SOME REMARKS ON PLECTIC MOTIVIC SPACES AND SPECTRA

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Abstract. We formulate a motivic homotopy theory version of the plectic conjecture of J.Nekovář and A.J.Scholl and give some initial discussion of it.

1. introduction

In [13], Section 18, J.Nekovář and A.J.Scholl asked if there is a *plectic* motivic category, which would be the target category of the cohomology of Shimura varieties in the case when the Shimura data come by Weil restriction from some totally real field. In this paper, we approach this question from the point of view of the motivic homotopy theory of Morel and Voevodsky [11]. We construct a category of r*-plectic motivic spaces and spectra*, and show that it verifies the basic properties of the target category conjectured in [13] in the case of pure Shimura data.

We also discuss a special guiding example of algebraic plectic cohomology classes on the canonical model of Hilbert-Blumenthal Shimura varieties for a totally real number field F. In this case, Nekovář and Scholl [14] 3.3 analyze the Hodge case of their conjecture. The answer is involved, using the calculation of cohomology via the toroidal compactification [2, 16]. Following Harder [7], they consider the decomposition of the cohomology of the moduli space over $\mathbb{C}$ into the boundary part and the interior part. The interior part consists further of the algebraic part and the cuspidal part. There results a mixed Hodge structure which can be determined in part using the Manin-Drinfeld principle, stating that the cuspidal part does not contribute to the non-triviality of the extension. The authors of [14] use this computational information to define a plectic structure on the real cohomology of the Hilbert-Blumenthal moduli space.

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One of course hopes that a direct manipulation on the $\mathbb{A}^1$-space corresponding to a Shimura variety obtained by Weil restriction of Shimura data on a totally real field F would result in a plectic structure on its motivic suspension spectrum, pulled back to a suitable extension of F. The computations of [14], however, suggest that a completely new idea would be necessary to carry out such a program. In the present note, we specialize only to the algebraic part of the cohomology in the Hilbert-Blumenthal case, which is already known to have a motivic origin ([16]), and we spell out how to exhibit the motivic plectic structure on those cohomology classes. Even this basic example is valuable in showing that to get a plectic structure, we have to pull back to the Galois closure of F over $\mathbb{Q}$. In fact, the motivic setting allows directly extending this discussion to the case of the Gp_{2g}-Shimura data ([4], 4.22). In this case, we are not dealing with ordinary motivic cohomology, but with algebraic vector bundles. Nevertheless, the discussion is the same.

The present note is organized as follows: The majority of our discussion is devoted to introducing the relevant preliminaries from $\mathbb{A}^1$-homotopy theory. In Section 2, we explain the plectic version of the stable and unstable motivic categories of Morel and Voevodsky [11]. In Section 3, we explain how this concept interacts with Hodge and étale realizations. In Section 3, we review the motivic multiplicative norm of Hu [8] and Bachman-Hoyois [1], and we state our version of the plectic conjecture. In Section 5, we discuss the examples of the algebraic bundles on Hilbert-Blumenthal Shimura varieties, and Gp_{2g}.

2. The basic construction

Let k be a field. Recall that *motivic spaces* Spc_k^{Mot} are simplicial objects in the category of sheaves of sets over the category Sm_k of finite type separated smooth schemes over $Spec(k)$ in the Nisnevich topology. Here by a Nisnevich cover, we mean an étale cover where over every Zariski point (not necessarily closed), there exists a point with the same residue field. One proves [11] that a presheaf F on Sm_k is a Nisnevich sheaf if and only if it takes diagrams of the form

$$(1) \qquad \begin{array}{ccc} V & \xrightarrow{j} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j'} & Y, \end{array}$$

where j, j' are Zariski open inclusion, f is étale and restricts to an isomorphism

$$X \smallsetminus V \cong Y \smallsetminus U,$$

into pullbacks. The diagram (1) is then called a *Nisnevich square.*

One defines *simplicial equivalence* on motivic spaces as a morphism which induces an equivalence on Nisnevich stalks. A *simplicial model structure* is then put on Spc_k^{Mot} using the technique of model structures on categories of simplicial sheaves [9]. One obtains the motivic homotopy category $DSpc_k^{Mot}$ by localizing the homotopy category of the simplicial model structure on Spc_k^{Mot} with respect to morphisms of the form

$$\mathbb{A}^1_X \to X$$

for a separated smooth scheme X of finite type over $Spec(k)$. The category of *motivic spectra* is defined as the category of sequences of motivic spaces $(Z_{(n)})_{n\in\mathbb{N}_0}$ together with maps

$$\mathbb{P}^1 \wedge Z_{(n)} \to Z_{(n+1)}.$$

A model structure is put on this category using the techniques of [11] (or alternatively, [10]), and the resulting category is called the *motivic stable homotopy category Sh_k^{Mot} over k.*

