EQUIVARIANT OPERATIONS IN TOPOLOGICAL HOCHSCHILD HOMOLOGY

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Abstract. We observe a new equivariant relationship between topological Hochschild homology and cohomology. We also calculate the topological Hochschild homology of the topological Hochschild cohomology of a finite prime field, which can be viewed as a certain ring of structured operations in this case.

1. Introduction

Topological Hochschild homology, with its structure of a genuine S^1-equivariant spectrum, is a remarkably strong tool, which has surprising applications. Examples include the construction of topological cyclic cohomology TC, [5, 12, 13, 14, 1], which is an effective tool for computing algebraic K-theory of complete rings [8]. Constructed in the process, another form of topological cyclic cohomology, TR was found, which was later used by Bhatt, Morrow, and Scholze, [3, 4] (see also [27, 19]) to give an approach to prismatic cohomology, unifying several known cohomology theories in p-adic Hodge theory.

There is a remarkable asymmetry between topological Hochschild homology $THH(R)$ and topological Hochschild cohomology $THC(R)$, which is known to be an algebra over the little 2-cube operad (a structure originally conjectured by Deligne), but there is no known counterpart of a genuine S^1-equivariant structure, which seems odd.

There is an old suggestion that the topological Hochschild homology $THH(R)$ and topological Hochschild cohomology $THC(R)$ for an associative S-algebra R should somehow be dual (see e.g. [29]). This stems from the idea of Koszul duality for operads [11] which, even though its statements do not apply here literally (due to the fact that associative S-algebras do not form a based category), should still have some manifestation.

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Yet the behaviors of the constructions of topological Hochschild homology and cohomology are quite different. As already remarked, $THH(R)$ forms a genuine S^1-spectrum, while, $THC(R)$ has a structure of a $\mathcal{C}_2$-algebra in the category of S-modules. This is the spectral form of the Deligne conjecture, which appears to have been first proved in [25] (even though one can also interpret spectrally the proof for chain complexes given by Hu and Kaufmann [16, 22]). How are these two structures dual, or, more broadly, what do they have in common?

It is of course not uncommon in algebraic topology for one of the partners of homology and cohomology to have more structure. For example, the ring-valued cohomology of a space is a ring, while its homology is not. Still, when learning about the structures of algebraic topology, we realize that homology of a space is a module over cohomology, a structure which works well with duality, so the disparity between the homology and cohomology of a space is largely resolved. From this point of view, the asymmetry between $THH(R)$ and $THC(R)$ for ring (or A_∞ ring spectrum R) seems much more profound, with little work in this direction done up to this point.

The purpose of this paper is to give a step in this direction by considering topological Hochschild homology of topological Hochschild cohomology:

1. **Theorem.** *Let R be an associative S-algebra. There exists a natural S^1-equivariant associative S-algebra $THHC(R)_{S^1}$ (indexed over the complete universe) equivalent to $THH(THC(R))$ over which there exists an S^1-equivariant module equivalent to $THH(R)$.*

Formulating the requisite constructions requires solving numerous technical problems, and none of the formalisms available in the literature seemed to fit our purposes completely. For this reason, we actually develop new (or modified) approaches to parts of the THH story from scratch, using S-modules [9].

It is worth noting that considering topological Hochschild homology of $THC(R)$ is, in many ways, a novel direction. While THH can be applied to any A_∞ (i.e. coherently associative) ring, $THC(R)$ is not a type of ring one would usually think of in this context. It is typically highly non-connective (as we shall see), which makes, for example, many of the methods of Nikolaus and Scholze [27] not applicable. Because of this, it seemed to make sense to do a calculation at least in one basic case. This is provided by the following result:

2. **Theorem.** *We have*

$$(1)\qquad (THH(THC(H\mathbb{F}_p)))_*^{\mathbb{Z}/p^{r-1}} = F(H\mathbb{Z}/p^r, H\mathbb{Z}/p^r)_* \otimes \Gamma_{\mathbb{Z}/p^r}(\rho)$$

where ρ is in homological degree -2 and Γ denotes the divided power algebra. Additionally,

$$(2)\qquad (TR(THC(H\mathbb{F}_p)))_* = (F(H\mathbb{Z}, H\mathbb{Z})_*)_p^\wedge \otimes \Gamma_{\mathbb{Z}}(\rho) \otimes \Lambda_{\mathbb{Z}}(q)$$

where TR is the homotopy limit of $THH^{\mathbb{Z}/p^{r-1}}$ with respect to the map R of [12] *and q has homological degree -1.*

Theorem 2 can also be interpreted as a calculation of a type of structured THH operations, in the basic case of the perfect field $\mathbb{F}_p$. The answer is remarkably small, reminding us of the result of Caruso [7] on the lack of $\mathbb{Z}/p$-equivariant cohomology operations. That, of course, was later explained in [21, 28, 20], where it was shown that to get all the expected operations, one needs to consider a twist. This in fact suggest a connection between [20] and the present paper, which is the question of the first k-invariant of THH, which will be pursued in subsequent work.

The present paper is organized as follows: Section 2 recalls some important preliminary constructions, namely the multiplicative norm and the unframed cactus operad. In Section 3, we reformulate the construction of THH in a way which is compatible with the constructions needed to prove Theorem 1. Theorem 1 is proved in Section 4. Sections 5 and 6 serve to recall some preliminary material needed in the proof of Theorem 2. In Section 5, we recall some facts about the dual Steenrod algebra and about integral Steenrod operations. In Section 6, we recall the calculation of the equivariant homotopy groups of $THH(\mathbb{F}_p)$ from our present point of view. In Section 7, we prove Theorem 2.

