The Game of Pass the Buck and the Markov Chain Tree Theorem

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Abstract. We describe a new approach for calculating the winning probabilities in the game of *Pass the Buck* on arbitrary graphs. It is based on the Markov chain tree theorem, and reduces the problem to counting arborescences in directed graphs. We investigate the game on several classes of graphs, provide short derivations of existing results, and obtain several new ones.

1. INTRODUCTION. The game called *Pass the Buck* is played on an arbitrary graph, whose vertices represent players. There is a designated first player, who is holding a prize referred to as 'the buck'. In each round of the game, one of the following events is chosen uniformly at random: Either the player holding the buck retains the prize and becomes the winner of the game, or passes the buck to one of their neighbors. Hence, if the current player has k neighbors, there are $k+1$ possible moves, each of which takes place with probability $\frac{1}{k+1}$. Although this does not sound like the most enjoyable game in the world, it leads to beautiful mathematics.

The problem of calculating the winning probabilities of all players for various classes of graphs was investigated in several papers. Bruce Torrence and Robert Torrence have analyzed the game on path graphs, and expressed the winning probabilities in terms of the Fibonacci and Lucas numbers [**8**]. An alternative method involving a chip-firing process called the "stochastic abacus" was developed by B. Torrence, who considered a much larger class of graphs [**9**]. Kenneth Levasseur's analysis of the game on complete trees [**6**] and rooted trees [**7**] is also based on the use of the stochastic abacus.

In the present paper, we propose a new way of calculating the winning probabilities. The idea is to view the game as a Markov chain, and calculate the absorbing probabilities using the Markov chain tree theorem, which reduces the task to counting arborescences (directed analogues of spanning trees) in a directed graph.

Our exposition is self-contained. We do not assume an a priori knowledge of the Markov chain tree theorem, and explain all necessary details in Sections 2 and 3. In Section 4, we calculate the numbers of arborescences of certain graphs that will be needed later. The remaining sections, which form the core of the paper, demonstrate our approach to *Pass the Buck* by calculating the winning probabilities for several classes of graphs. For path graphs (Section 5), we extend the results available in the literature by calculating the winning probabilities for all players (not just those at the endpoints). The results for cycle graphs (Section 6) are known, but we present a unified approach where there is no need to distinguish whether the number of players is even or odd. The results for complete graphs (Section 7) and complete k-ary trees (Section 9) are also known, but our derivations are new and short. The solutions for complete bipartite graphs (Section 8) and connected cycles (Section 10) are new. For spoke graphs (Section 11), we extend the solution which was previously known only if the starting player is located in the central vertex. Finally, we point out that similar ideas can be used to deal with windmill and Dutch windmill graphs (Section 12).

2. PASS THE BUCK AS A MARKOV CHAIN. The game of *Pass the Buck* involving n players is played on a connected undirected graph with vertex set $\{1,\dots,n\}$. It is equivalent to a Markov chain represented by a directed graph, which is obtained as follows: Each undirected edge $i \leftrightarrow j$ in the original undirected graph is replaced by a pair of directed edges $i \to j$ and $j \to i$. Moreover, for each vertex $i \in \{1, \ldots, n\}$ in the original graph, we create a new vertex S_i, add a directed edge $i \to S_i$, and a loop $S_i \to S_i$. Figure 1 illustrates this construction for a path graph.

Each vertex $i \in \{1, \ldots, n\}$ corresponds to the situation when the game is in progress, and player i holds the buck. The victory of a player i corresponds to the move from i to S_i. Vertices $S_1, \ldots, S_n$ represent absorbing states of the Markov chain, since it is impossible to leave S_i. The transition from S_i to S_i occurs with probability 1. On the other hand, vertices $i \in \{1, \ldots, n\}$ represent transient states, and all their neighbors can be reached with the same probability.

The winning probability for player i equals the probability that the Markov chain will reach the absorbing state S_i, and our goal is to calculate these probabilities.

