)^2 = \left(\frac{2}{1}\right)^2 = 4$

   (h) $\lim_{n\to\infty} \frac{n^3-3n}{3n^2+7n-5} = \frac{n-\frac{3}{n}}{3+\frac{7}{n}-\frac{5}{n^2}} = \frac{\infty}{3} = \infty$

   (i) $\lim_{n\to\infty} \frac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}} = \lim_{n\to\infty} \frac{3^n}{3^n} \cdot \frac{(-2/3)^n+1}{3\cdot(-2/3)^{n+1}+3} = \lim_{n\to\infty} \frac{(-2/3)^n+1}{3\cdot(-2/3)^{n+1}+3} = \frac{0+1}{0+3} = \frac{1}{3}$

3. (a) $\lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + ... + \frac{n-1}{n^2}\right) = \lim_{n\to\infty} \left(\frac{1+2+...+n-1}{n^2}\right) = \lim_{n\to\infty} \frac{\frac{n(n-1)}{2}}{n^2} = \lim_{n\to\infty} \frac{1-\frac{1}{n}}{2} = \frac{1}{2}$

   (b) $\lim_{n\to\infty} \left(\frac{1+2+...+n}{n+2} - \frac{n}{2}\right) = \lim_{n\to\infty} \left(\frac{n(n+1)/2}{n+2} - \frac{n}{2}\right) = \lim_{n\to\infty} \left(\frac{n(n+1)-n(n+2)}{(n+2)2}\right) = \lim_{n\to\infty} \frac{-n}{2n+4} = \frac{-1}{2}$

   (c) $\lim_{n\to\infty} \frac{1^2+2^2+...+n^2}{n^2} = \lim_{n\to\infty} \frac{\frac{n(n+1)(2n+1)}{6}}{n^2} = \lim_{n\to\infty} \frac{2n+3+\frac{1}{n}}{6} = \infty$

   (d) $\lim_{n\to\infty} \frac{1^3+2^3+...+n^3}{n^4} = \lim_{n\to\infty} \frac{\left(\frac{n(n+1)}{2}\right)^2}{n^4} = \lim_{n\to\infty} \frac{n^2(n+1)^2}{4n^4} = \frac{1}{4}$

   (e) $\lim_{n\to\infty} \frac{(2n-3)^{20}(3n+2)^{30}}{(2n+1)^{50}} = \lim_{n\to\infty} \frac{(2-\frac{3}{n})^{20}(3+\frac{2}{n})^{30}}{(2+\frac{1}{n})^{50}} = \frac{2^{20}\cdot 3^{30}}{2^{50}} = \frac{3^{30}}{2^{30}}$

4. (a) Jelikož je $n!$ sudé číslo pro všechna $n > 1$, tak $(-1)^{n!} = 1$ pro $n > 1$, a tedy $\lim_{n\to\infty}(-1)^{n!} = 1$.

Důležité limity:

- $\lim_{n\to\infty} \frac{n^k}{a^n} = 0, \quad a > 1, k \in \mathbb{R}$
- $\lim_{n\to\infty} \frac{a^n}{n!} = 0$
- $\lim_{n\to\infty} \frac{\log_a n}{n^k} = 0 \quad a > 0, k \in \mathbb{N}$
- $\lim_{n\to\infty} \sqrt[n]{a} = 1 \quad a > 0$
- $\lim_{n\to\infty} \sqrt[n]{n} = 1$
- $\lim_{n\to\infty} \sqrt[n]{n!} = \infty.$