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2. Preliminaries

The main purpose of this section is to recall, and partially reformulate for the purposes of this paper, two important preliminary constructions: the multiplicative norm and the unframed cactus operad.

2.1. **The multiplicative norm.** One of the subjects to address is the multiplicative norm of equivariant S-modules. This was introduced by Hill, Hopkins, and Ravenel in [15] in the context of finite groups. (It had been previously introduced by Hu [17] in the context of motivic spectra, which was later studied in detail by Bachmann and Hyois [2].) For the basics on equivariant spectra, we refer the reader to [24].

Let G be a compact Lie group. Suppose $H \subseteq G$ is a subgroup of finite index. Let $\mathcal{U}$ be a complete H-universe. Then we have a complete G-universe

$$\mathcal{V} = Ind_H^G(\mathcal{U}).$$

Enumerating, once and for all, the cosets in G/H, we obtain an inner product space isomorphism

$$\mathcal{V} \cong \bigoplus_{|G/H|} \mathcal{U}. \tag{3}$$

(We use $g \in G/H$ to identify the gth copy of the H-universe $\mathcal{U}$ with a gHg^{-1}-universe.) Now, for a $\mathcal{U}$-indexed Lewis-May H-spectrum X, (3) gives the external smash product

$$\underbrace{X \wedge \cdots \wedge X}_{|G/H| \text{ copies}}$$

a structure of a $\mathcal{V}$-indexed G-spectrum. This construction become, in an obvious way, a functor from $\mathcal{U}$-indexed H-spectra to $\mathcal{V}$-indexed G-spectra, which we denote by N_H^G.

Racell from [9] the construction extends to $\mathbb{L}$-spectra (and hence to S-modules), which goes as follows: Recall that an $\mathbb{L}$-spectrum is a spectrum X with a map

$$\mathcal{I}(\mathcal{U},\mathcal{U}) \rtimes X \to X$$

satisfying the obvious associativity and unit properties. Now consider the coequalizer of

$$\mathcal{I}(\mathcal{U}^n, \mathcal{V}) \rtimes (\bigwedge_{|G/H|} \mathcal{I}(\mathcal{U},\mathcal{U}) \rtimes X) \rightrightarrows \mathcal{I}(\mathcal{U}^n, \mathcal{V}) \rtimes (\bigwedge_{|G/H|} X). \tag{4}$$

The two arrows are by composing of linear isometries, or by applying the action on X. Now on (4), both maps are actually morphisms of G-equivariant spectra indexed over $\mathcal{V}$, if we use the G-action on $\mathcal{V}$, conjugation action on isometries and coset action on the smash components.

This defines a functor from $\mathcal{U}$-indexed $\mathbb{L}$-H-spectra to $\mathcal{V}$-indexed $\mathbb{L}$-G-spectra, which further passes to S-modules. We denote all these functors by N_H^G. By construction, we have an isomorphism

$$N_{\{e\}}^{\mathbb{Z}/n}(X) \wedge N_{\{e\}}^{\mathbb{Z}/n}(Y) \cong N_{\{e\}}^{\mathbb{Z}/n}(X \wedge Y) \tag{5}$$

(where $\wedge$ denotes the symmetric monoidal smash product of S-modules), satisfying the obvious associativity and commutativity properties. The construction also preserves cell objects.

2.2. **The unframed cactus operad.** For our purposes, it is also appropriate to describe in detail the *unframed cactus operad* introduced by Voronov [30]. To start out, by a *cactus datum*, we shall mean a pair

$$(T, \mathcal{E})$$

where

$$T = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$$

is a partition of the unit interval and $\mathcal{E}$ is an equivalence relation on T such that

- Every equivalence class of $\mathcal{E}$ has > 1 element
- $0 \sim 1$
- If $E_1 = \{t_{i_1} < \cdots < t_{i_k}\}$, $E_2 = \{t_{j_1} < \cdots < t_{j_\ell}\}$ are equivalence classes of $\mathcal{E}$, then one of the following occurs:
  (a) There exist a $1 \leq s < \ell$ such that
  $$t_{j_s} < t_{i_1} < t_{i_k} < t_{j_{s+1}}$$
  or
  (b) There exist a $1 \leq s < k$ such that
  $$t_{i_s} < t_{j_1} < t_{j_\ell} < t_{i_{s+1}}.$$
  or
  (c) $t_{i_k} < t_{j_1}$
  or
  (d) $t_{j_\ell} < t_{i_1}$.

The topology on the set of all cactus data is given by defining a sequence to converge if it converges in the Hausdorff topology on the set of equivalence classes of $\mathcal{E}$, and no two points in an equivalence class are identified in the limit.

The *cactus graph* $\Gamma(T, \mathcal{E})$ associated with a cactus datum $(T, \mathcal{E})$ is obtained by identifying the elements of T which belong to the same equivalence class of $\mathcal{E}$.

A *cactus loop* of the graph $\Gamma(T, \mathcal{E})$ associated with the cactus datum $(T, \mathcal{E})$ is determined by choosing an $\mathcal{E}$-equivalence class

$$E = \{t_{i_1} < \cdots < t_{i_k}\},$$

a number $1 \leq s < k$, and taking the union of the images of all intervals $[t_j, t_{j+1}]$, $i_s \leq j < i_{s+1}$ such that for all $i_s \leq p \leq j$, $j+1 \leq q \leq i_{s+1}$,

$$t_p \not\sim t_q.$$

One readily sees that a cactus loop is indeed a loop in the graph $\Gamma(T, \mathcal{E})$. Further, form a 2-dimensional CW-complex $\Gamma_2(T, \mathcal{E})$ by attaching a 2-cell to each cactus loop homeomorphically on the bounary. Further, choosing an orientation of the 2-cell so that its boundary intervals $[t_j, t_{j+1}]$ with increasing j appear, say, in clockwise order, there is a unique (up to homeomorphism) oriented embedding $\Gamma_2(T, \mathcal{E}) \subset \mathbb{R}^2$.

