THE DIRICHLET PROBLEM ON COMPACT CONVEX SETS

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Abstract. Let X be a compact convex set with the set $\operatorname{ext} X$ of extreme points being Lindelöf and $f\colon \operatorname{ext} X \to F$ be a bounded Baire mapping with values in a Fréchet space F. We present a necessary and sufficient condition for f to be extended to a strongly affine Baire function on the whole set X.

1. Introduction

If X is a compact convex set in a locally convex (Hausdorff) space and $\operatorname{ext} X$ is the set of all extreme points of X, the Dirichlet problem deals with the question of extending a function f defined on $\operatorname{ext} X$ to an affine function h on X. The aim of this extension, provided it exists, is to preserve as many properties of f as possible. The characterization of the possibility of an affine continuous extension was given by [1, Theorem II.4.5] (see also [2, Theorem], [4, Theorem 2.4] or [12, Theorem 2.2]). For functions of higher Baire classes, the necessary and sufficient condition was presented in [20, Theorem 3.3].

The aim of this paper is to generalize these results to the context of vector-valued Baire functions and to get rid in the characterization theorem of the assumption on envelopes, which was essential both in [1] and [20]. So our main result (see Theorem 2.1) asserts that, under some mild topological assumption imposed on the set $\operatorname{ext} X$, a bounded Baire function f defined on $\operatorname{ext} X$ is extendable to a so-called strongly affine Baire mapping on X if and only if f is annihilated by any boundary measure perpendicular to the space of affine continuous real functions on X. It is easy to see that this condition is necessary and thus the most important part of the proof is concerned with the sufficiency of this condition.

The paper is organized as follows. After introducing the necessary notions and definitions, we present our main theorems in Section 2. The next section is devoted to the proofs, and the examples witnessing the sharpness of our results are collected in the last section.

1.1. **Compact convex sets.** We will deal both with real and complex spaces. To shorten the notation we will use the symbol $\mathbb{F}$ to denote the respective field $\mathbb{R}$ or $\mathbb{C}$.

If X is a compact (Hausdorff) topological space, we denote by $\mathcal{C}(X,\mathbb{F})$ the Banach space of all $\mathbb{F}$-valued continuous functions on X equipped with the sup-norm. The dual of $\mathcal{C}(X,\mathbb{F})$ will be identified (by the Riesz representation theorem) with $\mathcal{M}(X,\mathbb{F})$, the space of $\mathbb{F}$-valued (complete) Radon measures on X equipped with

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the total variation norm and the respective weak* topology. Let $\mathcal{M}^1(X)$ stand for the set of all probability Radon measures on X. A set $B \subset X$ is *universally measurable* if it is measurable with respect to any probability Radon measure on X. If $B \subset X$ is universally measurable, we write $\mathcal{M}^1(B)$ for the set of all $\mu \in \mathcal{M}^1(X)$ with $\mu(X \setminus B) = 0$.

Let X be a convex subset of a (real or complex) vector space E and F be another (real or complex) vector space. Recall that a mapping $f\colon X \to F$ is said to be *affine* if $f(tx + (1-t)y) = tf(x) + (1-t)f(y)$ whenever $x, y \in X$ and $t \in [0,1]$. We stress that the notion of an affine function uses only the underlying structure of real vector spaces.

Let X be a compact convex set in a locally convex (Hausdorff) topological vector space. We write $\mathfrak{A}(X, \mathbb{F})$ for the space of all $\mathbb{F}$-valued continuous affine functions on X. This space is equipped with the inherited sup-norm from $\mathcal{C}(X, \mathbb{F})$ and it is its closed subspace. Given a Radon probability measure μ on X, we write $r(\mu)$ for the *barycenter of* μ, i.e., the unique point $x \in X$ satisfying $a(x) = \int_X a \, \mathrm{d}\mu$ for each affine continuous function on X (see [1, Proposition I.2.1] or [11, Chapter 7, § 20]; note that it does not matter whether we consider real or complex affine functions). Conversely, for a point $x \in X$, we denote by $\mathcal{M}_x(X)$ the set of all Radon probability measures on X with the barycenter x (i.e., the set of all probabilities *representing* x).

The usual dilation order $\prec$ on the set $\mathcal{M}^1(X)$ of Radon probability measures on X is defined as $\mu \prec \nu$ if and only if $\mu(f) \le \nu(f)$ for any real-valued convex continuous function f on X. (Recall that $\mu(f)$ is a shortcut for $\int f \, \mathrm{d}\mu$.) A measure $\mu \in \mathcal{M}^1(X)$ is said to be *maximal* if it is maximal with respect to the dilation order. In case X is metrizable, maximal measures are exactly the probabilities carried by the G_δ set $\operatorname{ext} X$ of extreme points of X (see, e.g., [1, p. 35] or [14, Corollary 3.62]). Further, if $B \supset \operatorname{ext} X$ is a Baire set, then $\mu(B) = 1$ for any maximal measure $\mu \in \mathcal{M}^1(X)$ (this follows from [1, Corollary I.4.12]).

By the Choquet representation theorem, for any $x \in X$ there exists a maximal representing measure (see [11, p. 192, Corollary] or [1, Theorem I.4.8]). A compact convex set X is termed *simplex* if this maximal measure is uniquely determined for each $x \in X$.

1.2. **Vector integration.** We will deal with vector-valued strongly affine mappings. To be able to do that we need some vector integral. We will use the Pettis approach.

Let μ be an $\mathbb{F}$-valued σ-additive measure defined on an abstract measurable space $(X, \mathcal{A})$ (i.e., X is a set and $\mathcal{A}$ is a σ-algebra of subsets of X) and F a locally convex (Hausdorff) space over $\mathbb{F}$. (To avoid confusion we stress that we will consider only finite measures.) A mapping $f\colon X \to F$ is said to be *μ-measurable* if $f^{-1}(U)$ is μ-measurable for any $U \subset F$ open. The map f is called *weakly μ-measurable* if $\tau \circ f$ is μ-measurable for each $\tau \in F^*$.

A mapping $f\colon X \to F$ is said to be *μ-integrable* over a μ-measurable set $A \subset X$ if

- $\tau \circ f \in L^1(|\mu|)$ for each $\tau \in F^*$,
- for each $B \subset A$ μ-measurable there exists an element $x_B \in F$ such that

$$\tau(x_B) = \int_B \tau \circ f \, \mathrm{d}\mu, \quad \tau \in F^*.$$

It is clear that the element x_B is uniquely determined, we denote it as $\int_B f\,\mathrm{d}\mu$. If μ is clearly determined, we say only that f is integrable.

Lemma 1.1. *Let μ be an $\mathbb{F}$-valued measure defined on a measurable space $(X,\mathcal{A})$ and F be a Fréchet space over $\mathbb{F}$. Suppose that $f\colon X\to F$ is a bounded weakly μ-measurable mapping with (essentially) separable range. Then the following assertions hold.*

(a) *The mapping f is μ-integrable.*
(b) *If μ is a probability measure and $L\subset F$ is a closed convex set such that $f(X)\subset L$, then $\mu(f)\in L$.*
(c) *Let $f_n, g\colon X\to F$ be mappings such that*
  – *f_n are weakly μ-measurable and have separable range,*
  – *the sequence $\{f_n\}$ is bounded in F (i.e., $\bigcup_{n=1}^{\infty} f_n(X)$ is bounded in F),*
  – *$f_n(x)\to g(x)$ in F for $x\in X$.*

  Then g is bounded and μ-measurable. Moreover, all the involved functions are μ-integrable and $\int_X f_n\,\mathrm{d}\mu\to\int_X g\,\mathrm{d}\mu$ in F.