We define the category of *r-plectic motivic spaces $P_rSpc_k^{Mot}$* as the category of simplicial objects in the category of sheaves of sets over

$$Sm_k^r = \underbrace{Sm_k \times \cdots \times Sm_k}_{r \text{ times}}$$

with respect to the Nisnevich topology, where a Nisnevich cover is defined as an r-tuple of Nisnevich covers. Once again, a presheaf of sets on Sm_k^r is a Nisnevich sheaf if and only if it turns into pullbacks square diagrams in Sm_k^r which are Nisnevich squares of the form (1) in each coordinate.

Again, we define equivalence by equivalence on stalks, and we define the simplicial model structure using the techniques of [9]. The $\mathbb{A}^1$-model structure is obtained by localizing with respect to all morphisms of the form

$$\underline{(\mathbb{A}^1_{X_1}, \dots, \mathbb{A}^1_{X_r})} \to \underline{(X_1, \dots, X_r)}. \tag{2}$$

The resulting homotopy category is the *r-plectic motivic homotopy category $DP_rSpc_k^{Mot}$.*

A *based plectic motivic space* is a morphism

$$\underline{(Spec(k), \dots, Spec(k))} \to X$$

where the source denotes the representable sheaf, and X is an r-plectic motivic space. Thus, we have a category $P_rBased_k^{Mot}$, and a corresponding derived category $DP_rBased_k^{Mot}$. One can then define a *smash product* of r-plectic spaces as a colimit from products of r-plectic spaces. One defines an *r-plectic motivic spectrum* as a sequence of r-plectic motivic spaces $(Z_{(n)})$, $n \in \mathbb{N}_0$, and a sequence of based maps

$$\underline{\bigwedge}(\mathbb{P}^1_k, \dots, \mathbb{P}^1_k) \wedge Z_{(n)} \to Z_{(n+1)}$$

where

$$\underline{\bigwedge}(X_1, \dots, X_r)$$

denotes the *plectic smash product*, defined as follows:

Let, for $I \subsetneq \{1, \dots, r\}$,

$$X_I = (Y_1, \dots, Y_r)$$

where

$$Y_i = \begin{cases} Spec(k) & \text{if } i \in I \\ X_i & \text{if } i \notin I. \end{cases}$$

For $I \subseteq J$, there are inclusions

$$X_I \to X_J \to (X_1, \dots, X_r).$$

Thus, we have a morphism

$$\operatorname*{colim}_I X_I \to (X_1, \dots, X_r).$$

We let $\underline{\bigwedge}(X_1, \dots, X_r)$ be the pushout of the diagram

$$\begin{array}{ccc} \operatorname{colim}_I X_I & \longrightarrow & (X_1, \dots, X_r) \\ \downarrow & & \\ (Spec(k), \dots, Spec(k)). & & \end{array}$$

Again, one puts a model structure on r-plectic motivic spectra using the techniques of [11, 10], and the homotopy category is called the r-plectic motivic stable homotopy category $P_rSh_k^{Mot}$.

3. Realizations

Now assume $\mathcal{D}$ is an ∞-category (by which we mean a category with limits and colimits where the Hom-sets are given the structure of simplicial sets, where composition and identities are morphisms of simplicial sets). Suppose we have a functor

$$F : Sm_k^r \to \mathcal{D}$$

which takes Nisnevich covers to homotopy limits and morphisms of the form (2) into equivalences. Then we obtain *realization functors* on the level of derived categories

$$DF : DP_rSpc_k^{Mot} \to D\mathcal{D},$$

$$DF : DP_rBased_k^{Mot} \to D\mathcal{D}.$$

Assume further that there is a shift operator

$$\sigma : \mathcal{D} \to \mathcal{D}$$

which is an ∞-equivalence (i.e. has an inverse up to equivalence). Assuming we have a natural equivalence

$$F(\underline{\bigwedge}(\mathbb{P}^1_k, \dots, \mathbb{P}^1_k) \wedge Z) \sim \sigma F(Z),$$

we also obtain a realization functor

$$D_{st}F : P_rSh_k^{Mot} \to D\mathcal{D}.$$

The plectic étale and Hodge realizations in the case when k is a number field fall under this pattern.