Denote by $L(T, \mathcal{E})$ the set of loops of the cactus datum $(T, \mathcal{E})$. One can check that the set of pairs

$$((T, \mathcal{E}), x),\ x \in L(T, \mathcal{E})$$

forms a covering space $\widetilde{X}$ over the space X of all cactus data.

A *labelled cactus* consists of the data

$$(T, \mathcal{E}), \quad \sigma : L(T, \mathcal{E}) \xrightarrow{\cong} \{1, \dots N\}$$

where $(T, \mathcal{E})$ is a cactus datum. The *unframed cactus operad* is the set of labelled cacti with topology induced from the covering space $\widetilde{X}$.

To define the operad structure, letting a loop $\ell \in L(T, \mathcal{E})$ consists of edges

$$[t_{j_1}, t_{j_1+1}], \dots, [t_{j_m}, t_{j_m+1}], \tag{6}$$

$j_1 < j_2 < \dots < j_m$, we have

$$t_{j_s+1} \sim t_{j_{s+1}}, \tag{7}$$

$s = 1, \dots, m-1$.

Identifying (7) in (6), we obtain an interval congruent (by an increasing map) to

$$J = [0, \sum_{s=1}^{m} (t_{j_s+1} - t_{j_s})]$$

where (7) goes to

$$\sum_{p=1}^{s} (t_{j_p+1} - t_{j_p}).$$

Let h be the homothety mapping J homeomorphically onto $[0, 1]$.

Then a cactus datum

$$(S, \mathcal{F})$$

is inserted into the loop ℓ by taking the partition

$$T \cup h^{-1}(S)$$

with the equivalence relation generated by $\mathcal{E} \cup h^{-1}(\mathcal{F})$. One checks that this endows the space of all labelled cacti with an operad structure and that, moreover, the resulting operad is equivalent to the little 2-cube operad.

3. A description of $THH(R)$

By a *cyclically ordered finite set*, we mean a finite set embedded to S^1. Two embeddings are considered the same when they are related by an orientation-preserving diffeomorphism of S^1. A *cyclically ordered set* is a set Q each of whose finite subsets are cyclically ordered in a fashion compatible with inclusion. A cyclic ordering on a set Q is determined by a ternary relation of being "in the anti-clockwise order." A *morphisms of cyclically ordered sets* $f : Q \to Q'$ is a map of sets where whenever x, y, z are in the anticlockwise order in Q, then either $f(x), f(y), f(z)$ are in the anticlockwise order, or $|\{f(x), f(y), f(z)\}| < 3$. The category of finite cyclically ordered sets will be denoted by Φ.

Denoting by Top the category of topological spaces, we have a canonical functor

$$T : \Phi^{Op} \to Top$$

where $T(Q)$ is the space of morphisms of cyclic sets

$$Q \to S^1$$

(with the subspace topology of $(S^1)^Q$).

Let R be a cell associative S-algebra. Then there is a natural functor TH^R from Φ into S-modules given by setting

$$TH^R(Q) = R^{\wedge Q}$$

where the action of surjective (resp. injective) morphisms of cyclically ordered finite sets is by multiplication in R (resp. by insertion of units). Here $\wedge$ denotes the symmetric monoidal smash product of S-modules.

We consider the coend

$$THH(R) = T_+ \wedge_\Phi TH^R(Q).$$

To give $THH(R)$ a genuine $\mathbb{Z}/n$-equivariant structure, we denote by Φ_n the category whose objects are $\mathbb{Z}/n$-equivariant cyclically ordered sets Q such that $\mathbb{Z}/n$ acts freely on Q in a way which preserves cyclic ordering and multiplication by any element $x \in Q$ defines a morphism of cyclically ordered sets

$$\mathbb{Z}/n \to Q.$$

Morphisms in Φ_n are morphisms of cyclically ordered sets which preserve the $\mathbb{Z}/n$-action. We have, again, a canonical functor

$$T_n : \Phi_n^{Op} \to \mathbb{Z}/n\text{-}Top$$

where $T_n(Q)$ is the space of morphisms of $\mathbb{Z}/n$-equivariant cyclically ordered sets

$$Q \to S^1$$

(where we consider the standard $\mathbb{Z}/n$-action on S^1). Denoting for $Q \in Obj(\Phi_n)$ by $Q/(\mathbb{Z}/n)$ its set of orbits, we also have a functor from Φ_n to genuine $\mathbb{Z}/n$-equivariant spectra (i.e. indexed by the complete universe)

$$TH_n^R(Q) = (N_{\{e\}}^{\mathbb{Z}/n} R)^{\wedge Q/(\mathbb{Z}/n)}.$$

(This uses (5).) We set

$$THH(R)_{\mathbb{Z}/n} = (T_n)_+ \wedge_{\Phi_n} TH_n^R.$$

By (5), for $\mathbb{Z}/m \subset \mathbb{Z}/n$, we have a natural morphism

$$res_{\mathbb{Z}/m}^{\mathbb{Z}/n} THH(R)_{\mathbb{Z}/n} \to THH(R)_{\mathbb{Z}/m}$$

which is an equivalence. (This last property uses non-trivialy the unitality of R.) Thus, by the considerations of [18], we obtain a genuine S^1-equivariant spectrum $THH(R)_{S^1}$.