Proof. See [9, Lemma 3.8 and Theorem 3.10]. □

An important class of integrable functions are Baire measurable functions. We recall that if X is a topological space, a *zero set* in X is the inverse image of a closed set in $\mathbb{R}$ under a continuous function $f\colon X\to\mathbb{R}$. The complement of a zero set is a *cozero set*. If X is normal, it follows from Tietze's theorem that a closed set is a zero set if and only if it is also a G_δ set, i.e., a countable intersection of open sets. The complement of a G_δ set is called an F_σ set. We recall that *Borel sets* are members of the σ-algebra generated by the family of all open subset of X and *Baire sets* are members of the σ-algebra generated by the family of all cozero sets in X. Thus the *Baire measurable mappings* are the functions measurable with respect to Baire sets (we recall that, given a family $\mathcal{F}$ of sets in a set X and a topological space F, a mapping $f\colon X\to F$ is called *$\mathcal{F}$-measurable* if $f^{-1}(U)\in\mathcal{F}$ for every $U\subset F$ open).

Lemma 1.2. *Let X be a compact space, $\mu\in\mathcal{M}(X,\mathbb{F})$ and $f\colon X\to F$ be a bounded Baire measurable mapping from X to a Fréchet space F over $\mathbb{F}$. Then the mapping f is μ-integrable.*

Proof. See [9, Lemma 3.9]. □

If X is a compact space, $\mu\in\mathcal{M}(X,\mathbb{F})$ and $f\colon B\to F$ is a bounded Baire measurable mapping from a Baire set $B\subset X$ to a Fréchet space F over $\mathbb{F}$, we may integrate f with respect to μ simply by setting $\widetilde{f}(x)=\begin{cases} f(x), & x\in B,\\ 0, & x\in X\setminus B,\end{cases}$ and $\int_X f\,\mathrm{d}\,\mu=\int_X\widetilde{f}\,\mathrm{d}\,\mu$.

If $\varphi\colon X\to Y$ is a continuous mapping of a compact space X onto a compact space Y and $\mu\in\mathcal{M}^1(X)$, we denote by $\varphi_\sharp\mu\in\mathcal{M}^1(Y)$ the *image* of the measure μ.

1.3. **Baire mappings.** Given a set K, a topological space L and a family of mappings $\mathcal{F}$ from K to L, we define the *Baire classes* of mappings as follows. Let $(\mathcal{F})_0=\mathcal{F}$. Assuming that $\alpha\in[1,\omega_1)$ is given and that $(\mathcal{F})_\beta$ have been already defined for each $\beta < \alpha$, we set

$$(\mathcal{F})_\alpha = \{f\colon K \to L;\ \text{there exists a sequence } (f_n) \text{ in } \bigcup_{\beta<\alpha} (\mathcal{F})_\beta \\ \text{such that } f_n \to f \text{ pointwise}\}.$$

Among other hierarchies (see Paragraph 1.5) we will use the following ones:

- If K and L are topological spaces, by $\mathcal{C}_\alpha(K, L)$ we denote the set $(\mathcal{C}(K, L))_\alpha$, where $\mathcal{C}(K, L)$ is the set of all continuous functions from K to L.
- If K is a compact convex set and L is a convex subset of a locally convex space, by $\mathfrak{A}_\alpha(K, L)$ we denote $(\mathfrak{A}(K, L))_\alpha$, where $\mathfrak{A}(K, L)$ is the set of all affine continuous functions defined on K with values in L.

1.4. **Strongly affine mappings.** If X is a compact convex set and F is a Fréchet space, a mapping $f\colon X \to F$ is called *strongly affine* if, for any measure $\mu \in \mathcal{M}^1(X)$, f is μ-integrable and $\int_X f \,\mathrm{d}\mu = f(r(\mu))$. Note that this is a strengthening of the notion of an affine mapping. Indeed, f is affine if and only if the formula holds for any finitely supported probability μ.

By [9, Fact 1.2], the mapping f is strongly affine if and only if $\tau \circ f$ is strongly affine for each $\tau \in F^*$. It is known that any affine function $f \in \mathcal{C}_1(X, \mathbb{F})$ is strongly affine (see e.g., [1, Theorem I.2.6], [15, Section 14], [19] or [14, Corollary 4.22]) and, moreover, $f \in \mathfrak{A}_1(X, \mathbb{F})$ by a result of Mokobodzki (see, e.g., [16, Théorème 80] or [14, Theorem 4.24]).

If F is a Fréchet space and $f \in \mathcal{C}_1(X, F)$ is affine then it is strongly affine (see [9, Theorem 2.1]). If F is a Banach space with the bounded approximation property, any affine function $f \in \mathcal{C}_1(X, F)$ is in $\mathfrak{A}_1(X, F)$. However, this does not hold in general. Indeed, if F is a separable reflexive Banach space which fails the compact approximation property, $X = (B_F, \text{weak})$ and $f\colon X \to F$ is the identity embedding, then f is affine, $f \in \mathcal{C}_1(X, F)$ and $f \notin \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha(X, F)$ (see [9, Example 2.3].)

For affine functions of higher Baire classes the situation is different even in the scalar case. Firstly, an affine function of the second Baire class on a compact convex set X need not be strongly affine even if X is simplex (the example is due to Choquet, see, e.g., [1, Example I.2.10], [15, Section 14] or [14, Proposition 2.63]). Further, by [23] there is a compact convex set X and a strongly affine function $f\colon X \to \mathbb{R}$ of the second Baire class which does not belong to $\bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha(X, \mathbb{R})$.

1.5. **Baire mappings and Baire measurable mappings.** Further we need to recall Baire classes of functions in topological spaces. We follow the notation of [21]. If X is a set and $\mathcal{F}$ is a family of subsets of X, $\mathcal{F}$ is an *algebra* if $\emptyset, X \in \mathcal{F}$ and $\mathcal{F}$ is closed with respect to complements and finite unions.

If X is a topological space, we write $\operatorname{Bas}(X)$ for the algebra generated by cozero sets in X and $\Sigma_2(\operatorname{Bas}(X))$ for countable unions of sets from $\operatorname{Bas}(X)$. Let F be a topological space and let

$$\operatorname{Baf}_1(X, F) = \{f\colon X \to F;\ f^{-1}(U) \in \Sigma_2(\operatorname{Bas}(X)), U \subset F \text{ open}\}.$$

Now we use pointwise limits to create higher hierarchies of functions as in Section 1.3. Starting the procedure with $\operatorname{Baf}_1(X, F)$ and creating higher families $\operatorname{Baf}_\alpha(X, F)$ as pointwise limits of sequences contained in $\bigcup_{1\le\beta<\alpha} \operatorname{Baf}_\beta(X, F)$, we obtain the hierarchy of *Baire measurable* functions.

The algebra $\mathrm{Bas}(X)$ serves as a starting point of an inductive definition of descriptive classes of sets. More precisely, $\Sigma_2(\mathrm{Bas}(X))$ consists of all countable unions of sets from $\mathrm{Bas}(X)$ and $\Pi_2(\mathrm{Bas}(X))$ of all countable intersections of sets from $\mathrm{Bas}(X)$. Proceeding inductively, for any $\alpha \in (2, \omega_1)$ we let $\Sigma_\alpha(\mathrm{Bas}(X))$ to be made of all countable unions of sets from $\bigcup_{1\le\beta<\alpha} \Pi_\beta(\mathrm{Bas}(X))$ and $\Pi_\alpha(\mathrm{Bas}(X))$ is made of all countable intersections of sets from $\bigcup_{1\le\beta<\alpha} \Sigma_\beta(\mathrm{Bas}(X))$. The union of all created additive (or multiplicative) classes is then the σ-algebra generated by $\mathrm{Bas}(X)$. These classes characterize in terms of measurability the classes $\mathrm{Baf}_\alpha(X, F)$ defined above. Precisely, the following proposition is proved in [21, Theorem 5.2].

Proposition 1.3. *Let $f\colon X \to F$ be a function on a Tychonoff space X with values in a separable metrizable space F and $\alpha \in [1, \omega_1)$. Then $f \in \mathrm{Baf}_\alpha(X)$ if and only if f is $\Sigma_{\alpha+1}(\mathrm{Bas}(X))$-measurable.*

If we take $\Phi_0 = \mathcal{C}(X, F)$, i.e., the family of all continuous mapping from X to F, and create the hierarchy of functions using pointwise limits, we have the following result (see [24, Theorem 3.7(i)]).