Zadání

1. Vypočtěte limity:
   i) $\lim_{n\to\infty} \frac{\sqrt{n+\sqrt{n+\sqrt{n}}}}{\sqrt{n+1}}$
   ii) $\lim_{n\to\infty} \frac{\sqrt{n}+\sqrt[3]{n}+\sqrt[4]{n}}{\sqrt{2n+1}}$
   iii) $\lim_{n\to\infty} \frac{\sqrt{n^2+1}-n}{n}$
   iv) $\lim_{n\to\infty} \sqrt{n(n+a)} - n, \quad a \in \mathbb{R}$
   v) $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$
   vi) $\lim_{n\to\infty} (-1)^n \sqrt{n}(\sqrt{n+1} - \sqrt{n}).$
2. Vypočtěte limity:
   i) $\lim_{n\to\infty} \sqrt[n]{A^n + B^n + C^n}, \quad A, B, C > 0$
   ii) $\lim_{n\to\infty} \frac{\operatorname{sgn}(n^3-1000n^2+1)}{n}$
   iii) $\lim_{n\to\infty} \frac{\ln n+n^3+\frac{1}{n}+e^n+5^n}{\log n+n^4+5^n+n^3 4^n}$
   iv) $\lim_{n\to\infty} \frac{(n+2)!+(n+1)!}{(n+2)!-(n+1)!}.$
3. Vypočtěte limity:
   i) $\lim_{n\to\infty} \frac{(n+1)(n^2+1)\dots(n^m+1)}{[(mn)^m+1]^{\frac{m+1}{2}}}$
   ii) $\lim_{n\to\infty} \frac{\ln(n+1)-\ln(n)}{2}$
   iii) $\lim_{n\to\infty} (\sin(\ln(n+1)) - \sin(\ln n))$
   iv) $\lim_{n\to\infty} \frac{\ln(2+e^{3n})}{\ln(3+e^{3n})}.$
4. Vypočtěte limity:
   i) $\lim_{n\to\infty} \sum_{k=1}^n \frac{1}{k(k+1)}$
   ii) $\lim_{n\to\infty} \sum_{k=1}^n \frac{1}{\sqrt{n^2+k}}.$

Řešení

1. i) $\lim_{n\to\infty} \frac{\sqrt{n+\sqrt{n+\sqrt{n}}}}{\sqrt{n+1}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}} \cdot \frac{\sqrt{1+\sqrt{\frac{1}{n}+\sqrt{\frac{1}{n^3}}}}}{1} = \lim_{n\to\infty} \sqrt{1+\sqrt{\frac{1}{n}+\sqrt{\frac{1}{n^3}}}} = 1.$

   ii) $\lim_{n\to\infty} \frac{\sqrt{n}+\sqrt[3]{n}+\sqrt[4]{n}}{\sqrt{2n+1}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}} \cdot \frac{1+n^{-1/6}+n^{-1/4}}{\sqrt{2+\frac{1}{n}}} = \frac{1}{\sqrt{2}}.$

   iii) $\lim_{n\to\infty} \frac{\sqrt{n^2+1}-n}{n} = \lim_{n\to\infty} \left(\frac{\sqrt{n^2+1}}{n} - 1\right) = 1 - 1 = 0.$

   iv)

$$\lim_{n\to\infty} \sqrt{n(n+a)} - n = \lim_{n\to\infty} (\sqrt{n(n+a)} - n) \cdot \frac{\sqrt{n(n+a)}+n}{\sqrt{n(n+a)}+n} = \lim_{n\to\infty} \frac{n(n+a)-n^2}{n(\sqrt{1(1+\frac{a}{n})}+1)}$$
$$= \lim_{n\to\infty} \frac{a}{\sqrt{1(1+\frac{a}{n})}+1} = \frac{a}{2}.$$

   v) $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{n+1-n}{\sqrt{n}(\sqrt{1+\frac{1}{n}}+1)} = 0.$

   vi) S využitím úprav z minulého příkladu dostaneme $\lim_{n\to\infty} (-1)^n \sqrt{n}(\sqrt{n+1} - \sqrt{n}) = \lim_{n\to\infty} \frac{(-1)^n}{\sqrt{1+\frac{1}{n}}+1}$. Tedy pro sudé členy posloupnosti $\{\frac{(-1)^n}{\sqrt{1+\frac{1}{n}}+1}\}$ dostaneme limitu $\frac{1}{2}$, zatímco pro liché členy limitu $-\frac{1}{2}$. Proto posloupnost $\{(-1)^n\sqrt{n}(\sqrt{n+1} - \sqrt{n})\}$ nemá limitu.