4. The action of $THH(THC(R))$

In this section, we construct the S^1-equivariant S-algebra

$$THHC(R)_{S^1}$$

of Theorem 1, and show that it is equivalent to $THH(THC(R))$. Now let Ξ be the category whose objects are finite ordered sets and morphisms are non-strictly increasing maps. We have a functor

$$J : \Xi^{Op} \to Top$$

where $J(Q)$ is the set of non-decreasing maps

$$Q \to [0,1]$$

(with the subspace topology of $[0,1]^Q$).

For an associative S-algebra R, we also have a functor TJ^R from Ξ to $(R^{Op} \wedge R)$-modules given where

$$TJ^R(Q) = R^{\wedge Q}.$$

(Again, surjective morphism act by multiplication in R, while injective morphism act by inserction of units.) We let

$$TI(R) = J_+ \wedge_\Xi TJ^R.$$

One notices that a pair of strictly monotone onto maps $[0,1] \to [0,s]$, $[0,1] \to [s,1]$ induces an isomorphism

$$TI(R) \wedge_R TI(R) \cong TI(R). \tag{8}$$

Now we may define

$$THC(R) = F_{R^{Op}\wedge R}(TI(R), R). \tag{9}$$

By (8), we obtain an action of the unframed cactus operad (see Section 2 above) on $THC(R)$ where $(T, \mathcal{E})$ acts by subdividing $[0,1]$ along the partition T, using the inverse of (8), and then composing by using (8) again on the edges (6) of a given loop. In particular, the data giving (8) thereby induce a morphism

$$THC(R) \wedge THC(R) \to THC(R). \tag{10}$$

We will now replace $THC(R)$ with its cell approximation in the category of S-algebras over the unframed cactus operad. For simplicity, from now on, we suppress the approximation from the notation.

Now consider the space TJH of finite sets Q of closed subintervals of S^1 with disjoint interiors whose union has a non-empty complement. Then Q is a cyclically ordered set. Letting Φ^0 be the category of finite cyclically ordered sets and isomorphisms, then we may consider the contravariant functor

$$TJH^0 : \Phi^0 \to Top$$

given by sending Q to the space of isomorphims from Q to an element of TJH. Then we may define the spectrum

$$THHC^0(R) = (TJH^0)_+ \wedge_{\Phi^0} TH^{THC(R)}$$

where $TH^{THC(R)}$ is defined as in Section 3. Additionally, using (10), we may construct from $THHC^0(R)$ an S-module $THHC(R)$ by imposing a colimit identification where configurations in TJH^0 which contain a pair of intervals sharing a boundary point are identified with the configuration where the two intervals are merged.

More precisely, let Φ^1 denote the set of cyclically ordered sets with a distinguished element. TJH^1 denote the set of all isomorphism of cyclically ordered sets $Q \in \Phi^1$ with an element of TJH where the end point of the distinguished interval J is equal to the beginning point of the next interval J', counted counter-clockwise. In addition to the forgetful map $TJH^1 \to TJH^0$, we then have another map ϕ^1 :

$TJH^1 \to TJH^0$ given by replacing the intervals J, J' with their union. Then consider

$$(TJH^1)_+ \wedge_{\Phi^1} TH^{THC(R)} \tag{11}$$

Then there are two morphisms from (11) to $THHC^0(R)$, one given by applying inclusion to the first coordinate, the other by applying ϕ_1 to the first coordinate and the map (10) to the second. We define $THHC(R)$ as the coequalizer of these two morphisms.

Then $THHC(R)$ is equivalent to $THH(THC(R))$ and is an associative S-algebra by using (8). Also, by the same principle, $THHC(R)$ acts on $THH(R)$ (by contracting all the embedded intervals to a point).

To obtain a $\mathbb{Z}/n$-equivariant version, we consider similarly the space TJH_n of collections in TJH which are invariant under the standard $\mathbb{Z}/n$-action. We have a category Φ_n^0 with the objects $Obj(\Phi_n)$ and morphisms the isomorphisms in Φ_n. We define TJH_n^0 as the category of all isomorphisms from an object of Φ_n^0 to an element of TJH_n. We can then define

$$THHC^0(R)_{\mathbb{Z}/n} = (TJH_n)_+ \wedge_{\Phi_n^0} TH_n^{THC(R)}.$$

Again, identifications can be imposed when a $\mathbb{Z}/n$-invariant n-tuple of pairs of intervals sharing a boundary point is present.

More precisely, we define Φ_n^1 as the set of pairs (Q, x) where $Q \in Obj(\Phi_n) = Obj(\Phi_n^0)$ and $x \in Q$. Then let TJH_n^1 be the set of Φ_n^0-morphisms from Q with $(Q, x) \in \Phi_n^1$ to an element of TJH_n, where, again, the end point of the distinguished interval J is equal to the beginning point of the next interval J', counted counter-clockwise. In addition to the forgetful map $TJH_n^1 \to TJH_n^0$, we then, again, have a map $\phi_n^1 : TJH_n^1 \to TJH_n^0$ given by replacing J and J' with their union, and similarly for all the $\mathbb{Z}/n$ images of the pair J, J'. Again, we then have two morphism from

$$(TJH_n^1)_+ \wedge_{\Phi_n^1} TH_n^{THC(R)}$$

to $THHC_n^0(R)$ given by the forgetful map and by applying ϕ_n^1 in the first coordinate and (5), (10) in the second. We denote by $THHC_n(R)$ the coequalizer of these morphisms.