Proposition 1.4. *If X is a normal topological space and L is a convex subset of a separable Fréchet space, then $\mathcal{C}_1(X, L) = \mathrm{Baf}_1(X, L)$. Thus $\mathcal{C}_\alpha(X, L) = \mathrm{Baf}_\alpha(X, L)$ for each $\alpha \in [1, \omega_1)$.*

A set $A \subset X$, where X is a topological space, is *resolvable* (or an *H-set*) if for any nonempty $B \subset X$ (equivalently, for any nonempty closed $B \subset X$) there exists a relatively open $U \subset B$ such that either $U \subset A$ or $U \cap A = \emptyset$. By [14, Proposition A.117], a subset of a completely metrizable space is resolvable if and only if it is of type F_σ and G_δ.

We recall that a topological space is *Lindelöf*, if its any open cover has a countable subcover.

2. Main results

Now we can formulate our main results which are vector-valued variant of the mentioned result [1, Theorem II.4.5] due to E.M. Alfsen for continuous functions, which reads as follows:

Let X be a compact convex set and $f\colon \operatorname{ext} X \to \mathbb{R}$ be a bounded continuous function. Then f can be extended to a function in $\mathfrak{A}(X, \mathbb{R})$ if and only if the following two conditions are satisfied:

(i) *The upper and lower envelope of f coincide on $\overline{\operatorname{ext} X}$.*
(ii) *The restriction g of the upper envelope of f to $\overline{\operatorname{ext} X}$ satisfies $\mu(g) = \nu(g)$ for every $x \in X$ and any pair μ, ν of maximal measure in $\mathcal{M}_x(X)$.*

Theorem 2.1 below gets rid of the condition (i) and deals only with the condition (ii). Also, it generalizes known results on extension of functions on simplices (see e.g. [9]) because when X is a simplex, the condition (ii) is vacuously satisfied (see [1, p. 106]).

We mention that if X is a compact convex set with $\operatorname{ext} X$ being Lindelöf, any bounded Baire mapping on $\operatorname{ext} X$ with values in a Fréchet space F admits a bounded Baire extension to X (see [9, Lemma 7.1]).

Theorem 2.1. *Let X be a compact convex set with $\operatorname{ext} X$ being Lindelöf and let F be a Fréchet space. Let $f \in \mathcal{C}_\alpha(\operatorname{ext} X, F)$ be a bounded function for some $\alpha \in [0, \omega_1)$. Then the following conditions are equivalent:*

(i) *There exists a strongly affine function* $h \in \mathcal{C}_{1+\alpha}(X, F)$ *extending* f.
(ii) *For any* $x \in X$*, any pair of maximal measures* $\mu, \nu \in \mathcal{M}_x(X)$ *and any bounded Baire extension* $g\colon X \to F$ *of* f *we have* $\mu(g) = \nu(g)$.

We hasten to add that a strongly affine Baire extension of f is unique.

Indeed, if g_1, g_2 are bounded Baire strongly affine extensions of f and $x \in X$, let us assume that F is a real vector space. Then for each functional $\tau \in F^*$, the functions $\tau \circ g_1$ and $\tau \circ g_2$ are bounded Baire strongly affine real functions, which coincide on $\operatorname{ext} X$. Thus for a maximal measure $\mu \in \mathcal{M}_x(X)$, the set $B = \{y \in X;\, \tau(g_1(y)) = \tau(g_2(y))\}$ is Baire and contains $\operatorname{ext} X$. Thus $\mu(B) = 1$, which implies

$$\tau(g_1(x)) = \mu(\tau \circ g_1) = \int_B \tau \circ g_1 \,\mathrm{d}\,\mu = \int_B \tau \circ g_2 \,\mathrm{d}\,\mu = \mu(\tau \circ g_2) = \tau(g_2(x)).$$

Since F^* separates points of F, $g_1(x) = g_2(x)$.

In case X is metrizable, maximal measures are carried by $\operatorname{ext} X$, and thus we obtain the following corollary.

Corollary 2.2. *Let* X *be a metrizable compact convex set and let* F *be a Fréchet space. Let* $f \in \mathcal{C}_\alpha(\operatorname{ext} X, F)$ *be a bounded function for some* $\alpha \in [0, \omega_1)$*. Then the following conditions are equivalent:*

(i) *There exists a strongly affine function* $h \in \mathcal{C}_{1+\alpha}(X, F)$ *extending* f.
(ii) *For any* $x \in X$ *and any pair of maximal measures* $\mu, \nu \in \mathcal{M}_x(X)$ *we have* $\mu(f) = \nu(f)$.

If $\operatorname{ext} X$ is moreover a resolvable set, we can improve the shift of the class for $\alpha \geq 1$ to be the best possible.

Theorem 2.3. *Let* X *be a compact convex set with* $\operatorname{ext} X$ *being a Lindelöf resolvable set and let* F *be a Fréchet space. Let* $f \in \mathcal{C}_\alpha(\operatorname{ext} X, F)$ *be a bounded function for some* $\alpha \in [1, \omega_1)$*. Then the following conditions are equivalent:*

(i) *There exists a strongly affine function* $h \in \mathcal{C}_\alpha(X, F)$ *extending* f.
(ii) *For any* $x \in X$*, any pair of maximal measures* $\mu, \nu \in \mathcal{M}_x(X)$ *and any bounded Baire extension* $g\colon X \to F$ *of* f *we have* $\mu(g) = \nu(g)$.

Since $\operatorname{ext} X$ is a G_δ subset of a compact convex set X provided X is metrizable, it is resolvable if and only if it is of type F_σ. Hence Theorem 2.3 implies the following corollary.

Corollary 2.4. *Let* X *be a metrizable compact convex set with* $\operatorname{ext} X$ *being an* F_σ *set and let* F *be a Fréchet space. Let* $f \in \mathcal{C}_\alpha(\operatorname{ext} X, F)$ *be a bounded function for some* $\alpha \in [1, \omega_1)$*. Then the following conditions are equivalent:*

(i) *There exists a strongly affine function* $h \in \mathcal{C}_\alpha(X, F)$ *extending* f.
(ii) *For any* $x \in X$ *and any pair of maximal measures* $\mu, \nu \in \mathcal{M}_x(X)$ *we have* $\mu(f) = \nu(f)$.

Extending continuous mappings requires the set $\operatorname{ext} X$ to be closed. In this case, maximal measures are carried by $\operatorname{ext} X$ and thus we obtain the following result.

Theorem 2.5. *Let* X *be a compact convex set with* $\operatorname{ext} X$ *being a closed set and let* F *be a Fréchet space. Let* $f \in \mathcal{C}(\operatorname{ext} X, F)$ *be a bounded function. Then the following conditions are equivalent:*

(i) *There exists a continuous affine function* $h \in \mathfrak{A}(X, F)$ *extending* f.

(ii) *For any* $x \in X$ *and any pair of maximal measures* $\mu, \nu \in \mathcal{M}_x(X)$ *we have* $\mu(f) = \nu(f)$.

If we consider the Fréchet space F to be a Banach space with the bounded approximation property, we can improve the class of the extension in the following way. (This generalizes the result from [7]. Recall that a Banach space F is said to have *the bounded approximation property* if there exists $\lambda \geq 1$ such that for every $\varepsilon > 0$ and every compact set $K \subset F$ there is a finite-rank operator L on F such that $\sup_{x \in K} \|Lx - x\| \leq \varepsilon$ and $\|L\| \leq \lambda$.)

Theorem 2.6. *Let* X *be a compact convex set with* $\operatorname{ext} X$ *being Lindelöf. Let* $f \in \mathcal{C}(\operatorname{ext} X, F)$ *be a bounded mapping with values in a Banach space* F *with the bounded approximation property. Then the following conditions are equivalent:*

(i) *There exists a function* $h \in \mathfrak{A}_1(X, F)$ *extending* f.
(ii) *For any* $x \in X$, *any pair of maximal measures* $\mu, \nu \in \mathcal{M}_x(X)$ *and any bounded Baire extension* $g \colon X \to F$ *of* f *we have* $\mu(g) = \nu(g)$.