2. i) Uvažujme nejdříve, že $A \geq B$ a $A \geq C$. Pak $A = \sqrt[n]{A^n} \leq \sqrt[n]{A^n + B^n + C^n} \leq \sqrt[n]{A^n + A^n + A^n} = A \cdot \sqrt[n]{3}$. Jelikož $\lim_{n\to\infty} \sqrt[n]{A} = A$ a $\lim_{n\to\infty} A \cdot \sqrt[n]{3} = A$, tak i $\lim_{n\to\infty} \sqrt[n]{A^n + B^n + C^n} = A$ (věta o dvou policajtech). Tedy obecně dostaneme $\lim_{n\to\infty} \sqrt[n]{A^n + B^n + C^n} = \max\{A, B, C\}$.

   ii) Jelikož $-1 \leq \operatorname{sgn} \leq 1$ a $\lim_{n\to\infty} \frac{-1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$, tak je $\lim_{n\to\infty} \frac{\operatorname{sgn}(n^3-1000n^2+1)}{n} = 0$.

   iii)

$$\lim_{n\to\infty} \frac{\ln n + n^3 + \frac{1}{n} + e^n + 5^n}{\log n + n^4 + 5^n + n^3 4^n} = \lim_{n\to\infty} \frac{5^n}{5^n} \cdot \frac{\frac{\ln n}{5^n} + \frac{n^3}{5^n} + \frac{1}{n \cdot 5^n} + \frac{e^n}{5^n} + 1}{\frac{\log n}{5^n} + \frac{n^4}{5^n} + 1 + \frac{n^3 4^n}{5^n}}$$
$$= \lim_{n\to\infty} \frac{\frac{\ln n}{5^n} + \frac{n^3}{5^n} + \frac{1}{n \cdot 5^n} + \frac{e^n}{5^n} + 1}{\frac{\log n}{5^n} + \frac{n^4}{5^n} + 1 + \frac{n^3 4^n}{5^n}} = \frac{0+0+0+0+1}{0+0+1+0} = 1.$$

   iv) $\lim_{n\to\infty} \frac{(n+2)!+(n+1)!}{(n+2)!-(n+1)!} = \lim_{n\to\infty} \frac{(n+1)!(n+2+1)}{(n+1)!(n+2-1)} = \lim_{n\to\infty} \frac{n+3}{n+1} = 1.$

3. i) $\lim_{n\to\infty} \frac{(n+1)(n^2+1)\ldots(n^m+1)}{[(mn)^m+1]^{\frac{m+1}{2}}} = \lim_{n\to\infty} \frac{n^{\frac{m(m+1)}{2}}(1+\frac{1}{n})(1+\frac{1}{n^2})..(1+\frac{1}{n^m})}{n^{\frac{m(m+1)}{2}}[m^m+\frac{1}{n^m}]^{\frac{m+1}{2}}} = \frac{(1+0)(1+0)..(1+0)}{[m^m+0]^{\frac{m+1}{2}}} = \frac{1}{m^{m(m+1)/2}}.$

   ii) $\lim_{n\to\infty} \frac{\ln(n+1)-\ln(n)}{2} = \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n}\right)}{2} = \frac{0}{2} = 0.$

   iii)

$$\lim_{n\to\infty} (\sin(\ln(n+1)) - \sin(\ln n)) = \lim_{n\to\infty} (\sin(\ln(n+1) - \ln n + \ln n) - \sin(\ln n))$$
$$= \lim_{n\to\infty} (\sin(\ln(n+1) - \ln n)\cos(\ln n) + \cos(\ln(n+1) - \ln n)\sin(\ln n) - \sin(\ln n))$$
$$= \lim_{n\to\infty} (\sin(\ln(n+1) - \ln n)\cos(\ln n) + \lim_{n\to\infty} \sin(\ln n)(\cos(\ln(n+1) - \ln n) - 1) = 0.$$

   U první limity jsme využili faktu, že $\lim_{n\to\infty} \ln(n+1) - \ln(n) = 0$, tedy i limita $\lim_{n\to\infty} \sin(\ln(n+1) - \ln(n)) = 0$ z věty o limitě složené funkce. Dále už stačí jen využít faktu, že $\cos(\ln n)$ je omezený. U druhé limity jsme využili toho, že $\lim_{n\to\infty}(\cos(\ln(n+1) - \ln n) - 1) = 0$ (z limity složené funkce a z faktu, že $\cos(0) = 1$) a omezenosti funkce sinus.

