QUANDLE PROBLEMS FOR GROUP THEORISTS

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1. Translation Maps

Definition (Left Translation)

Let $X = (X, \triangleright)$ be a binary algebra, and let $a \in X$. The **left translation** by a is the map $\mathsf{L}_a \colon X \to X$ such that

$$\mathsf{L}_a(x) = a \triangleright x.$$

2. Quandles

Definition (Quandle)

Let $X = (X, \triangleright)$ be a binary algebra. X is a **quandle** if for all $a \in X$

1. L_a is an automorphism of X
2. $\mathsf{L}_a(a) = a$.

Example. The following structures are quandles that can be constructed from a group G.

- **Conjugation quandle**:

$$\mathsf{Conj}(G) = (G, x \triangleright y = xyx^{-1}).$$

- **Coset quandle**: $\alpha \in \mathsf{Aut}(G)$, $H \leq \mathsf{Fix}(\alpha)$

$$\mathcal{Q}(G, H, \alpha) = (G/H, xH \triangleright yH = x\alpha(x^{-1}y)H).$$

- **Affine quandle**: $\alpha \in \mathsf{Aut}(G)$, G abelian

$$\mathsf{Aff}(G, \alpha) = (G, x \triangleright y = \alpha(x) + (\mathsf{id} - \alpha)(y)).$$

Definition (Inner Group)

The **inner group** of a quandle X is

$$\mathsf{Inn}(X) = \langle \mathsf{L}_a \colon a \in X \rangle.$$

3. Connectedness

Definition (Connectedness)

Let X be a quandle, and $k \in \mathbb{N}$.

1. X is ***k*−connected** if for all $x, y \in X$ there are $a_1, \dots, a_k \in X$ such that

$$\mathsf{L}_{a_1} \dots \mathsf{L}_{a_k}(x) = y.$$

2. X is **connected** if it is k−connected for some $k \in \mathbb{N}$.

Remark. If a quandle X is $(k+1)$−connected, then it is also k−connected.

4. Regular Cycles

Definition (Regular Cycle)

Let $\sigma \in \mathsf{S}_n$ be a permutation with cycle structure $(l_1^{e_1}, \dots, l_k^{e_k})$. Then σ has a **regular cycle** if

$$l_i \mid l_k \quad \text{for } i = 1, \dots, k.$$

Example. $(1,2)(3,4)(5,6,7)(8,9,10,11,12,13)$.

5. Hayashi's Conjecture

Conjecture (Hayashi's, Original Formulation [1])

Let X be a finite connected quandle. Then every left translation of X has a regular cycle.

Remark. This conjecture has been proven true for several classes ([2], [3], [4]) Moreover, we can reformulate it in Group Theoretic terms (see [5]), providing a different approach.

Conjecture (Hayashi's, Group Theoretic Reformulation [5])

Let G be a transitive permutation group over a finite set X, and let $e \in X$. If $\zeta \in \mathsf{Z}(G_e)$ and $\langle \zeta^G \rangle = G$, then ζ has a regular cycle.

6. Some Techniques

Proposition (lcm Constraint) [6]

Let X be a finite connected quandle, and let $\Lambda(X)$ be the set of cycle lengths of any left translation. If $\Lambda(X) = A \cup B$ for some $A, B \neq \emptyset$, then $\mathsf{lcm}(A)$ divides $\mathsf{lcm}(B)$, or viceversa.

Proposition [7]

Let X be a finite connected quandle such that

1. $\mathsf{Z}(\mathsf{Inn}(X)) = 1$
2. There are $x, y \in X$ such that $\langle \{\mathsf{L}_x\} \cup \mathsf{Inn}(X)_y \rangle = \mathsf{Inn}(X)$.

Then X has a regular cycle.

Remark. The previous proposition seems to apply to the class of simple quandles.

7. A Conjecture on Connectedness Degree

Conjecture (Bound in Connectedness Degree)

Let X be a finite connected quandle. Then X is 3−connected.

Remark. This conjecture says that given any connected quandle X, and any pair of its elements x, y, we can go to y starting from x by means of three left translations at most. Of course, for $k \geq 4$ no examples of k−connected quandles are known.

Theorem (k−connectedness for Coset Quandles)

Consider a finite coset quandle $\mathcal{Q}(G, H, \alpha)$, where $\alpha \in \mathsf{Aut}(G)$ is the conjugation map by some $\zeta \in H$. Then $\mathcal{Q}(G, H, \alpha)$ is k−connected if and only if for every $x \in G$ there are $x_1, \dots, x_k \in G$ such that

$$xH = \zeta^{x_1} \dots \zeta^{x_k} H.$$

Conjecture (Group Theoretic Reformulation for Coset Quandles)

Let G be a group, $H \leq G$, and $\zeta \in H$. Then for every $x \in G$ there are at most 3 elements x_1, x_2, x_3 such that

$$xH = \zeta^{x_1} \zeta^{x_2} \zeta^{x_3} H.$$

Remark. Another special case where $X = \mathsf{Conj}(\sigma^{\mathsf{S}_n})$ and σ is an odd cycle has been intensively studied. The tests show that this quandle is even 2−connected, this means that starting from any element of σ^{S_n} we can reach any other with two conjugations.

8. References

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