3. Proofs

The strategy of the proofs is based on the ideas from [22]. The main idea is to extend a given Baire function on $\operatorname{ext} X$ to a Baire set $B \supset \operatorname{ext} X$ in an "affine" way and then to use the assumption to define the extension by the only possible way, namely by the integral with respect to a maximal measure. The most difficult part of the proof is to show that this extension is Baire and strongly affine.

Lemma 3.1. *Let* K *be a compact Hausdorff topological space,* F *be a Fréchet space and* $f \colon K \to F$ *be a bounded function in* $\mathcal{C}_\alpha(K, F)$ *for some* $\alpha \in [0, \omega_1)$. *Let* $L = \overline{\operatorname{co}} f(K)$. *Then the function* $\widetilde{f} \colon \mathcal{M}^1(K) \to F$ *defined as* $\widetilde{f}(\mu) = \mu(f)$, $\mu \in \mathcal{M}^1(K)$, *is well defined and contained in* $\mathfrak{A}_\alpha(\mathcal{M}^1(K), L)$. *In particular, for* $\Lambda \in \mathcal{M}^1(\mathcal{M}^1(K))$ *with* $r(\Lambda) = \lambda \in \mathcal{M}^1(K)$ *we have* $\Lambda(\widetilde{f}) = \widetilde{f}(\lambda)$.

Proof. Lemma 1.2 implies that $\widetilde{f}$ is well defined and, by Lemma 1.1(b), $\widetilde{f}(\mu) \in L$ for each $\mu \in \mathcal{M}^1(K)$.

We consider first the case $\alpha = 0$, i.e., the case when f is continuous. First we want to show that $\widetilde{f}$ is continuous. Since L is a compact convex subset of F, the original topology coincides with the weak topology on L. For any $\tau \in F^*$, the function $\tau \circ f$ is continuous on K, and thus the mapping

$$\mu \mapsto \mu(\tau \circ f) = \tau(\mu(f)) = (\tau \circ \widetilde{f})(\mu), \quad \mu \in \mathcal{M}^1(K),$$

is continuous. Hence $\widetilde{f} \colon \mathcal{M}^1(K) \to (L, \text{weak})$ is continuous, and thus continuous with respect to the original topology.

Secondly, the function $\widetilde{f}$ is affine on $\mathcal{M}^1(K)$, since for a convex combination $c\mu_1 + (1-c)\mu_2$ ($c \in [0,1]$, $\mu_1, \mu_2 \in \mathcal{M}^1(K)$) we have

$$\begin{aligned}\widetilde{f}(c\mu_1 + (1-c)\mu_2) &= (c\mu_1 + (1-c)\mu_2)(f) = c\mu_1(f) + (1-c)\mu_2(f) = \\ &= c\widetilde{f}(\mu_1) + (1-c)\widetilde{f}(\mu_2).\end{aligned}$$

Hence $\widetilde{f} \in \mathfrak{A}(\mathcal{M}^1(K), L) \subset \mathfrak{A}(\mathcal{M}^1(K), F)$, and thus it is strongly affine by [9, Fact 1.2]. Hence $\Lambda(\widetilde{f}) = \widetilde{f}(r(\Lambda))$ for each $\Lambda \in \mathcal{M}^1(\mathcal{M}^1(K))$.

If $\alpha > 0$, we can finish the proof by transfinite induction using Lemma 1.1(c). Indeed, assume that the assertion is true for all β smaller then some $\alpha \in [1, \omega_1)$.

Let $f\colon K \to F$ be a bounded function in $\mathcal{C}_\alpha(K,F)$. Then $L = \overline{\mathrm{co}} f(K)$ is a bounded separable closed subset of F and f is in $\mathcal{C}_\alpha(K,L)$. Indeed, f is $\Sigma_{\alpha+1}(\mathrm{Bas}(K))$-measurable as a mapping to F. Obviously, f is $\Sigma_{\alpha+1}(\mathrm{Bas}(K))$-measurable as a mapping to L. By Proposition 1.3, $f \in \mathrm{Baf}_\alpha(K,L)$. Thus by Proposition 1.4, $f \in \mathcal{C}_\alpha(K,L)$. Hence there exist ordinals $\alpha_n < \alpha$ and functions $f_n \in \mathcal{C}_{\alpha_n}(K,L)$ such that $f_n \to f$. Then $\widetilde{f_n} \in \mathfrak{A}_{\alpha_n}(\mathcal{M}^1(K),L)$ by the inductive assumption and Lemma 1.1(b). Moreover, $\widetilde{f}$ is well defined and $\widetilde{f_n} \to \widetilde{f}$ by Lemma 1.1(c). Thus $\widetilde{f} \in \mathfrak{A}_\alpha(\mathcal{M}^1(K),L)$ and again by Lemma 1.1(c), $\Lambda(\widetilde{f}) = \widetilde{f}(r(\Lambda))$ for each $\Lambda \in \mathcal{M}^1(\mathcal{M}^1(K))$. $\square$

We recall that a universally measurable subset F of a compact convex set X is *measure extremal*, provided $\lambda(F) = 1$ whenever $\lambda \in \mathcal{M}^1(X)$ with $r(\lambda) \in F$. It is *measure convex* if $r(\lambda) \in F$ for any $\lambda \in \mathcal{M}^1(F)$. A convex subset F of X is a *face*, if it is *extremal*, i.e. if $cx + (1-c)y \in F$ with $x, y \in X$ and $c \in (0,1)$ implies $x, y \in F$.

Lemma 3.2. *Let K be a compact Hausdorff topological space and $B \subset K$ be a universally measurable set. Then the set $\mathcal{M}^1(B) = \{\mu \in \mathcal{M}^1(K);\, \mu(B) = 1\}$ is a measure extremal and measure convex face of $\mathcal{M}^1(K)$.*

Proof. By [14, Proposition 5.30], the function $\widetilde{f}(\mu) = \mu(B)$, $\mu \in \mathcal{M}^1(K)$, is a strongly affine function on $\mathcal{M}^1(K)$ with values in $[0,1]$. Thus $\widetilde{B} = \{\mu \in \mathcal{M}^1(K);\, \widetilde{f}(\mu) = 1\}$ is a universally measurable face of $\mathcal{M}^1(K)$. Let $\Lambda \in \mathcal{M}^1(\mathcal{M}^1(K))$ satisfy $\Lambda(\widetilde{B}) = 1$. Then

$$1 = \int_{\widetilde{B}} \widetilde{f} \,\mathrm{d}\,\Lambda = \Lambda(\widetilde{f}) = \widetilde{f}(r(\Lambda)) = (r(\Lambda))(B).$$

Hence $r(\Lambda) \in \widetilde{B}$ and $\widetilde{B}$ is measure convex.

If $\Lambda \in \mathcal{M}^1(\mathcal{M}^1(K))$ satisfies $r(\Lambda) \in \widetilde{B}$, then we have

$$\begin{aligned} 1 = r(\Lambda)(B) = \widetilde{f}(r(\Lambda)) &= \int_{\mathcal{M}^1(K)} \widetilde{f} \,\mathrm{d}\,\Lambda = \int_{\widetilde{B}} \widetilde{f} \,\mathrm{d}\,\Lambda + \int_{\mathcal{M}^1(K)\setminus\widetilde{B}} \widetilde{f} \,\mathrm{d}\,\Lambda \\ &= \int_{\widetilde{B}} 1 \,\mathrm{d}\,\Lambda + \int_{\mathcal{M}^1(K)\setminus\widetilde{B}} \widetilde{f} \,\mathrm{d}\,\Lambda = \Lambda(\widetilde{B}) + \int_{\mathcal{M}^1(K)\setminus\widetilde{B}} \widetilde{f} \,\mathrm{d}\,\Lambda. \end{aligned}$$

Thus if $\Lambda(\widetilde{B}) < 1$, we get $\Lambda(\mathcal{M}^1(K) \setminus \widetilde{B}) > 0$, and hence the following inequality

$$1 = \Lambda(\widetilde{B}) + \int_{\mathcal{M}^1(K)\setminus\widetilde{B}} \widetilde{f} \,\mathrm{d}\,\Lambda < \Lambda(\widetilde{B}) + \Lambda(\mathcal{M}^1(K) \setminus \widetilde{B}) = 1$$

gives a contradiction. Thus $\Lambda(\widetilde{B}) = 1$ and $\widetilde{B}$ is measure extremal. $\square$