For example, let $MHS_{\mathbb{R}}$-$Chain$, resp. $P_rMHS_{\mathbb{R}}$-$Chain$, be the category of unbounded chain complexes of $\mathbb{R}$-mixed Hodge structures resp. r-plectic $\mathbb{R}$-mixed Hodge structures defined by Deligne [3] and Nekovář-Scholl [14]. Then we have the *plectic tensor product functor*

$$\underline{\bigotimes} : (MHS_{\mathbb{R}}\text{-}Chain)^r \to P_rMHS_{\mathbb{R}}\text{-}Chain. \tag{3}$$

For a smooth scheme X over $\mathbb{R}$, given a smooth projective compactification $\overline{X}$ of $X_{\mathbb{C}}$ where the complement is a union of divisors with normal crossings, we produce an object of $MHS_{\mathbb{R}}$-$Chain$. This is not functorial, since the choice of $\overline{X}$ isn't, but by considering the pro-system of all such resolutions (with arrows also given by resolutions, if we wish), we can produce a functor. Applying the plectic tensor product (3), we can produce an ∞-functor F which gives an input of the machine described above.

Comment: Ordinarily, a (say pure) Hodge structure has a real structure coming from the period lattice. When the scheme X is defined over $\mathbb{R}$, we get another real structure coming from algebraic real De Rham cohomology. The Hodge structure is real with respect to both. In [14], periods are not considered, so the real structure comes from X, which is why we assume X is defined over $\mathbb{R}$.

The case of the plectic étale realization is treated similarly.

4. Norms and the motivic plectic conjecture

Let $k \subset F$ be a finite separable extension of fields. Then the morphism

$$i : Spec(F) \to Spec(k)$$

induces a pullback functor

$$i^* : Spc_k^{Mot} \to Spc_F^{Mot}$$

which has a left adoint $i_\sharp$ (the "forgetful functor") and a right adjoint i_* (the "Weil restriction.") These functors naturally pass to the based context and to spectra, and also to the corresponding derived categories. Hu [8] proved that

$$i_* \sim i_\sharp : Sh_F^{Mot} \to Sh_k^{Mot}.$$

Hu [8] also defined a *multiplicative norm*

$$N : Sh_F^{Mot} \to Sh_k^{Mot}$$

with the property that for a motivic space X over $Spec(F)$,

$$N\Sigma^\infty X_+ \sim \Sigma^\infty (i_* X)_+$$

where $?_+$ denotes adding a disjoint base point. This concept of the norm was further generalized and applied by Bachmann and Hoyois [1].

Let L be a Galois extension of k containing F and let $G = Gal(L/k)$, $H = Gal(L/F)$. For a based motivic space X over $Spec(F)$, one defines the motivic space NX over $Spec(k)$ by factoring out a colimit of motivic spaces X_S indexed over subsets $S \subsetneq G$ satisfying $HS = S$ defined as follows: Let

$$J_S = \{g \in G \mid Sg = S\}.$$

Then write

$$S = \coprod_{g \in I_S} HgJ_S$$

for some map $I_S \to H\backslash G/J_S$. One can write

$$HgJ_s = H \times_{J_g} J_S$$

where J_g is the stabilizer of g in $H \times J_S$. Then J_g injects both into H and J_S. Let $F_g = L^{J_g}$, $F_S = L^{J_S}$ and let

$$i_g : F \subseteq F_g,$$

$$i_{g,S} : F_S \subseteq F_g,$$

$$i_S : k \subseteq F_S.$$

Then one puts

$$X_S := i_{S\sharp} \prod_{(g) \in I_S} i_{g,S*} i_g^* X. \tag{4}$$

To get functoriality

$$X_S \subset X_T \text{ for } S \subseteq T,$$

one defines

$$J_{S,T} = \{g \in G \mid S_g = S, Tg = T\},$$

$$S = \coprod_{g \in J_{S,T}} HgJ_{S,T}$$

for some $I_{S,T} \to H \backslash G / J_{S,T}$. Letting $J_{g,T}$ be the stabilizer of g in $H \times J_{S,T}$, we define $F_{g,T} = L^{J_{g,T}}$, $F_{S,T} = L^{J_{S,T}}$ and let