By construction, then, we thus obtain a genuine $\mathbb{Z}/n$-equivariant associative S-algebra $THHC_n(R)$ which acts $\mathbb{Z}/n$-equivariantly on

$$THH_n(R).$$

Again, this data is compatible under restriction, thus creating a genuine S^1-equivariant S-algebra $THHC(R)_{S^1}$ acting on $THH(R)_{S^1}$. This completes our proof of the last statement of Theorem 1.

5. Some recollections on the Steenrod algebra

In this section, we will recall some facts about cohomological operations which will be needed in the next section. We refer the reader to Kochman [23] as a general reference. First of all, we recall that on

$$A_p^* = F(H\mathbb{Z}/p, H\mathbb{Z}/p)_*,$$

multiplication by the Bockstein Q_0 from the left (or the right) is exact in the sense that its kernel is equal to its image. These two maps are induced by maps

$$Q_0^L, Q_0^R : F(H\mathbb{Z}/p, H\mathbb{Z}/p) \to \Sigma F(H\mathbb{Z}/p, H\mathbb{Z}/p),$$

given by applying the Bockstein map in the first resp. the second coordinate. Thus, the Bockstein spectral sequence for $F(H\mathbb{Z}, H\mathbb{Z}/p)_*$ collapses to $E_2 = 0$, and we have

$$0 = p : F(H\mathbb{Z}, H\mathbb{Z}/p) \to F(H\mathbb{Z}, H\mathbb{Z}/p),$$

since it is 0 on coefficients and both the source and the target are generalized Eilenberg-MacLane spectra. The cofibration sequence

$$H\mathbb{Z} \xrightarrow{p} H\mathbb{Z} \longrightarrow H\mathbb{Z}/p$$

then gives a splitting

$$F(H\mathbb{Z}/p, H\mathbb{Z}/p) = F(H\mathbb{Z}, H\mathbb{Z}/p) \vee \Sigma^{-1} F(H\mathbb{Z}, H\mathbb{Z}/p).$$

Now the E_1 of the Bockstein spectral sequence from $F(H\mathbb{Z}, H\mathbb{Z}/p)_*$ to $F(H\mathbb{Z}, H\mathbb{Z})_*$ is a Koszul complex (by Milnor's relation [26]). On coefficients, the image of the Bockstein on the second coordinate of $F(H\mathbb{Z}, H\mathbb{Z}/p)$ is therefore equal to the coefficients of some generalized Eilenberg-MacLane spectrum, which we will denote by P. We therefore conclude that

$$F(H\mathbb{Z}, H\mathbb{Z}) = H\mathbb{Z} \vee P. \tag{12}$$

We may now work backward, studying the effect of p^k on eiher coordinate, finding for example that

$$F(H\mathbb{Z}, H\mathbb{Z}/p^k) = H\mathbb{Z}/p^k \vee P \vee \Sigma P. \tag{13}$$

Therefore, p^k is 0 on (13), and we obtain, in general,

$$F(H\mathbb{Z}/p^k, H\mathbb{Z}/p^k) = F(H\mathbb{Z}, H\mathbb{Z}/p^k) \vee \Sigma^{-1} F(H\mathbb{Z}, H\mathbb{Z}/p^k), \tag{14}$$

where the splitting is induced by the cofibration sequence

$$H\mathbb{Z} \xrightarrow{p^k} H\mathbb{Z} \longrightarrow H\mathbb{Z}/p^k$$

in the first coordinate. Symmetric statements also hold in the other coordinate.

3. **Lemma.** *The fiber of the morphism*

$$F(H\mathbb{Z}/p^k, H\mathbb{Z}/p^k) \to F(H\mathbb{Z}/p^\ell, H\mathbb{Z}/p^\ell) \tag{15}$$

given by the difference of the Bockstein maps in both coordinates is canonically equivalent to

$$F(H\mathbb{Z}/p^{k+\ell}, H\mathbb{Z}/p^{k+\ell}).$$

Proof. Using (14), we may write (15) on coefficients as

$$\begin{array}{c} \mathbb{Z}/p^k_0 \oplus P^*_0 \oplus P^*[1]_0 \oplus \mathbb{Z}/p^k[-1]_{-1} \oplus P^*[-1]_{-1} \oplus P^*_{-1} \\ \downarrow \\ \mathbb{Z}/p^\ell[1]_0 \oplus P^*[1]_0 \oplus P^*[2]_0 \oplus \mathbb{Z}/p^\ell_{-1} \oplus P^*_{-1} \oplus P^*[1]_{-1} \end{array} \tag{16}$$

(Here the subscript 0 resp. -1 denotes a part of the coefficients of the term (13) which is unsuspended resp. suspended by -1.)