Lemma 3.3. *Let $\varphi\colon X \to Y$ be an affine continuous surjection of a compact convex set X onto a compact convex set Y. If $\lambda \in \mathcal{M}^1(X)$, then $r(\varphi_\sharp(\lambda)) = \varphi(r(\lambda))$.*

Proof. For each $h \in \mathfrak{A}(Y,\mathbb{R})$ we have

$$h(r(\varphi_\sharp(\lambda))) = (\varphi_\sharp(\lambda))(h) = \lambda(h \circ \varphi) = (h \circ \varphi)(r(\lambda)) = h(\varphi(r(\lambda))).$$

Thus $r(\varphi_\sharp(\lambda)) = \varphi(r(\lambda))$. $\square$

Lemma 3.4. *For $i = 1, 2$, let F_i be a measure extremal subset of a compact convex set X_i. Then $F_1 \times F_2$ is a measure extremal subset of $X_1 \times X_2$.*

Proof. First we observe that $F_1 \times F_2$ is λ-measurable for each $\lambda \in \mathcal{M}^1(X_1 \times X_2)$. To this end, it is enough to verify that $F_1 \times X_2$ is λ-measurable, since

$$F_1 \times F_2 = (F_1 \times X_2) \cap (X_1 \times F_2).$$

If $\varepsilon > 0$ and $\pi_1\colon X_1 \times X_2 \to X_1$ is the projection, let $K \subset F_1 \subset G$ be sets such that K is compact, G open and $(\pi_1)_\sharp(\lambda)(G \setminus K) < \varepsilon$. Then $K \times X_2 \subset F_1 \times X_2 \subset G \times X_2$ and

$$\lambda((G \times X_2) \setminus (K \times X_2)) = \lambda((G \setminus K) \times X_2) = (\pi_1)_\sharp(\lambda)(G \setminus K) < \varepsilon.$$

Hence $F_1 \times X_2$ is λ-measurable.

Let now $\lambda \in \mathcal{M}^1(X_1 \times X_2)$ with $r(\lambda) = (x_1, x_2) \in F_1 \times F_2$ be given. By Lemma 3.3, $r((\pi_1)_\sharp(\lambda)) = \pi_1(r(\lambda)) = \pi_1((x_1, x_2)) = x_1 \in F_1$. By the measure extremality of F_1, $(\pi_1)_\sharp(\lambda)(F_1) = 1$. Hence

$$\lambda(F_1 \times X_2) = \lambda(\pi_1^{-1}(F_1)) = (\pi_1)_\sharp(\lambda)(F_1) = 1.$$

Similarly we obtain $\lambda(X_1 \times F_2) = 1$, and thus $\lambda(F_1 \times F_2) = 1$. □

The next Lemma 3.5 is a preliminary version of Lemma 3.6.

Lemma 3.5. *Let X be a compact convex set with $\operatorname{ext} X$ being Lindelöf, F be a Fréchet space and let $f\colon \operatorname{ext} X \to F$ be a bounded mapping in $\mathcal{C}_\alpha(\operatorname{ext} X, F)$ for some $\alpha \in [1, \omega_1)$. Let $L = \overline{\text{co}} f(\operatorname{ext} X)$. Then there exists a Baire set $B \supset \operatorname{ext} X$ and a bounded mapping $g \in \mathcal{C}_\alpha(B, L)$ such that*

- *$g = f$ on $\operatorname{ext} X$,*
- *the function $\widetilde{g}\colon \mathcal{M}^1(B) \to F$ defined $\widetilde{g}(\mu) = \mu(g)$, $\mu \in \mathcal{M}^1(B)$, is in $\mathcal{C}_\alpha(\mathcal{M}^1(B), L)$,*
- *for each $\Lambda \in \mathcal{M}^1(\mathcal{M}^1(X))$ with $\Lambda(\mathcal{M}^1(B)) = 1$ we have $\Lambda(\widetilde{g}) = \widetilde{g}(r(\Lambda))$ (we remark that $r(\Lambda) \in \mathcal{M}^1(B)$ by Lemma 3.2).*

Proof. We will prove the result by transfinite induction on α. Suppose first that $\alpha = 1$, i.e., that $f \in \mathcal{C}_1(\operatorname{ext} X, F)$. Since L is separable and completely metrizable (indeed, if f is continuous, the range $f(\operatorname{ext} X)$ is Lindelöf, and thus separable. For functions of higher Baire classes the assertion easily follows by transfinite induction), by [8, Theorem 30 and Proposition 28] there is an extension $g\colon X \to L$ which is $\Sigma_2(\operatorname{Bas}(X))$-measurable. Proposition 1.4 now implies that $g \in \mathcal{C}_1(X, L)$. By setting $B = X$ we obtain from Lemma 3.1 that $\widetilde{g} \in \mathfrak{A}_1(\mathcal{M}^1(X), L)$. Thus for each $\Lambda \in \mathcal{M}^1(\mathcal{M}^1(X))$ we have $\Lambda(\widetilde{g}) = \widetilde{g}(r(\Lambda))$.

Assume now that $\alpha > 1$ and the assertion is valid for all $\beta \in [1, \alpha)$. Suppose that $f \in \mathcal{C}_\alpha(\operatorname{ext} X, F)$ is a bounded mapping. Then $f \in \mathcal{C}_\alpha(\operatorname{ext} X, L)$ by Proposition 1.4 (note that $\operatorname{ext} X$ is normal, being Lindelöf and regular), and thus there exist mappings $f_n \in \bigcup_{\beta<\alpha} \mathcal{C}_\beta(\operatorname{ext} X, L)$, $n \in \mathbb{N}$, converging pointwise to f on $\operatorname{ext} X$. For each $n \in \mathbb{N}$, let $B_n \supset \operatorname{ext} X$ be a Baire set and

$$g_n \in \mathcal{C}_{\beta_n}(B_n, \overline{\text{co}} f_n(\operatorname{ext} X)) \subset \mathcal{C}_{\beta_n}(B_n, L)$$

for some $\beta_n < \alpha$ be such that

- $g_n = f_n$ on $\operatorname{ext} X$,
- the function $\widetilde{g_n}\colon \mathcal{M}^1(B_n) \to F$ defined by $\widetilde{g_n}(\mu) = \mu(g_n)$, $\mu \in \mathcal{M}^1(B_n)$, is in $\mathcal{C}_{\beta_n}(\mathcal{M}^1(B_n), L)$,
- for each $\Lambda \in \mathcal{M}^1(\mathcal{M}^1(X))$ with $\Lambda(\mathcal{M}^1(B_n)) = 1$ we have $r(\Lambda) \in \mathcal{M}^1(B_n)$ and $\Lambda(\widetilde{g_n}) = \widetilde{g_n}(r(\Lambda))$.

Let

$$B = \{x \in \bigcap_{n=1}^{\infty} B_n;\ (g_n(x)) \text{ converges}\}.$$

Let ρ be a compatible complete metric on F. Then

$$B = \{x \in \bigcap_{n=1}^{\infty} B_n;\ \forall k \in \mathbb{N}\ \exists l \in \mathbb{N}\ \forall m_1, m_2 \geq l\colon \rho(g_{m_1}(x), g_{m_2}(x)) < \frac{1}{k}\},$$

which gives that B is a Baire subset of X. Obviously, $B \supset \operatorname{ext} X$ and the function $g(x) = \lim g_n(x)$, $x \in B$, satisfies $g = f$ on $\operatorname{ext} X$. Let $\mu \in \mathcal{M}^1(B)$ be arbitrary. Since $g_n \to g$ on B, from Lemma 1.1(c) we obtain $\widetilde{g}(\mu) = \lim \widetilde{g_n}(\mu)$. By the inductive assumption, $\widetilde{g_n} \in \mathcal{C}_{\beta_n}(\mathcal{M}^1(B_n), L)$. Thus $\widetilde{g} \in \mathcal{C}_\alpha(\mathcal{M}^1(B), L)$. If $\Lambda \in \mathcal{M}^1(\mathcal{M}^1(K))$ with $\Lambda(\mathcal{M}^1(B)) = 1$, then $r(\Lambda) \in \mathcal{M}^1(B)$ by Lemma 3.2. Thus by the inductive assumption and Lemma 1.1(c) we obtain

$$\begin{aligned}\Lambda(\widetilde{g}) &= \int_{\mathcal{M}^1(B)} \widetilde{g}\,\mathrm{d}\,\Lambda = \lim_{n\to\infty} \int_{\mathcal{M}^1(B)} \widetilde{g_n}\,\mathrm{d}\,\Lambda = \lim_{n\to\infty} \int_{\mathcal{M}^1(B_n)} \widetilde{g_n}\,\mathrm{d}\,\Lambda \\ &= \lim_{n\to\infty} \widetilde{g_n}(r(\Lambda)) = \widetilde{g}(r(\Lambda)).\end{aligned}$$

□

The following "affine partial extension" result is the decisive step to the proof of Theorem 2.1.