iv)

$$\lim_{n\to\infty}\frac{\ln(2+e^{3n})}{\ln(3+e^{3n})}=\lim_{n\to\infty}\frac{\ln(e^{3n}(\frac{2}{e^{3n}}+1))}{\ln(e^{3n}(\frac{3}{e^{3n}}+1))}=\lim_{n\to\infty}\frac{\ln(e^{3n})+\ln(\frac{2}{e^{3n}}+1)}{\ln(e^{3n})+\ln(\frac{3}{e^{3n}}+1)}=\lim_{n\to\infty}\frac{\ln(e^{3n})}{\ln(e^{3n})}\cdot\frac{1+\frac{\ln(\frac{2}{e^{3n}}+1)}{\ln(e^{3n})}}{1+\frac{\ln(\frac{3}{e^{3n}}+1)}{\ln(e^{3n})}}$$

$$=\lim_{n\to\infty}\frac{1+\frac{\ln(\frac{2}{e^{3n}}+1)}{\ln(e^{3n})}}{1+\frac{\ln(\frac{3}{e^{3n}}+1)}{\ln(e^{3n})}}=\frac{1+0}{1+0}=1.$$

4. i)

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)}=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\lim_{n\to\infty}\left(1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+...+\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1.$$

ii) Využijeme toho, že $\forall k=1,...,n$ platí $\frac{1}{\sqrt{n^2+n}}\le\frac{1}{\sqrt{n^2+k}}\le\frac{1}{\sqrt{n^2+1}}$.
Dále $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+1}}=\lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}}=1$ a podobně $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+n}}=1$.
Tedy i $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}}=1$.

Zadání

1. Vypočtěte limity:
   a) $\lim_{n\to\infty} \frac{\ln(n^2+\sqrt{n})}{\ln(\sqrt[3]{n})}$
   b) $\lim_{n\to\infty} \left(\sqrt[3]{n^3+2n^2}-n\right)$
   c) $\lim_{n\to\infty} \left(1+\frac{1}{2n}\right)^{8n+7}$
2. Vypočtěte limity:
   a) $\lim_{n\to\infty} \frac{\ln(1+\sqrt{n}+\sqrt[3]{n})}{\ln(1+\sqrt[3]{n}+\sqrt[4]{n})}$
   b) $\lim_{n\to\infty} \frac{\sqrt{n^4+3n-1}-n^2}{\sqrt[3]{n^3+1}-\sqrt{n^2-1}}$
   c) $\lim_{n\to\infty} \left(\sqrt[3]{n^3+\sqrt{n}}-\sqrt[3]{n^3-1}\right)\sqrt{3n^3+1}$
3. Vypočtěte limity:
   a) $\lim_{n\to\infty} \frac{3n^3\cos(2n)}{2n^3+1}$
   b) $\lim_{n\to\infty}(\sqrt{3}-\sin n-\cos n)\frac{n^5}{\sqrt[n]{2}-1}$
   c) $\lim_{n\to\infty} \frac{(2n^2+1)\sin n}{3n^3}$.
   d) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\lfloor xk\rfloor}{n^2}, \quad x\in\mathbb{R}$.
4. Vypočtěte limity:
   a) $\lim_{n\to\infty} a_0\left(1+0.85\cdot\frac{p}{100}\cdot\frac{1}{n}\right)^{kn}, \quad k\in\mathbb{N}, a_0\in\mathbb{R}$
   b) $\lim_{n\to\infty}\left(365+\left|\cos\frac{2NR}{4}\right|^n-\left|\cos\frac{2NR}{100}\right|^n+\left|\cos\frac{2NR}{400}\right|^n\right), \quad R=90, N>1582.$

Řešení

1. a)

$$\lim_{n\to\infty}\frac{\ln(n^2+\sqrt{n})}{\ln(\sqrt[3]{n})}=\lim_{n\to\infty}\frac{\ln(n^2(1+n^{-\frac{3}{2}}))}{\frac{1}{3}\ln n}=\lim_{n\to\infty}\frac{\ln n^2+\ln(1+n^{-\frac{3}{2}})}{\frac{1}{3}\ln n}=\lim_{n\to\infty}\frac{2\ln n+\ln(1+n^{-\frac{3}{2}})}{\frac{1}{3}\ln n}$$
$$=\lim_{n\to\infty}\frac{\ln n\left(2+\frac{\ln(1+n^{-\frac{3}{2}})}{\ln n}\right)}{\frac{1}{3}\ln n}=\lim_{n\to\infty}\frac{2+\frac{\ln(1+n^{-\frac{3}{2}})}{\ln n}}{\frac{1}{3}}=6$$

b)