Now one of the Bocksteins sends $P^*[1]$ isomorphically to $P^*[1]_0$, while the other sends it isomorphically to $P^*[1]_{-1}$. On the other hand, one of the Bocksteins sends P^*_0 to P^*_{-1}, while the other sends P^*_{-1} isomorphically to P^*_{-1}. The surviving terms give the answer, with the surviving target terms desuspended by 1. Extensions are present between the copies of $\mathbb{Z}/p^k$ and $\mathbb{Z}/p^\ell$, due to the definition of the Bockstein. Bookkeeping completes the result. $\square$

Also, using the splitting (14), we obtain a canonical (up to homotopy) map

$$F(H\mathbb{Z}/p^k, H\mathbb{Z}/p^k) \to F(H\mathbb{Z}/p^{k-1}, H\mathbb{Z}/p^{k-1})$$

and we have

(17)
$$\operatorname*{holim}_k(\dots \to F(H\mathbb{Z}/p^k, H\mathbb{Z}/p^k) \to F(H\mathbb{Z}/p^{k-1}, H\mathbb{Z}/p^{k-1}) \to \dots)$$
$$= F(H\mathbb{Z}, H\mathbb{Z})^\wedge_p \vee \Sigma^{-1} F(H\mathbb{Z}, H\mathbb{Z})^\wedge_p.$$

6. A recollection of $THH(H\mathbb{F}_p)$

We begin with recalling the calculation of $THH(H\mathbb{F}_p)^{\mathbb{Z}/p^{r-1}}$ of [12]. One has

$$THH(H\mathbb{F}_p) = B_{H\mathbb{F}_p}(H\mathbb{F}_p, HF_p \wedge H\mathbb{F}_p, H\mathbb{F}_p), \tag{18}$$

(where the bar construction is in the category of $H\mathbb{F}_p$-algebras and everywhere we assume cofibrant models). The coefficients of (18) can be calculated via the Eilenberg-MacLane spectral sequence which collapses to E^2 for $p = 2$ and has a Kudo differential for $p > 2$. In both cases, we can conclude that

$$THH(H\mathbb{F}_p)_* = \mathbb{Z}/p[\sigma] \tag{19}$$

where σ is in homological degree 2. Now the Tate spectral sequence for $THH(H\mathbb{F}_p)$ is

$$\mathbb{Z}/p[\sigma][x, x^{-1}] \otimes \Lambda_{\mathbb{F}_p}(u) \Rightarrow \widehat{THH(H\mathbb{F}_p)}^{\mathbb{Z}/p^{r-1}} \tag{20}$$

where x is the Tate periodicity element of homological degree -2 and u has homological degree -1. Then [12] prove that x is a permanent cycle, while

$$d^{2r-1}u = x^{2r}\sigma^{r-1}. \tag{21}$$

Furthermore, there is a multiplicative extension

$$p = x\sigma, \tag{22}$$

therefore giving

$$\widehat{THH(H\mathbb{F}_p)}_*^{\mathbb{Z}/p^{r-1}} = \mathbb{Z}/p^{r-1}[x, x^{-1}]. \tag{23}$$

The $\mathbb{Z}/p^{r-1}$-Borel cohomology spectral sequence for $THH(H\mathbb{F}_p)$ results from taking the part of (20) with non-negative powers of x (and all the differentials contained entirely in that part), while the $\mathbb{Z}/p^{r-1}$-Borel homology spectral sequence for $THH(H\mathbb{F}_p)$ results from taking the part of (20) with negative powers of x, shifted by -1 (graded homologically).

One gets:

$$\begin{aligned} (E\mathbb{Z}/p^{r-1}_+ \wedge THH(H\mathbb{F}_p))^{\mathbb{Z}/p^{r-1}}_{2i} &= \mathbb{Z}/p^{\max(i+1,r)} \\ (E\mathbb{Z}/p^{r-1}_+ \wedge THH(H\mathbb{F}_p))^{\mathbb{Z}/p^{r-1}}_{2i+1} &= \mathbb{Z}/p^{\max(i+1,r-1)} \end{aligned} \tag{24}$$

for $i \geq 0$ (it is 0 for $i < 0$).

The calculation of $THH(H\mathbb{F}_p)^{\mathbb{Z}/p^{r-1}}$ is then completed by induction: For any $\mathbb{Z}/p^{r-1}$-equivariant spectrum E, we have a cofibration sequence

$$(E\mathbb{Z}/p^{r-1}_+ \wedge E)^{\mathbb{Z}/p^{r-1}} \to E^{\mathbb{Z}/p^{r-1}} \to (\Phi^{\mathbb{Z}/p}E)^{\mathbb{Z}/p^{r-2}}. \tag{25}$$

Since $THH(H\mathbb{F}_p)$ is a cyclotomic spectrum, its coefficients are the coefficients of the homotopy fiber of the connecting map

$$THH(H\mathbb{F}_p)^{\mathbb{Z}/p^{r-2}} \to \Sigma(E\mathbb{Z}/p^{r-1}_+ \wedge THH(H\mathbb{F}_p))^{\mathbb{Z}/p^{r-1}}. \tag{26}$$

We know the target by (24). The induction gives

$$THH(H\mathbb{F}_p)_*^{\mathbb{Z}/p^{r-1}} = \mathbb{Z}/p^r[y] \tag{27}$$

where y has homological degree 2. Assuming this inductively with r replaced by $r-1$, the connecting map (26) on coefficients (which decreases homological degree by 1) is onto in odd degrees, and the even terms have an extension, creating the answer (27) additively. The multiplicative answer then also follows inductively from the fact that the second map (25) is a ring map when E is a ring spectrum.

We also recall the fact that the map R (which is defined in [12] by composing the second map (25) with the cyclotomic structure map) sends y to py, while the map F (defined as the forgetful map) sends y to y. This implies that, letting TR be the microscope of the map R, we have

$$TR(H\mathbb{F}_p) = H\mathbb{Z}_p. \tag{28}$$

All the spectra discussed in the process of the calculation are module spectra over (28), and thus are generalized Eilenberg-MacLane spectra.