Lemma 3.6. *Let X be a compact convex set with $\operatorname{ext} X$ being Lindelöf, F be a Fréchet space and f be a bounded function in $\mathcal{C}_\alpha(\operatorname{ext} X, F)$ for some $\alpha \in [1, \omega_1)$. Let $L = \overline{\operatorname{co}} f(\operatorname{ext} X)$. Then there exist a Baire set $B \supset \operatorname{ext} X$ and a mapping $g\colon B \to L$ such that*

- *$g \in \mathcal{C}_\alpha(B, L)$,*
- *$g = f$ on $\operatorname{ext} X$,*
- *for each $\mu \in \mathcal{M}^1(B)$ with $r(\mu) \in B$ holds $g(r(\mu)) = \mu(g)$,*
- *the function $\widetilde{g}\colon \mathcal{M}^1(B) \to F$ defined as $\widetilde{g}(\mu) = \mu(g)$, $\mu \in \mathcal{M}^1(B)$, is in $\mathcal{C}_\alpha(\mathcal{M}^1(B), L)$,*
- *for each $\Lambda \in \mathcal{M}^1(\mathcal{M}^1(X))$ with $\Lambda(\mathcal{M}^1(B)) = 1$ we have $\Lambda(\widetilde{g}) = \widetilde{g}(r(\Lambda))$ (we remark that $r(\Lambda) \in \mathcal{M}^1(B)$ by Lemma 3.2).*

Proof. Without loss of generality we may assume that F is real because otherwise we would consider on F only multiplication by real numbers.

Since $f \in \mathcal{C}_\alpha(\operatorname{ext} X, F)$ and $\operatorname{ext} X$ is Lindelöf, $f(\operatorname{ext} X)$ is separable. Thus we may assume that F itself is separable. Let (τ_n) in F^* be a sequence separating points of F (see [18, Chapter 3, Exercise 28]).

By [13, Lemma 4.6], for each $n \in \mathbb{N}$ there exist a Baire set $B_n \supset \operatorname{ext} X$ and a bounded Baire function $f_n\colon B_n \to \mathbb{R}$ such that

- $f_n = \tau_n \circ f$ on $\operatorname{ext} X$,
- for each $\mu \in \mathcal{M}^1(B_n)$ with $r(\mu) \in B_n$ it holds $f_n(r(\mu)) = \mu(f_n)$.

By Lemma 3.5 there exist a Baire set $C \supset \operatorname{ext} X$ and a Baire function $h \in \mathcal{C}_\alpha(C, L)$ extending f such that

- the function $\widetilde{h}\colon \mathcal{M}^1(C) \to F$ defined by $\widetilde{h}(\mu) = \mu(h)$, $\mu \in \mathcal{M}^1(C)$, is in $\mathcal{C}_\alpha(\mathcal{M}^1(C), L)$,
- for each $\Lambda \in \mathcal{M}^1(\mathcal{M}^1(X))$ with $\Lambda(\mathcal{M}^1(C)) = 1$ we have $r(\Lambda) \in \mathcal{M}^1(C)$ and $\Lambda(\widetilde{h}) = \widetilde{h}(r(\Lambda))$.

Set

$$B = \{x \in C \cap \bigcap_{n=1}^{\infty} B_n ;\, \tau_n(h(x)) = f_n(x), n \in \mathbb{N}\} \quad \text{and} \quad g = h|_B.$$

Then B is a Baire set containing $\operatorname{ext} X$. Let $\mu \in \mathcal{M}^1(B)$ with $r(\mu) \in B$ be given. Then for each $n \in \mathbb{N}$ we have

$$\begin{aligned}\tau_n(\mu(g)) &= \int_B \tau_n(h(x)) \,\mathrm{d}\mu(x) = \int_B f_n(x) \,\mathrm{d}\mu(x) = f_n(r(\mu)) \\ &= \tau_n(h(r(\mu))) = \tau_n(g(r(\mu))).\end{aligned}$$

Thus $\mu(g) = g(r(\mu))$. Since $\widetilde{g}(\mu) = \widetilde{h}(\mu)$ for $\mu \in \mathcal{M}^1(B)$, we obtain that $\widetilde{g} \in \mathcal{C}_\alpha(\mathcal{M}^1(B), L)$.

Finally, let $\Lambda \in \mathcal{M}^1(\mathcal{M}^1(X))$ with $\Lambda(\mathcal{M}^1(B)) = 1$ be given. Then $r(\Lambda) \in \mathcal{M}^1(B)$ by Lemma 3.2 and

$$\Lambda(\widetilde{g}) = \int_{\mathcal{M}^1(B)} \widetilde{g} \,\mathrm{d}\,\Lambda = \int_{\mathcal{M}^1(B)} \widetilde{h} \,\mathrm{d}\,\Lambda = \int_{\mathcal{M}^1(C)} \widetilde{h} \,\mathrm{d}\,\Lambda = \widetilde{h}(r(\Lambda)) = \widetilde{g}(r(\Lambda)).$$

This finishes the proof. □

Before the proof of Theorem 2.1 we recall that a topological space is K-*analytic* if it is an image of a Polish space under an upper semicontinuous compact-valued mapping (see [17, Section 2.1]).

Proof of Theorem 2.1. (ii) $\Longrightarrow$ (i) Let $f \colon \operatorname{ext} X \to F$ be a bounded mapping in $\mathcal{C}_{\alpha'}(\operatorname{ext} X, F)$ for some $\alpha' \in [0, \omega_1)$ satisfying that $\mu(g) = \nu(g)$ for any pair of maximal measures μ, ν with the same barycenter and any bounded Baire mapping $g \colon X \to F$ extending f. Our aim is to find a strongly affine function $h \colon X \to F$ extending f. Let $L = \overline{\operatorname{co}} f(\operatorname{ext} X)$ and $\alpha = \max\{1, \alpha'\}$. Then $f \in \mathcal{C}_\alpha(\operatorname{ext} X, F)$ and $\alpha \geq 1$.

Using Lemma 3.6 we find a Baire set $B \supset \operatorname{ext} X$ along with a mapping $g \colon B \to L$ such that

- $g \in \mathcal{C}_\alpha(B, L)$,
- $g = f$ on $\operatorname{ext} X$,
- for each $\mu \in \mathcal{M}^1(B)$ with $r(\mu) \in B$ holds $g(r(\mu)) = \mu(g)$,
- the function $\widetilde{g} \colon \mathcal{M}^1(B) \to F$ defined as $\widetilde{g}(\mu) = \mu(g)$, $\mu \in \mathcal{M}^1(B)$, is in $\mathcal{C}_\alpha(\mathcal{M}^1(B), L)$,
- for each $\Lambda \in \mathcal{M}^1(\mathcal{M}^1(X))$ with $\Lambda(\mathcal{M}^1(B)) = 1$ we have $r(\Lambda) \in \mathcal{M}^1(B)$ and $\Lambda(\widetilde{g}) = \widetilde{g}(r(\Lambda))$.