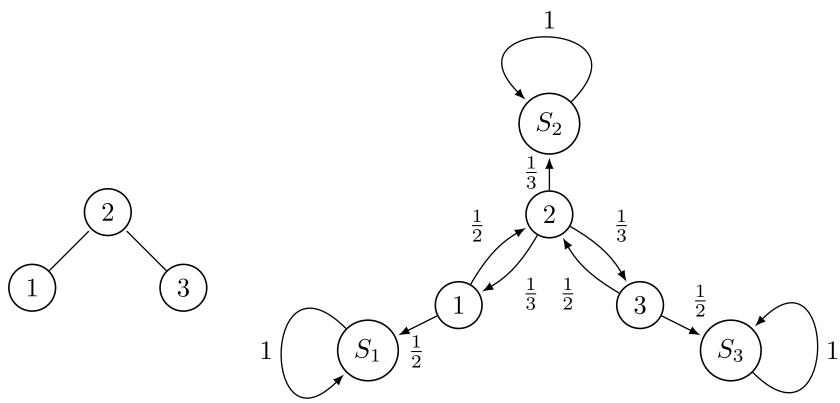

Figure 1. A graph for *Pass the Buck* (left), and the corresponding Markov chain represented by a weighted directed graph (right).

3. THE MARKOV CHAIN TREE THEOREM. The standard way of calculating the absorbing probabilities in a Markov chain is based on the concept of a fundamental matrix (see [**2**, Chapter 3]). Consider the matrices $Q = (q_{ij})_{i,j}$ and $R = (r_{ij})_{i,j}$, where q_{ij} is the transition probability from a transient state i to a transient state j, while r_{ij} is the transition probability from a transient state i to an absorbing state j. The fundamental matrix is $N = (I - Q)^{-1}$, and the sum of the values in its i-th row is the expected number of steps before absorption, provided that it started in state i. Finally, the elements b_{ij} of the matrix $B = NR$ give the probabilities that the Markov chain will end in the absorbing state j, assuming that we started from the transient state i. For the Markov chain depicted in the right part of Figure 1, we get

$$Q = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, \quad R = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{5}{8} & \frac{2}{8} & \frac{1}{8} \\ \frac{2}{8} & \frac{4}{8} & \frac{2}{8} \\ \frac{1}{8} & \frac{2}{8} & \frac{5}{8} \end{pmatrix}. \tag{1}$$

The same results can be obtained using a tool called the probabilistic or stochastic abacus. The idea is to run a certain chip-firing process on the vertices of the directed graph from Figure 1, and count the number of chips accumulated in the absorbing vertices as soon as the process terminates. This approach to calculating the winning probabilities for *Pass the Buck* on various graphs was employed in the papers [**6, 8, 9**].

But there is another completely different way of calculating the absorbing probabilities, which is based on counting certain arborescences in the graph representing the Markov chain. The result is known as the Markov chain tree theorem, and its general version makes it possible to calculate the long-run average probabilities that the chain will be in state j, provided that it started from state i. In our case, the long-run probabilities for transient states are zero, and the long-run probabilities for absorbing states are exactly the absorbing probabilities we want to calculate.

We will describe only a special case of the Markov chain tree theorem that is directly applicable to *Pass the Buck*; it follows from a more general statement that is available in [**1, 4, 5**]. Suppose that we have a finite Markov chain represented by a weighted directed graph $G = (V, E)$. Its transient states are $1, \dots, n$, and absorbing states are $S_1, \dots, S_n$, where each state S_i can be reached only from i. If necessary, we allow multiple edges between pairs of vertices, but we assume that all edges leaving an arbitrary vertex have the same weight (i.e., transition probability). Hence, the weight of all edges leaving a vertex $v \in V$ is the reciprocal of the outdegree of v (and there is no need to specify these weights whenever we draw the corresponding graph).

A collection of edges $A \subset E$ will be called an arborescence if it contains no cycles, and each vertex from $\{1, \dots, n\}$ has a unique outgoing edge contained in A. Denote by $\mathcal{A}(G)$ the set of all arborescences in G, and by $\mathcal{A}_{ij}(G)$ the set of all arborescences in G containing a directed path from i to S whose last edge is $j \to S$. By the Markov chain tree theorem, if $\mathcal{A}(G) \neq \emptyset$, then the probability of reaching an absorbing state S_j, provided that we started in a transient state i, is

$$p_{ij} = \frac{|\mathcal{A}_{ij}(G)|}{|\mathcal{A}(G)|}. \tag{2}$$

This is the crucial formula for the rest of the paper.