$$\lim_{n\to\infty}\left(\sqrt[3]{n^3+2n^2}-n\right)=\lim_{n\to\infty}\left(\sqrt[3]{n^3+2n^2}-n\right)\cdot\frac{\sqrt[3]{n^3+2n^2}^2+\sqrt[3]{n^3+2n^2}\cdot n+n^2}{\sqrt[3]{n^3+2n^2}^2+\sqrt[3]{n^3+2n^2}\cdot n+n^2}$$
$$=\lim_{n\to\infty}\frac{n^3+2n^2-n^3}{\sqrt[3]{n^3+2n^2}^2+\sqrt[3]{n^3+2n^2}\cdot n+n^2}$$
$$=\lim_{n\to\infty}\frac{2n^2}{n^2\left(\sqrt[3]{1+\frac{2}{n}}^2+\sqrt[3]{1+\frac{2}{n}}+1\right)}=\lim_{n\to\infty}\frac{2}{\sqrt[3]{1+\frac{2}{n}}^2+\sqrt[3]{1+\frac{2}{n}}+1}=\frac{2}{3}$$

c)

$$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{8n+7}=\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{2n}\cdot\left(1+\frac{1}{2n}\right)^{2n}\cdot\left(1+\frac{1}{2n}\right)^{2n}\cdot\left(1+\frac{1}{2n}\right)^{2n}\cdot\left(1+\frac{1}{2n}\right)^{7}$$
$$=e\cdot e\cdot e\cdot e\cdot 1=e^4.$$

2. a)

$$\lim_{n\to\infty}\frac{\ln(1+\sqrt{n}+\sqrt[3]{n})}{\ln(1+\sqrt[3]{n}+\sqrt[4]{n})}=\lim_{n\to\infty}\frac{\ln(\sqrt{n}(\frac{1}{\sqrt{n}}+1+\frac{1}{\sqrt[6]{n}}))}{\ln(\sqrt[3]{n}(\frac{1}{\sqrt[3]{n}}+1+\frac{1}{\sqrt[12]{n}}))}=\lim_{n\to\infty}\frac{\ln(\sqrt{n})+\ln(\frac{1}{\sqrt{n}}+1+\frac{1}{\sqrt[6]{n}})}{\ln(\sqrt[3]{n})+\ln(\frac{1}{\sqrt[3]{n}}+1+\frac{1}{\sqrt[12]{n}})}$$
$$=\lim_{n\to\infty}\frac{\ln(\sqrt{n})}{\ln(\sqrt[3]{n})}\cdot\frac{1+\frac{\ln(\frac{1}{\sqrt{n}}+1+\frac{1}{\sqrt[6]{n}})}{\ln(\sqrt{n})}}{1+\frac{\ln(\frac{1}{\sqrt[3]{n}}+1+\frac{1}{\sqrt[12]{n}})}{\ln(\sqrt[3]{n})}}=\lim_{n\to\infty}\frac{\frac{1}{2}\ln n}{\frac{1}{3}\ln n}\,\frac{1+\frac{\ln(\frac{1}{\sqrt{n}}+1+\frac{1}{\sqrt[6]{n}})}{\ln(\sqrt{n})}}{1+\frac{\ln(\frac{1}{\sqrt[3]{n}}+1+\frac{1}{\sqrt[12]{n}})}{\ln(\sqrt[3]{n})}}$$
$$=\frac{3}{2}\cdot\frac{1+0}{1+0}=\frac{3}{2}.$$

b)