We consider g to be extended by 0 on $X \setminus B$ and define

$$h(x) = \mu(g), \quad \mu \in \mathcal{M}_x(X) \text{ maximal}, \quad x \in X.$$

By our assumption (ii), the definition is correct. We aim to prove that h is a Baire function on X. To this end we show that

$$h(r(\mu)) = \widetilde{g}(\mu), \quad \mu \in \mathcal{M}^1(B). \tag{3.1}$$

So let $\mu \in \mathcal{M}^1(B)$ be given. Let $\nu \in \mathcal{M}^1(X)$ be a maximal measure with $\mu \prec \nu$ ([1, Lemma I.4.7]). Then $r(\mu) = r(\nu)$ and $\nu \in \mathcal{M}^1(B)$. By [14, Theorem 3.92], there exists a measure $\Lambda \in \mathcal{M}^1(M) \subset \mathcal{M}^1(\mathcal{M}^1(X) \times \mathcal{M}^1(X))$ such that $r(\Lambda) = (\mu, \nu)$ where $M \subset \mathcal{M}^1(X) \times \mathcal{M}^1(X)$ is a compact set defined as

$$M = \{(\varepsilon_x, \lambda) \in \mathcal{M}^1(X) \times \mathcal{M}^1(X);\, \varepsilon_x \prec \lambda\}.$$

Since $r(\Lambda) = (\mu, \nu) \in \mathcal{M}^1(B) \times \mathcal{M}^1(B)$ and the latter set is measure extremal (see Lemma 3.2 and 3.4), Λ is carried by $\mathcal{M}^1(B) \times \mathcal{M}^1(B)$.

Let $(\varepsilon_x, \lambda) \in M \cap (\mathcal{M}^1(B) \times \mathcal{M}^1(B))$. Then $r(\lambda) = x \in B$ and by the third property of g we have

$$g(x) = g(r(\lambda)) = \lambda(g). \tag{3.2}$$

Further, let $f_1, f_2 \colon B \to F$ be bounded Baire mappings. We extend f_1, f_2 by 0 on $X \setminus B$, denote them likewise, and let $N = \overline{\text{co}}(f_1(X) \cup f_2(X))$. We consider the mapping $F \colon \mathcal{M}^1(X) \times \mathcal{M}^1(X) \to F$ defined as

$$F((\lambda_1, \lambda_2)) = \lambda_1(f_1) + \lambda_2(f_2), \quad (\lambda_1, \lambda_2) \in \mathcal{M}^1(X) \times \mathcal{M}^1(X).$$

We claim that F is Λ-integrable and $\Lambda(F) = F(r(\Lambda))$. Firstly we realise that F is a bounded Baire mapping on a compact set $\mathcal{M}^1(X) \times \mathcal{M}^1(X)$, and thus it is Λ-integrable by Lemma 1.2. Indeed, the extended mappings f_1, f_2 are Baire measurable on X, and thus they are of some Baire class (see [9, Lemma 3.3]), i.e., there exists $\alpha \in [0, \omega_1)$ such that $f_1, f_2 \in \mathcal{C}_\alpha(X, N)$. By Lemma 3.1, $\lambda \mapsto \lambda(f_i)$, $i = 1, 2$, is a bounded mapping in $\mathfrak{A}_\alpha(X, N)$. In particular, F is Baire measurable and bounded, and thus Λ-integrable by Lemma 1.2.

Further, let $\tau \in F^*$ be arbitrary. Since we may assume that F is real, we get using [14, Proposition 3.90] equalities

$$\begin{aligned}
\tau\left(\Lambda(F)\right) = \Lambda(\tau \circ F) &= \int_{\mathcal{M}^1(X) \times \mathcal{M}^1(X)} \left(\tau(\lambda_1(f_1)) + \tau(\lambda_2(f_2))\right) \mathrm{d}\,\Lambda = \\
&= \int_{\mathcal{M}^1(X) \times \mathcal{M}^1(X)} \left(\lambda_1(\tau \circ f_1) + \lambda_2(\tau \circ f_2)\right) \mathrm{d}\,\Lambda \\
&= \mu(\tau \circ f_1) + \nu(\tau \circ f_2) = \tau(\mu(f_1) + \nu(f_2)) \\
&= \tau(F(r(\Lambda))).
\end{aligned}$$

Hence $\Lambda(F) = F(r(\Lambda))$.

Now we use this equality for G_1, G_2 defined by the pairs $(g, 0)$ and $(0, g)$, i.e., $G_1((\lambda_1, \lambda_2)) = \lambda_1(g)$ and $G_2((\lambda_1, \lambda_2)) = \lambda_2(g)$ for $(\lambda_1, \lambda_2) \in \mathcal{M}^1(X) \times \mathcal{M}^1(X)$. We obtain from (3.2)

$$\begin{aligned}
\widetilde{g}(\mu) = \mu(g) = G_1(r(\Lambda)) = \Lambda(G_1) &= \int_{\{(\varepsilon_x, \lambda) \in \mathcal{M}^1(B) \times \mathcal{M}^1(B);\, \varepsilon_x \prec \lambda\}} \varepsilon_x(g)\, \mathrm{d}\,\Lambda \\
&= \int_{\{(\varepsilon_x, \lambda) \in \mathcal{M}^1(B) \times \mathcal{M}^1(B);\, \varepsilon_x \prec \lambda\}} \lambda(g)\, \mathrm{d}\,\Lambda \\
&= \Lambda(G_2) = G_2(r(\Lambda)) = \nu(g) = h(r(\nu)) = h(r(\mu)).
\end{aligned}$$

Thus (3.1) holds.

From (3.1) we infer that h is a Baire mapping. Indeed, we consider the barycenter mapping $r \colon \mathcal{M}^1(B) \to X$. Then r is a continuous surjection onto X. We want to show that h is Baire measurable. To this end, let $U \subset F$ be an open set. By (3.1), $\widetilde{g} = h \circ r$. Hence

$$h^{-1}(U) = r(\widetilde{g}^{-1}(U)) \quad \text{and} \quad h^{-1}(F \setminus U) = r(\widetilde{g}^{-1}(F \setminus U)).$$

Since $\widetilde{g}$ is Baire and $\mathcal{M}^1(B) = \{\mu \in \mathcal{M}^1(K);\, \mu(B) = 1\}$, as a Baire subset of a K-analytic space, is a K-analytic space, both the sets $\widetilde{g}^{-1}(U)$ and $\widetilde{g}^{-1}(F \setminus U)$ are K-analytic as well. Since r is continuous, both the sets $h^{-1}(U)$ and $h^{-1}(F \setminus U)$ are

K-analytic and thus, being disjoint, they are Baire (see [17, Theorem 3.3.1]). Thus h is Baire measurable.

Next we verify that h is strongly measurable. Let $\lambda \in \mathcal{M}^1(X)$ with $r(\lambda) = x$ be given. Since $r\colon \mathcal{M}^1(B) \to X$ is a continuous surjection and $\mathcal{M}^1(B)$ is a K-analytic space, by [6, Corollary 432G] there exists a Radon measure $\Lambda \in \mathcal{M}^1(\mathcal{M}^1(X))$ with $\Lambda(\mathcal{M}^1(B)) = 1$ and $r_\sharp \Lambda = \lambda$. Then the barycenter μ of Λ is in $\mathcal{M}^1(B)$ (see Lemma 3.2) and $r(\mu) = r(\lambda)$. Indeed, if $a \in \mathfrak{A}(X, \mathbb{R})$ is arbitrary and $\widetilde{a} \in \mathfrak{A}(\mathcal{M}^1(X), \mathbb{R})$ is defined as $\widetilde{a}(\omega) = \omega(a)$, $\omega \in \mathcal{M}^1(X)$, then $\widetilde{a} = a \circ r$, and thus

$$a(x) = \lambda(a) = (r_\sharp \Lambda)(a) = \Lambda(a \circ r) = \Lambda(\widetilde{a}) = \widetilde{a}(\mu) = \mu(a).$$

Hence by (3.1) we have

$$\lambda(h) = (r_\sharp \Lambda)(h) = \Lambda(h \circ r) = \Lambda(\widetilde{g}) = \widetilde{g}(\mu) = h(r(\mu)) = h(x).$$

Thus h is a strongly affine Baire mapping of X into F extending f. Using [22, Theorem 2.2] we infer that $h \in \mathcal{C}_{1+\alpha'}(X, F)$. This concludes the proof of the implication (ii) $\Longrightarrow$ (i).