Let us make two simple but useful observations. When counting arborescences, we can omit all loops, since they can never appear in an arborescence. Also, we can contract the vertices $S_1, \dots, S_n$ into a single vertex S as in Figure 2. This has no effect on the total number of arborescences, and instead of calculating the number of all arborescences containing a directed path from i to S_j (which necessarily pass through j), we calculate the number of all arborescences containing a directed path from i to S via the edge $j \to S$.

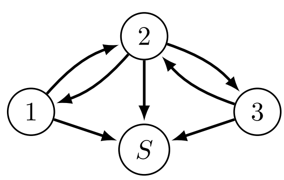

Figure 2. The graph obtained from the right part of Figure 1 by removing loops and contracting the absorbing states into a single vertex S.

To provide an illustration, let us calculate the winning probabilities in *Pass the Buck* for the graph from the left part of Figure 1 by counting arborescences of the graph in Figure 2. The latter graph has exactly 8 arborescences; they are shown in Figure 3. The winning probabilities depend on the starting player:

- *Player 1 starts.* There are 5 arborescences containing the edge $1 \to S$, 2 arborescences containing a path from 1 to S via $2 \to S$, and 1 arborescence containing a path from 1 to S via $3 \to S$. According to (2), the winning probabilities for the three players are $5/8$, $2/8$, and $1/8$.
- *Player 2 starts.* There are 2 arborescences containing a path from 2 to S via $1 \to S$, 4 arborescences containing the edge $2 \to S$, and 2 arborescences containing a path from 2 to S via $3 \to S$. The winning probabilities are $2/8$, $4/8$, and $2/8$.
- *Player 3 starts.* There is 1 arborescence containing a path from 3 to S via $1 \to S$, 2 arborescences containing a path from 3 to S via $2 \to S$, and 5 arborescences containing the edge $3 \to S$. The winning probabilities are $1/8$, $2/8$, and $5/8$.

The three triples of numbers correspond to the rows of the matrix B obtained in (1).

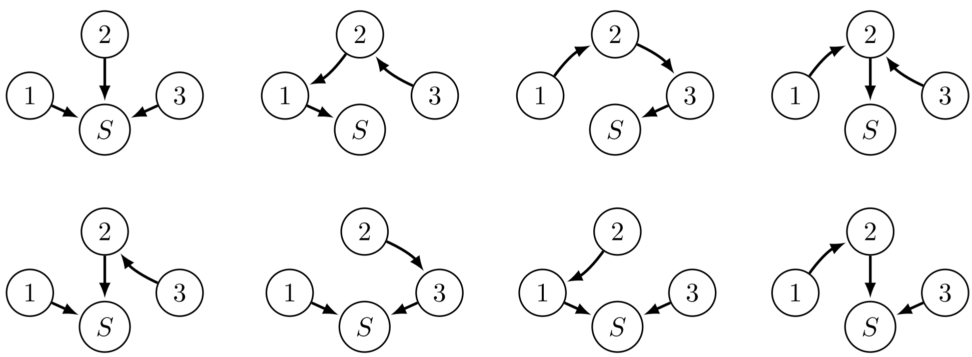

Figure 3. All arborescences of the graph from Figure 2.

The following remarks are intended primarily for readers who are already familiar with the Markov chain tree theorem, or those who are planning to read its proof in [**1, 4, 5**]:

- The standard formulation of the Markov chain tree theorem involves weights of arborescences. The weight of an arborescence is defined as the product of weights of all edges in A, and the original Markov chain tree theorem says that

$$p_{ij} = \frac{\|\mathcal{A}_{ij}(G)\|}{\|\mathcal{A}(G)\|},$$

  where $\|\mathcal{A}(G)\|$ denotes the sum of weights of all arborescences in $\mathcal{A}(G)$, and $\|\mathcal{A}_{ij}(G)\|$ is the sum of weights of all arborescences in $\mathcal{A}_{ij}(G)$. However, since we assume that all edges leaving an arbitrary vertex have the same weight, and since each arborescence contains exactly one outgoing edge for each transient vertex, it is clear that all arborescences have the same weight. Thus, we can forget about edge weights as well as arborescence weights, and simply count the numbers of arborescences as in (2).
- In [**1, 4, 5**], the definition of an arborescence is formulated in a slightly different way. An arborescence is an edge set of maximum possible cardinality in which there is at most one edge leaving every vertex, and there are no cycles. However, it is clear that maximum cardinality is achieved when all non-absorbing vertices have exactly one outgoing edge, while absorbing vertices have none.