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sqrt{n^4+3n-1}-n^2}{\sqrt[3]{n^3+1}-\sqrt{n^2-1}} &= \lim_{n\to\infty} \frac{\sqrt{n^4+3n-1}-n^2}{\sqrt[3]{n^3+1}-\sqrt{n^2-1}} \cdot \frac{\sqrt{n^4+3n-1}+n^2}{\sqrt{n^4+3n-1}+n^2} \\
&= \lim_{n\to\infty} \frac{n^4+3n-1-n^4}{\left(\sqrt[3]{n^3+1}-\sqrt{n^2-1}\right)\left(\sqrt{n^4+3n-1}+n^2\right)} \\
&= \lim_{n\to\infty} \frac{n\left(3-\frac{1}{n}\right)}{n^2\left(\sqrt{1+\frac{3}{n^3}-\frac{1}{n^4}}+1\right)\left(\sqrt[3]{n^3+1}-\sqrt{n^2-1}\right)} \\
&= \lim_{n\to\infty} \frac{3-\frac{1}{n}}{n\left(\sqrt{1+\frac{3}{n^3}-\frac{1}{n^4}}+1\right)\left(\sqrt[3]{n^3+1}-\sqrt{n^2-1}\right)} \cdot \frac{\sqrt[3]{n^3+1}^2+\sqrt[3]{n^3+1}\sqrt{n^2-1}+n^2-1}{\sqrt[3]{n^3+1}^2+\sqrt[3]{n^3+1}\sqrt{n^2-1}+n^2-1} \\
&= \lim_{n\to\infty} \frac{n^2\left(3-\frac{1}{n}\right)\left(\sqrt[3]{1+\frac{1}{n^3}}^2+\sqrt[3]{1+\frac{1}{n^3}}\sqrt{1-\frac{1}{n^2}}+1-\frac{1}{n^2}\right)}{n\left(\sqrt{1+\frac{3}{n^3}-\frac{1}{n^4}}+1\right)\left(n^3+1-\sqrt{n^2-1}^3\right)} \\
&= \lim_{n\to\infty} \frac{n\left(3-\frac{1}{n}\right)\left(\sqrt[3]{1+\frac{1}{n^3}}^2+\sqrt[3]{1+\frac{1}{n^3}}\sqrt{1-\frac{1}{n^2}}+1-\frac{1}{n^2}\right)}{\left(\sqrt{1+\frac{3}{n^3}-\frac{1}{n^4}}+1\right)\left(n^3+1-\sqrt{n^2-1}^3\right)} \cdot \frac{\left(n^3+1+\sqrt{n^2-1}^3\right)}{\left(n^3+1+\sqrt{n^2-1}^3\right)} \\
&= \lim_{n\to\infty} \frac{n^4\left(3-\frac{1}{n}\right)\left(\sqrt[3]{1+\frac{1}{n^3}}^2+\sqrt[3]{1+\frac{1}{n^3}}\sqrt{1-\frac{1}{n^2}}+1-\frac{1}{n^2}\right)\left(1+\frac{1}{n^3}+\sqrt{1-\frac{1}{n^2}}^3\right)}{\left(\sqrt{1+\frac{3}{n^3}-\frac{1}{n^4}}+1\right)\left((n^3+1)^2-(n^2-1)^3\right)} \\
&= \lim_{n\to\infty} \frac{n^4\left(3-\frac{1}{n}\right)\left(\sqrt[3]{1+\frac{1}{n^3}}^2+\sqrt[3]{1+\frac{1}{n^3}}\sqrt{1-\frac{1}{n^2}}+1-\frac{1}{n^2}\right)\left(1+\frac{1}{n^3}+\sqrt{1-\frac{1}{n^2}}^3\right)}{\left(\sqrt{1+\frac{3}{n^3}-\frac{1}{n^4}}+1\right)\left(n^6+2n^3+1-n^6+3n^4-3n^2+1\right)} \\
&= \lim_{n\to\infty} \frac{n^4\left(3-\frac{1}{n}\right)\left(\sqrt[3]{1+\frac{1}{n^3}}^2+\sqrt[3]{1+\frac{1}{n^3}}\sqrt{1-\frac{1}{n^2}}+1-\frac{1}{n^2}\right)\left(1+\frac{1}{n^3}+\sqrt{1-\frac{1}{n^2}}^3\right)}{n^4\left(\sqrt{1+\frac{3}{n^3}-\frac{1}{n^4}}+1\right)\left(3+\frac{2}{n}-\frac{3}{n^2}+\frac{2}{n^4}\right)} \\
&= \frac{3\cdot 3\cdot 2}{2\cdot 3} = 3.
\end{aligned}$$

c)