$$i_{g,T} : F \subseteq F_{g,T},$$

$$i_{g,S,T} : F_{S,T} \subseteq F_{g,T},$$

$$i_{S,T} : k \subseteq F_{S,T}.$$

One lets

$$X_{S,T} := i_{S,T\sharp} \prod_{(g) \in I_{S,T}} i_{g,S,T*} i_{g,T}^* X.$$

Then one has natural maps

$$\begin{array}{ccc} & X_{S,T} & \\ {}^{\pi_{S,T}}\swarrow & & \searrow \\ X_S & \dashrightarrow & X_T. \end{array}$$

The map $\pi_{S,T}$ is onto finite étale, so the dotted factorization arrow exists (and is determined by) étale descent.

One notes that

$$N\mathbb{P}^1 = \mathbb{P}^{[F:k]} / \mathbb{P}^{[F:k]-1},$$

so the construction stabilizes to give a norm functor from motivic spectra over F to motivic spectra over k ([8, 1]).

One notes that if one denotes by F' the Galois closure of F over k, so we have a diagram

$$\begin{array}{ccc} Spec(F') & \xrightarrow{\quad i' \quad} & Spec(k) \\ {}_{\iota}\searrow & & \nearrow_{i} \\ & Spec(F) & \end{array}$$

for a motivic spectrum E over $Spec(F)$, one has

$$i'^{*}N(E) \sim \underbrace{\iota^{*}E \wedge \cdots \wedge \iota^{*}E}_{r},$$

$$i'^{*}i_{\sharp}E \sim \underbrace{\iota^{*}E \vee \cdots \vee \iota^{*}E}_{r}$$

where $r = [F : k]$.

This statement can be used to construct refinements of both of the functors $i'^{*}N$, $i'^{*}i_{\sharp}$ from the stable homotopy category over $Spec(F)$ to the ("naive") Σ_r-equivariant r-plectic stable homotopy category

$$\Sigma_r\text{-}P_r Sh_k^{Mot}$$

(where Σ_r acts by permutation of factors).

One version of the motivic plectic conjecture (in the pure case) can be stated as follows:

Conjecture: Let F be a finite totally real extension of $\mathbb{Q}$. Consider a Shimura datum over $\mathbb{Q}$ which is the Weil restriction of a Shimura data over F, and let M be a canonical model for a given level structure, with reflex field E. Let

$$i : Spec(E') \to Spec(E)$$

where the field E' contains both E and F' where F' is the Galois closure of F. Then $i^{*}\Sigma^{\infty}(M_{+})$ has a natural structure of an object of Σ_r-$P_r Sh_k^{Mot}$.

5. Some comments on plectic structures from Symplectic Shimura data over a totally real field

Let F be a totally real number field, $r = [F : \mathbb{Q}]$. Let $Gp_{2g}(F)$ denote the group of symplectic similitudes of a symplectic F-vector space V of dimension $2g$ with an antisymmetric non-degenerate bilinear form ψ. Denote by $\mathbb{S}$ the Weil restriction of $\mathbb{G}_m$ from $\mathbb{C}$ to $\mathbb{R}$.

When $F = \mathbb{Q}$, following Deligne, [4], 1.6, one has the symplectic Shimura data given by the unique conjugacy class of homomorphisms

$$h : \mathbb{S} \to Gp(V) \tag{5}$$

which, restricted to $\mathbb{G}_{m,\mathbb{R}}$ sends x to the homotethy of scale x^{-1}. As described in [4], 4.16, one can construct a canonical model (over $\mathbb{Q}$) in this case as the solution of a moduli problem. Let $\mathcal{S}$ be a scheme. By *principally polarized schemes* A over $\mathcal{S}$ we mean an abelian scheme A over $\mathcal{S}$ together with an isomorphism

$$p : A \to A^*$$

where A^* is the dual abelian scheme, which is symmetric with respect to the canonical isomorphism of the identity functor with the double dual and such that the pullback

$$(Id \times p)^* P \tag{6}$$

of the Poincaré line bundle on $A \times A^*$ is ample. Denote by $V_{\mathbb{Z}}$ an integral lattice on which the symplectic form ψ takes integral values and has discriminant 1.