7. Calculation of $TR(THC(H\mathbb{F}_p))$

We now combine the material of the last two sections to calculate the coefficients of $THH(THC(H\mathbb{F}_p))^{\mathbb{Z}/p^{r-1}}$, and thereby prove Theorem 2. First of all, we can write

$$THC(H\mathbb{F}_p) = Cobar_{H\mathbb{F}_p}(H\mathbb{F}_p, H\mathbb{F}_p \wedge H\mathbb{F}_p, H\mathbb{F}_p), \tag{29}$$

so the coefficients are indeed dual to (18). Here we write

$$\begin{array}{l} Cobar_{H\mathbb{F}_p}(H\mathbb{F}_p, H\mathbb{F}_p \wedge H\mathbb{F}_p, H\mathbb{F}_p) = C_{H\mathbb{F}_p}(H\mathbb{F}_p \wedge H\mathbb{F}_p) = \\ F_{H\mathbb{F}_p \wedge H\mathbb{F}_p}(B_{H\mathbb{F}_p}(H\mathbb{F}_p \wedge H\mathbb{F}_p, H\mathbb{F}_p \wedge H\mathbb{F}_p, H\mathbb{F}_p), H\mathbb{F}_p). \end{array}$$

In fact, there is a Hopf algebra structure, which implies that we can write

$$THC(H\mathbb{F}_p)_* = \Gamma_{\mathbb{F}_p}(\rho) \tag{30}$$

where Γ denotes the divided polynomial power algebra and the homological degree of the element ρ is -2. Now by (29), $THC(H\mathbb{F}_p)$ is an $H\mathbb{F}_p$-algebra, and (implicitly assuming cofibrant replacements in every term), we can therefore, non-equivariantly, write

$$\begin{array}{l} THH(THC(H\mathbb{F}_p)) = \\ B_{H\mathbb{F}_p}(THC(H\mathbb{F}_p), THC(H\mathbb{F}_p) \wedge THC(H\mathbb{F}_p), THC(H\mathbb{F}_p)), \end{array} \tag{31}$$

indicating that the bar construction is performed in the category of $H\mathbb{F}_p$-algebras. This means we have again an Eilenberg-MacLane spectral sequence. In fact, we have a further filtration on (31), in the category of $H\mathbb{F}_p$-algebras, with associated graded object

(32)
$$\begin{aligned} &THC(H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} B_{H\mathbb{F}_p}(H\mathbb{F}_p, H\mathbb{F}_p \wedge THC(H\mathbb{F}_p), H\mathbb{F}_p) = \\ &THC(H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} THH(H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} B_{H\mathbb{F}_p}(C_{H\mathbb{F}_p}(H\mathbb{F}_p \wedge H\mathbb{F}_p)) = \\ &THC(H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} THH(H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} F(H\mathbb{F}_p, H\mathbb{F}_p) \end{aligned}$$

(by Koszul duality). The coefficients of (32) are

$$\mathbb{F}_p[\sigma] \otimes \Gamma_{\mathbb{F}_p}(\rho) \otimes A^* \tag{33}$$

where $A^* = F(H\mathbb{Z}/p, H\mathbb{Z}/p)_*$. On the other hand, by Theorem 1, we have a map

$$THH(THC(H\mathbb{F}_p)) \to F(THH(H\mathbb{F}_p), THH(H\mathbb{F}_p)). \tag{34}$$

(This is, in fact, even true $\mathbb{Z}/p^{r-1}$-equivariantly.) Non-equivariantly, however, all the elements (33) exist and are non-zero in the target of (34), and thus cannot support differentials, Therefore, we have proved that

$$THH(THC(H\mathbb{F}_p))_* = \mathbb{F}_p[\sigma] \otimes \Gamma_{\mathbb{F}_p}(\rho) \otimes A^*. \tag{35}$$

From this point on, the strategy for computing $THH(THC(H\mathbb{F}_p))_*^{\mathbb{Z}/p^{r-1}}$ mimics the strategy for $THH(H\mathbb{F}_p)_*$, described in Section 6. We begin by calculating the coefficients of the Borel homology spectrum

$$E\mathbb{Z}/p^{r-1}_+ \wedge THH(THC(H\mathbb{F}_p)) \tag{36}$$

via the Borel homology spectral sequence. In contrast with the Borel homology of $THH(H\mathbb{F}_p)$, the differentials are somewhat different. First, we may filter the E^2-term

$$\mathbb{F}_p[\sigma] \otimes \Gamma_{\mathbb{F}_p}(\rho) \otimes A^*\{e_0, e_1, \dots\} \tag{37}$$

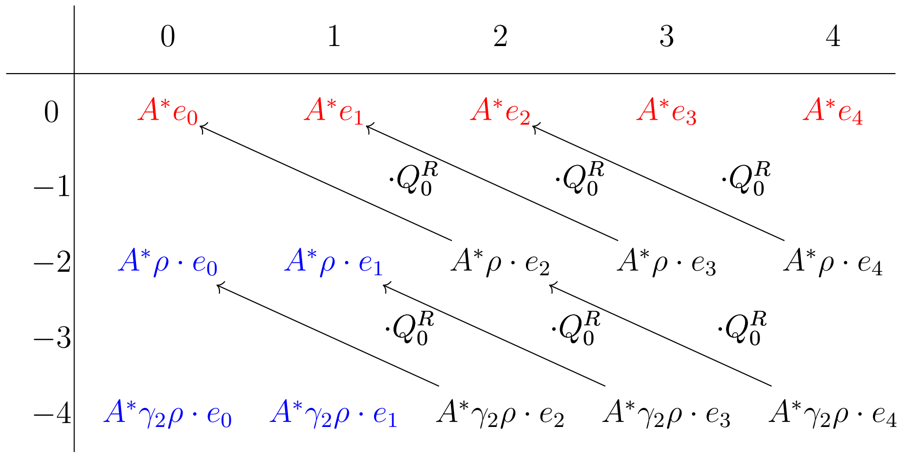
by powers of σ (where e_i is the generator of $H_i(\mathbb{Z}/p, \mathbb{Z}/p)$). This filtration is, in fact, also realized on the spectral level. Now the associated graded object is a polynomial algebra in one variable σ over