(i) $\Longrightarrow$ (ii) Let $h\colon X \to F$ be a strongly affine Baire extension of f. Let $g\colon X \to F$ be any bounded Baire extension of f and $\mu, \nu \in \mathcal{M}^1(X)$ be a pair of maximal measures with the same barycenter $x \in X$. Then the set $B = \{y \in X;\, h(y) = g(y)\}$ is a Baire subset of X and contains $\operatorname{ext} X$. Thus both μ, ν are carried by B and we obtain

$$\mu(g) = \int_B g \,\mathrm{d}\,\mu = \int_B h \,\mathrm{d}\,\mu = h(x) = \int_B h \,\mathrm{d}\,\nu = \int_B g \,\mathrm{d}\,\nu = \nu(g).$$

Hence (i) $\Longrightarrow$ (ii). □

Proof of Theorem 2.3. Let $f \in \mathcal{C}_\alpha(\operatorname{ext} X, F)$ be a bounded function with values in a Fréchet space F for $\alpha \in [1, \omega_1)$. By Theorem 2.1, f admits a strongly affine Baire extension h. By [22, Theorem 2.4], $h \in \mathcal{C}_\alpha(X, F)$. The reverse implication is the same as in the proof of Theorem 2.1. □

Proof of Theorem 2.5. Let $f \in \mathcal{C}(\operatorname{ext} X, F)$ be a continuous function with values in a Fréchet space F. By Theorem 2.1, f admits a strongly affine Baire extension h. By [22, Theorem 2.1(c)], $h \in \mathcal{C}(X, F)$, i.e., $h \in \mathfrak{A}(X, F)$. The reverse implication is the same as in the proof of Theorem 2.1. □

Proof of Theorem 2.6. Given a bounded function $f \in \mathcal{C}(\operatorname{ext} X, F)$, where F is a Banach space with the bounded approximation property, we extend f to a strongly affine function $h \in \mathcal{C}_1(X, F)$. By [9, Theorem 2.2], $h \in \mathfrak{A}_1(X, F)$. The reverse implication is the same as in the proof of Theorem 2.1. □

4. Examples

The first example shows that, without the assumption of the Lindelöf property of $\operatorname{ext} X$, the extension results do not hold in general.

Example 4.1. *There exist a compact convex set X and a bounded function $f \in \mathcal{C}(\operatorname{ext} X, \mathbb{R})$ such that*

- *$\mu(g) = \nu(g)$ for each pair $\mu, \nu \in \mathcal{M}^1(X)$ of maximal measures on X with the same barycenter and each bounded Borel extension g of f.*
- *there is no extension of f to an affine Borel function.*

Proof. Using the standard "porcupine" construction we consider the simplex X constructed in [9, Example 5.4] (see also [1, Proposition I.4.15]). Then if we take the set A used there to be the unit interval $[0,1]$, $\operatorname{ext} X$ can be identified with $[0,1]\times\{-1,1\}$. Moreover, the restriction of the topology of X to $\operatorname{ext} X$ is a discrete. Let $B\subset[0,1]$ be a non-Borel set and a bounded $f\in\mathcal{C}(\operatorname{ext} X,\mathbb{R})$ be defined as

$$f(x)=\begin{cases}1, & x\in B\times\{-1,1\},\\ 0, & x\in([0,1]\setminus B)\times\{-1,1\}.\end{cases}$$

Then the assumption on boundary measures is vacuously satisfied since X is a simplex and thus for each $x\in X$ there is only one maximal measure in $\mathcal{M}_x(X)$. Finally, any affine extension h of f satisfies

$$h(x)=\begin{cases}1, & x\in B,\\ 0, & x\in[0,1]\setminus B,\end{cases}\qquad x\in[0,1],$$

which gives that h is non-Borel. $\square$

The next example witnesses that the shift of classes in Theorem 2.1 may occur for each finite α.

Example 4.2. *There exists a metrizable compact convex set X such that for each $\alpha\in[0,\omega_0)$ there exists a bounded function $f\in\mathcal{C}_\alpha(\operatorname{ext} X,\mathbb{R})$ such that*

- *$\mu(f)=\nu(f)$ for each pair $\mu,\nu\in\mathcal{M}^1(X)$ of maximal measures on X with the same barycenter,*
- *there is no extension of f to a strongly affine function $h\in\mathcal{C}_\alpha(X,\mathbb{R})$.*

Proof. Let X be a metrizable simplex constructed in [10, Theorem 1.1(b)]. Then for each $\alpha\in[0,\omega_0)$ there exists a bounded function $f\in\mathcal{C}_\alpha(\operatorname{ext} X,\mathbb{R})$ such that the function $h(x)=\delta_x(f)$, $x\in X$ (here δ_x stands for the unique maximal measure in $\mathcal{M}_x(X)$) is not in $\mathcal{C}_\alpha(X,\mathbb{R})$. Since h is the only possible strongly affine extension of f, the proof is finished. $\square$

We may ask whether the strong affinity along with Baire class can be improved to an affine class. This holds for first class, i.e., any affine Baire-one real function on X is strongly affine and in $\mathfrak{A}_1(X,\mathbb{R})$. The following example shows that this is not longer true for higher Baire classes.

Example 4.3. *There exist a metrizable compact convex set X and a bounded function $f\in\mathcal{C}_2(\operatorname{ext} X,\mathbb{R})$ such that*

- *$\mu(f)=\nu(f)$ for each pair $\mu,\nu\in\mathcal{M}^1(X)$ of maximal measures on X with the same barycenter,*
- *there is no extension of f to a function $h\in\bigcup_{\alpha<\omega_1}\mathfrak{A}_\alpha(X,\mathbb{R})$.*

Proof. Let X be a metrizable compact convex set constructed in [23] (see also [14, Theorem 12.77]). Then there exists a strongly affine function $h\in\mathcal{C}_2(X,\mathbb{R})$ that is not in $\bigcup_{\alpha<\omega_1}\mathfrak{A}_\alpha(X,\mathbb{R})$. Hence $f=h|_{\operatorname{ext} X}\in\mathcal{C}_2(\operatorname{ext} X,\mathbb{R})$, but the only possible strongly affine extension of f, namely h, is not of any affine class. (To see this, consider a strongly affine extension g of f. Then for each $x\in X$, let $\mu\in\mathcal{M}_x(X)$ be maximal. Then

$$g(x)=\mu(g)=\int_{\operatorname{ext} X} g\,\mathrm{d}\,\mu=\int_{\operatorname{ext} X} f\,\mathrm{d}\,\mu=\int_{\operatorname{ext} X} h\,\mathrm{d}\,\mu=h(x).$$

$\square$

Example 4.4. *There exist a metrizable compact convex set X, a Banach space F and a function $f \in \mathcal{C}(\operatorname{ext} X, B_F)$ such that*

- $\mu(f) = \nu(f)$ *for each pair* $\mu, \nu \in \mathcal{M}^1(X)$ *of maximal measures on* X *with the same barycenter,*
- *there is no extension of* f *to a function* $h \in \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha(X, F)$.

Proof. Let F be a separable reflexive space which fails the compact approximation property (see [3, Proposition 2.12]). By [5, Theorem 8.1], F admits an equivalent locally uniformly rotund renorming. Let $X = (B_F, w)$ be the new unit ball with the weak topology. Let $h\colon X \to F$ be the identity mapping. Then h is strongly affine and in $\mathcal{C}_1(X, F)$, but $h \notin \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha(X, F)$ (this follows from [9, Example 2.3]). Since h is strongly affine, $f = h|_{\operatorname{ext} X}$ satisfies the assumption $\mu(f) = \nu(f)$ for any pair of maximal measure on X with the same barycenter and clearly the only strongly affine extension of f, namely h, is not of any affine class. Finally, since F has a locally uniformly rotund norm, the weak and the norm topology coincide on the sphere S_F (see [5, Exercise 8.45]). But $S_F \supset \operatorname{ext} B_F = \operatorname{ext} X$, and thus f is continuous on $\operatorname{ext} X$.

□

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