$$\begin{aligned}
&\lim_{n\to\infty}\left(\sqrt[3]{n^3+\sqrt{n}}-\sqrt[3]{n^3-1}\right)\sqrt{3n^3+1} \\
&= \lim_{n\to\infty}\sqrt{3n^3+1}\left(\sqrt[3]{n^3+\sqrt{n}}-\sqrt[3]{n^3-1}\right)\cdot\frac{\sqrt[3]{n^3+\sqrt{n}}^2+\sqrt[3]{n^3+\sqrt{n}}\cdot\sqrt[3]{n^3-1}+\sqrt[3]{n^3-1}^2}{\sqrt[3]{n^3+\sqrt{n}}^2+\sqrt[3]{n^3+\sqrt{n}}\cdot\sqrt[3]{n^3-1}+\sqrt[3]{n^3-1}^2} \\
&= \lim_{n\to\infty}\sqrt{3n^3+1}\cdot\frac{n^3+\sqrt{n}-(n^3-1)}{\sqrt[3]{n^3+\sqrt{n}}^2+\sqrt[3]{n^3+\sqrt{n}}\cdot\sqrt[3]{n^3-1}+\sqrt[3]{n^3-1}^2} \\
&= \lim_{n\to\infty}\frac{n^{\frac{3}{2}}\cdot\sqrt{3+\frac{1}{n^3}}\cdot\sqrt{n}\cdot\left(1-\frac{1}{\sqrt{n}}\right)}{n^2\left(\sqrt[3]{1+n^{-\frac{5}{2}}}^2+\sqrt[3]{1+n^{-\frac{5}{2}}}\cdot\sqrt[3]{1-\frac{1}{n^3}}+\sqrt[3]{1-\frac{1}{n^3}}^2\right)} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}.
\end{aligned}$$

3. a) Jelikož $\lim_{n\to\infty}\frac{3n^3}{2n^3+1}=\frac{3}{2}$ a limita $\lim_{n\to\infty}\cos(n)$ neexistuje, tak neexistuje ani limita $\lim_{n\to\infty}\frac{3n^3\cos(2n)}{2n^3+1}$.

b) Jelikož $\sqrt[n]{2}-1>0$, $\lim_{n\to\infty}(\sqrt[n]{2}-1)=0$ a $\lim_{n\to\infty}n^5=\infty$, tak $\lim_{n\to\infty}\frac{n^5}{\sqrt[n]{2}-1}=\infty$. Jelikož maximum funkce $\sin x+\cos x$ je $\sqrt{2}$, proto je $(\sqrt{3}-\sin n-\cos n)\geq\sqrt{3}-\sqrt{2}>0$. Tedy $\lim_{n\to\infty}(\sqrt{3}-\sin n-\cos n)\frac{n^5}{\sqrt[n]{2}-1}=\infty$.

c) Jelikož $\frac{-(2n^2+1)}{3n^3}\leq\frac{(2n^2+1)\sin n}{3n^3}\leq\frac{(2n^2+1)}{3n^3}$ a $\lim_{n\to\infty}\frac{-(2n^2+1)}{3n^3}=\lim_{n\to\infty}\frac{-\left(2+\frac{1}{n^2}\right)}{3n}=0=\lim_{n\to\infty}\frac{(2n^2+1)}{3n^3}$, tak $\lim_{n\to\infty}\frac{(2n^2+1)\sin n}{3n^3}=0$.

d) Využijeme faktu, že $xk-1 \leq \lfloor xk \rfloor \leq xk$, tedy $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{xk-1}{n^2} \leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\lfloor xk \rfloor}{n^2} \leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{xk}{n^2}$.
$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{xk-1}{n^2} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n}(xk-1)}{n^2} = \lim_{n\to\infty} \frac{x\frac{n(n+1)}{2}-n}{n^2} = \frac{x}{2}$.
Podobně $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{xk}{n^2} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} xk}{n^2} = \lim_{n\to\infty} \frac{x\frac{n(n+1)}{2}}{n^2} = \frac{x}{2}$. Tedy $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\lfloor xk \rfloor}{n^2} = \frac{x}{2}$.

4. a)

$$\lim_{n\to\infty} a_0 \left(1 + 0.85 \cdot \frac{p}{100} \cdot \frac{1}{n}\right)^{kn} = a_0 \lim_{n\to\infty} e^{kn \ln\left(1+0.85\cdot\frac{p}{100}\cdot\frac{1}{n}\right)} = a_0 \lim_{n\to\infty} e^{kn\frac{\ln\left(1+0.85\cdot\frac{p}{100}\cdot\frac{1}{n}\right)}{0.85\cdot\frac{p}{100}\cdot\frac{1}{n}}\cdot 0.85\cdot\frac{p}{100}\cdot\frac{1}{n}}$$
$$= a_0 e^{\frac{k\cdot 0.85\cdot p}{100}}.$$