When working at level n, one then considers the functor assigning to a scheme $\mathcal{S}$ the set of principally polarized abelian schemes over $\mathcal{S}$ together with a symplectic similitude

$$k_n : {}_nA \to (V_{\mathbb{Z}}/nV_{\mathbb{Z}})_{\mathcal{S}} \tag{7}$$

(where ${}_nA$ is the kernel of multiplication by n on A). For $n \geq 3$, this functor is representable by a scheme M ([12], Chapter 7), which defines a canonical model for the data (5).

As pointed out in Section 4.3 of [4], instead of principally polarized abelian schemes, we could alternatively also work "up to isogeny." At a stable level, we can describe an isogeny

$$A \to A_\phi \tag{8}$$

uniformly on the moduli space. For an isogeny ϕ in this sense, we could consider the moduli space M_ϕ of abelian schemes at the given level with the polarization induced by the principal polarization on A, which is represented by a point in M. We could then equivalently replace M by

$$\operatorname*{hocolim}_{\phi} M_\phi. \tag{9}$$

More precisely, the homotopy colimit is taken over the category whose objects are isogenies ϕ as in (8), and morphisms $\phi \to \phi'$ are isogenies ψ satisfying $\phi' = \psi \circ \phi$. The homotopy colimit (9) will give the same answer as M in the motivic homotopy category as we are taking a homotopy colimit of a functor from a contractible category, where morphisms go to isomorphisms.

Now for a general totally real number field F, the analogous construction constructing a canonical model (over F) is outlined in [4], Variante 4.22. In this case, we have

$$Gp_{2g}(F)_{\mathbb{R}} \cong Gp_{2g}(\mathbb{R})^n,$$

so we obtain Shimura data

$$h : \mathbb{S} \to Gp_{2g}(F)_{\mathbb{R}} \tag{10}$$

by taking a product of n copies of (5). The reflex field is again $\mathbb{Q}$.

In this case, one lets $V_{\mathbb{Z}}$ be a maximal free $\mathcal{O}_F$-submodule of V on which the form ψ takes values in $\mathcal{O}_F$ and has discriminant 1. For a scheme $\mathcal{S}$ over $Spec(\mathbb{Q})$, we consider abelian schemes A over $\mathcal{S}$ together with a homomorphism

$$\lambda : \mathcal{O}_F \to End(A)$$

and an isomorphism

$$p : A \to A^*$$

which satisfies

$$p \circ \lambda = \lambda^* \circ p,$$

is symmetrical with respect to the double duality, such that, again, (6) is ample.

We further consider a symplectic similitude (7) over $\mathcal{O}_F$ (i.e., in particular, a map of $\mathcal{O}_F$-modules). Again, for a sufficiently high level n, this moduli problem can be solved, i.e. the contravariant functor assigning to $\mathcal{S}$ abelian schemes with the above structure is representable by a scheme M^F over $Spec(\mathbb{Q})$, thus providing a canonical model for the Shimura data (10).

Now ideally, one would like to construct a natural plectic structure on

$$i'^* \Sigma^\infty M^F_+$$

where

$$i' : Spec(F') \to Spec(\mathbb{Q})$$

where F' is the Galois closure of F. Currently, such a construction is not known.

The case of Hilbert-Blumenthal modular varieties ($g = 1$) was studied in the article by Nekovář and Scholl [14], Section 3.3., see also [2]. We comment here on the “algebraic" part of the cohomology of M^F. Rapoport [16], Exemples 6.9, observed that, denoting

$$i : Spec(F) \to Spec(\mathbb{Q}),$$

we have an $i_*\mathbb{G}_m$-principal bundle on M^F coming from the $i_*\mathbb{G}_m$-action on the module of differentials $(\Omega_{X/\mathbb{Q}})_0$ on the weakly polarizable abelian variety at the point 0 (more precisely an abelian scheme over some parameter scheme $\mathcal{S}$, as in [16], Variante 6.8).