For directed graphs having only one vertex S with no outgoing edges except loops, there is yet another characterization of arborescences: From each vertex v different from S, there is a unique directed path from v to S.

4. COUNTING ARBORESCENCES. We now collect some auxiliary results on the numbers of arborescences for certain simple graphs. In later sections, we will utilize these results while solving *Pass the Buck* on more complicated graphs.

First, we take arbitrary $a, b \in \mathbb{N}$ and calculate the number of arborescences for a graph denoted by $P_{n,a,b}$, which has vertices $1, \dots, n$ and S, and its edges are as follows: For each $i \in \{1, \dots, n\}$, there is an edge $i \to S$. For each $i \in \{1, \dots, n-1\}$, there are a edges $i \to i+1$. Finally, for each $i \in \{2, \dots, n\}$, there are b edges $i \to i-1$; see Figure 4. Note that the graph from Figure 2 is a special case of this construction with $n = 3$ and $a = b = 1$.

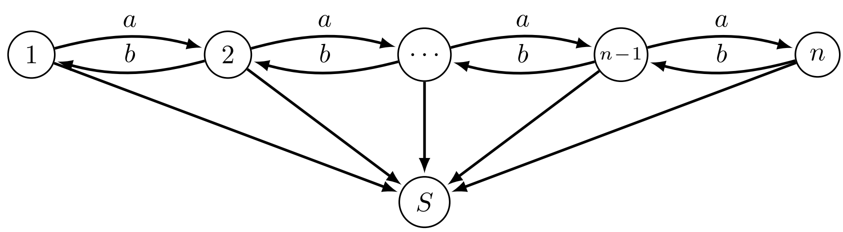

Figure 4. The graph $P_{n,a,b}$ (edge labels correspond to their multiplicities).

The next lemma provides a recurrence relation for the number of arborescences of $P_{n,a,b}$.

Lemma 1. *Let $\tau(P_{n,a,b})$ be the number of arborescences of the graph $P_{n,a,b}$. Then*

$$\tau(P_{n,a,b}) = (a+b+1)\tau(P_{n-1,a,b}) - ab\tau(P_{n-2,a,b}), \quad n \geq 3, \tag{3}$$

with $\tau(P_{1,a,b}) = 1$ and $\tau(P_{2,a,b}) = a+b+1$.

Proof. The cases $n = 1$ and $n = 2$ are straightforward, and we focus on $n \geq 3$. Denote by $\sigma(P_{n,a,b})$ the number of arborescences of the graph $P_{n,a,b}$ containing the edge $n \to S$ and one of the a edges $n-1 \to n$. Then we have

$$\tau(P_{n,a,b}) = b\tau(P_{n-1,a,b}) + \tau(P_{n-1,a,b}) + \sigma(P_{n,a,b}). \tag{4}$$

Indeed, the first term on the right-hand side counts arborescences containing one of the b edges $n \to n-1$ (therefore, the edge $n \to S$ is missing), the second term counts arborescences containing the edge $n \to S$ and no edge $n-1 \to n$, and by definition, the final term counts arborescences containing the edge $n \to S$ and one of the a edges $n-1 \to n$.

Also, we have the second recurrence relation

$$\sigma(P_{n,a,b}) = a\tau(P_{n-2,a,b}) + a\sigma(P_{n-1,a,b}). \tag{5}$$

In both cases, the factor a on the right-hand side corresponds to the choice of one of the a edges $n-1 \to n$ required by the definition of $\sigma(P_{n,a,b})$. The first summand counts arborescences that do not contain an edge $n-2 \to n-1$, while the second summand counts arborescences containing one of the edges $n-2 \to n-1$.