$$\Gamma_{\mathbb{F}_p}(\rho) \otimes A^*\{e_0, e_1, \dots\}. \tag{38}$$

On (38), however, there is a d^2-differential given by

$$d^2(e_i) = Q_0^R e_{i-2} \overline{\sigma} \tag{39}$$

(where $\overline{\sigma}$ acts on $\Gamma_{\mathbb{F}_p}(\rho)$ as the dual variable σ does using the Hopf algebra structure on $THC(H\mathbb{F}_p)_*$, i.e by $\overline{\sigma} : \gamma_i(\rho) \mapsto \gamma_{i-1}(\rho)$). This differential in fact comes from the extension (22):



By the observations of Section 5, however, Q_0^R is in fact exact on the Steenrod algebra. This means that the E^3-term of the slice (38) is a sum of

$$(\Gamma_{\mathbb{F}_p}(\rho))_{<0} \otimes A^*/Q_0^R\{e_0, e_1\} \tag{40}$$

where $(\Gamma_{\mathbb{F}_p}(\rho))_{<0}$ denotes the augmentation ideal of $\Gamma_{\mathbb{F}_p}(\rho)$ and

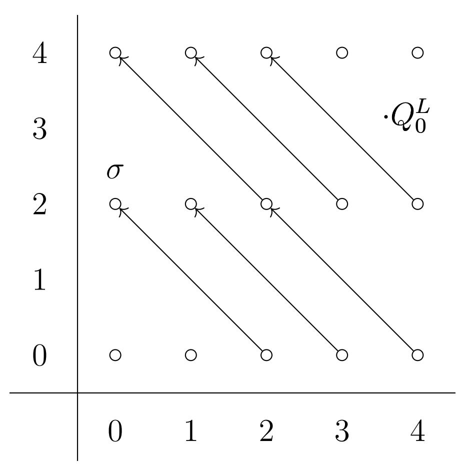
$$A^*/Q_0^R\{e_0, e_1, \dots\}. \tag{41}$$

When we put the slices together, we get a polynomial algebra on one generator σ tensored with (40) and (41). On

$$A^*/Q_0^R\{e_0, e_1, \dots\}[\sigma], \tag{42}$$

we then get another component of the d^2-differential

$$d^2(e_i) = Q_0^L e_{i-2}\sigma. \tag{43}$$



By the observations of Section 5, Q_0^L on A^*/Q_0^R is a Koszul complex, with image P_* and cokernel $\mathbb{Z}/p \oplus P_*[1]$. Therefore, the part of the E^3-term on (43) is a sum of

$$P_* \oplus P_*[1]\{e_0, e_1, \dots\} \tag{44}$$

and

$$\mathbb{Z}/p[\sigma]\{e_0, e_1, \dots\}. \tag{45}$$

On (45), we then have the differential (21).

In summary, we obtain

4. **Lemma.** *The coefficients of (36) are a direct sum of (41), (44), and (24).*

□

Next, we use (25) to obtain

7.1. **Proposition.** *We have*

$$THH(THC(H\mathbb{F}_p))_*^{\mathbb{Z}/p^{r-1}} = F(H\mathbb{Z}/p^r, H\mathbb{Z}/p^r)_*[y] \otimes \Gamma_{\mathbb{Z}/p^r}(\rho). \tag{46}$$

Additionally, the map R of [12] *sends y to py and $\gamma_i\rho$ to $\gamma_i\rho$.*

Proof. Analogously to (26), the coefficients of $THH(THC(H\mathbb{F}_p))^{\mathbb{Z}/p^{r-1}}$ are the coefficients of the spectrum

$$\begin{array}{c} THH(THC(H\mathbb{F}_p))^{\mathbb{Z}/p^{r-2}} \\ \downarrow \\ \Sigma(E\mathbb{Z}/p^{r-1}_+ \wedge THH(THC(H\mathbb{F}_p)))^{\mathbb{Z}/p^{r-1}}. \end{array} \tag{47}$$

The target is computed in Lemma 4. Thus, we proceed again by induction on r. Assuming the statement is true with r replaced by $r-1$, we have an inductive calculation of the source of (47). Thus, we need to compute the connecting map. To this end, use the computations (13), (14). Every copy of $P[1]$ in the source is isomorphically mapped to a copy of P in the target. At the augmentation ideal of $\Gamma(\rho)$, in fact, we have a sum of copies of the exension of Lemma 3 with $k = r-1$, $\ell = 1$. On the second component $F(H\mathbb{Z}, H\mathbb{Z}/p^{r-1})_*[y][-1]$ from (14) of the

$$F(H\mathbb{Z}/p^{r-1}, H\mathbb{Z}/p^{r-1})_*[y] \tag{48}$$

in the source of (47), we also have the standard Bockstein extension. On the $\mathbb{Z}/p^{r-1}$ part of the first component $F(H\mathbb{Z}, H\mathbb{Z}/p^{r-1})_*[y]$ from (14) of (48), we have, in fact, the same map and extension as in (26). Multiplicative considerations (which follow from the behavior of the unit) complete the proof. □

Proof of Theorem 2. Apply Proposition 7.1, and take the limit (17). □

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