Now we have

$$i'^{*}i_{*}\mathbb{G}_m = \prod_{[F:\mathbb{Q}]} \mathbb{G}_m,$$

where the product on the right hand side ranges functorially over the embeddings $F \subseteq F'$. Therefore, the right hand side is canonically lifted to an $[F:\mathbb{Q}]$-tuple, thereby showing that these cohomology classes (which make up a part of the cohomology of M^F, [14, 2, 5, 6]) have a canonical motivic plectic structure.

It is worth pointing out that if F is not Galois over $\mathbb{Q}$, then $i^{*}i_{*}\mathbb{G}_m$ does not canonically split into copies of $\mathbb{G}_m$. For example,

$$F = \mathbb{Q}[x]/(x^3 - 4x + 1)$$

is a totally real field with a non-square discriminant 229, and thus is not Galois. We see that accordingly, $i^{*}i_{*}\mathbb{G}_m$ splits as a product of $\mathbb{G}_m$ and a quadratically twisted $\mathbb{G}_m$. A variant of this effect on the multiplicative norm was also used by Hu in [8] to produce additional examples of non-trivial elements of the Picard groups of motivic spectra. This is the reason we need to go to the Galois closure of F to formulate a motivic plectic conjecture.

There is also an analogous discussion for $g > 1$. In this case, $i_{*}\mathbb{G}_m$ acts on the g-dimensional module of differentials $(\Omega_{X/\mathbb{Q}})_0$ of a weakly F-polarized abelian variety over $\mathbb{Q}$, thus giving rise to a $i_{*}GL_g$-bundle over the canonical model M^F. Again, we have

$$i'^{*}i_{*}GL_g = \prod_{[F:\mathbb{Q}]} GL_q,$$

thus giving a Σ_r-equivariant plectic structure on the classifying space $Bi'^{*}i_{*}GL_g$ of such bundles ([11]).

References

[1] T. Bachmann and M. Hoyois, Norms in motivic homotopy theory. *Astérisque* no. 425, AMS 2021. 208 pp.

[2] C. Davidescu, A.J. Scholl: Extensions in the cohomology of Hilbert modular varieties, arXi:2003.07827v1

[3] P. Deligne. Théorie de Hodge II, *Publ. Math. de IHES* vol. 40 (1971), 5-58

[4] P. Deligne: Travaux de Shimura, *Séminaire N. Bourbaki*, 1971, exp. no. 389, pp. 123-165

[5] G. Faltings: Arithmetische Kompaktifizierung des Modulraumes der abelschen Varietäten. In: *Arbeitstagung Bonn 1984*, Lecture Notes in Math. 1111, 321-383. Springer-Verlag 1985

[6] G. van der Geer: *Hilbert modular surfaces*, Springer-Verlag Berlin Heidelberg, 1988
[7] G. Harder: On the cohomology of $SL(2, O)$, *Lie groups and their representations*, In: Gelfand, I.M. (ed.). Proc. of the Summer School on Group Repres. London, A. Hilger, 1975, pp. 139-150
[8] P. Hu. Base change functors in the $\mathbb{A}^1$-stable homotopy category. Equivariant stable homotopy theory and related areas (Stanford, CA, 2000). *Homology, Homotopy App.* 3 (2001), no. 2, 417-451
[9] J. F. Jardine. Simplicial presheaves. *J. Pure. Appl. Algebra* 47 (1987), 35-87
[10] J. F. Jardine. Motivic symmetric spectra. *Doc. Math.* 5 (2000), 445-553
[11] F. Morel, V. Voevodsky: $\mathbb{A}^1$-homotopy theory of schemes, *Inst. Hautes Études Sci. Publ. Math.* (1999), no.90, 45-143
[12] D. Mumford: *Geometric invariant theory*, Ergebnisse 34, Springer-Verlag, 1965
[13] J. Nekovář, A.J. Scholl: Introduction to plectic cohomology, *Advances in the theory of automorphic forms and their L-functions*, 321-337, Contemp. Math., 664 American Mathematical Society, Providence, RI, 2016
[14] J.Nekovář, A.J.Scholl: Plectice Hodge theory I, preprint, 2017
[15] S. Patrikis, F. Voloch, Y. Zarhin: Anabelian geometry and descent obstructions on moduli spaces, Algebra Number Theory, Volume 10, 2016, no. 6.
[16] M. Rapoport: Compactifications de l'espace de modules de Hilbert-Blumenthal, *Composition Math.* 36 (1978) 255-335