From (4), we obtain

$$\sigma(P_{n,a,b}) = \tau(P_{n,a,b}) - (b+1)\tau(P_{n-1,a,b}).$$

Using this relation to replace both occurrences of σ in (5), we get

$$\begin{aligned}\tau(P_{n,a,b}) - (b+1)\tau(P_{n-1,a,b}) &= a\tau(P_{n-2,a,b}) \\ &\quad + a(\tau(P_{n-1,a,b}) - (b+1)\tau(P_{n-2,a,b})),\end{aligned}$$

and simplification leads to the desired recurrence (3). ■

Remark 2. An inspection of the previous proof shows that if we create a new graph, say $H_{n,a,b}$, by doubling the edge $1 \to S$ in $P_{n,a,b}$, then the number of arborescences of $H_{n,a,b}$ still satisfies the same recurrence relation

$$\tau(H_{n,a,b}) = (a+b+1)\tau(H_{n-1,a,b}) - ab\tau(H_{n-2,a,b}), \quad n \geq 3,$$

but the initial values are now $\tau(H_{1,a,b}) = 2$ and $\tau(P_{2,a,b}) = a + 2b + 2$.

The next corollary shows that if $a = b = 1$, the arborescences of $P_{n,a,b}$ are counted by the Fibonacci numbers. Throughout this paper, we deal with the Fibonacci sequence whose initial terms are $F_0 = 0$, $F_1 = 1$.

Corollary 3. *The number of arborescences in $P_{n,1,1}$ is F_{2n}, and the number of arborescences in $P_{n,1,1}$ containing the edge $1 \to S$ is F_{2n-1}.*

Proof. By Lemma 1, we have

$$\tau(P_{n,1,1}) = 3\tau(P_{n-1,1,1}) - \tau(P_{n-2,1,1}), \;\; n \geq 3, \;\; \tau(P_{1,1,1}) = 1, \;\; \tau(P_{2,1,1}) = 3.$$

The even-indexed Fibonacci numbers F_{2n} satisfy the same relations.[1] Therefore, $\tau(P_{n,1,1}) = F_{2n}$.

The number of arborescences that contain $1 \to S$ equals F_{2n} minus the number of arborescences that do not contain $1 \to S$. Arborescences of the latter type necessarily contain $1 \to 2$, and the remaining edges form an arborescence of a graph that is isomorphic to $P_{n-1,1,1}$; their number is F_{2n-2}. Hence, the result is $F_{2n} - F_{2n-2} = F_{2n-1}$. ■

In what follows, we calculate the number of arborescences for an additional two classes of graphs: T_n, which is obtained from $P_{n,1,1}$ by doubling the edge $1 \to S$, and D_n, which is obtained from $P_{n,1,1}$ by doubling the edges $1 \to S$ and $n \to S$ (see Figure 5).

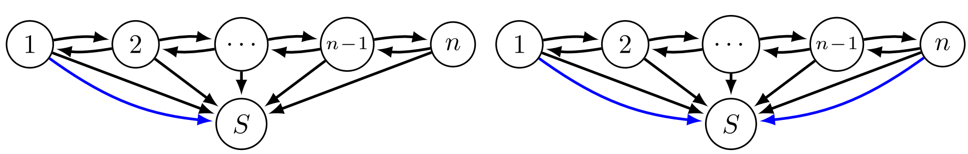

Figure 5. The graphs T_n (left) and D_n (right).

[1]See https://oeis.org/A001906.

Lemma 4. *For each $n \in \mathbb{N}$, the number of arborescences of T_n is F_{2n+1}, and the number of arborescences of D_n is F_{2n+2}.*

Proof. We begin with T_n. By Corollary 3, there are $2F_{2n-1}$ arborescences containing one of the edges $1 \to S$, and F_{2n-2} arborescences that do not contain an edge $1 \to S$ (i.e., containing the edge $1 \to 2$). The total number is

$$2F_{2n-1} + F_{2n-2} = F_{2n-1} + F_{2n} = F_{2n+1}.$$

Let us proceed to D_n. The number of arborescences that contain no edge $1 \to S$, and therefore contain the edge $1 \to 2$, is F_{2n-1}, because the remaining edges form an arborescence of a graph that is isomorphic to T_{n-1}.

Suppose for a moment that we remove one of the edges $1 \to S$. The number of arborescences of the new graph that contain the remaining edge $1 \to S$ is $F_{2n+1} - F_{2n-1}$ (i.e., the total number of arborescences of a graph isomorphic to T_n, minus the number of arborescences that do not contain $1 \to S$ and contain $1 \to 2$). This simplifies to F_{2n}. Multiplying this number by 2, we get the number of arborescences of D_n containing one of the two edges $1 \to S$.

Hence, the total number of arborescences of D_n is $F_{2n-1} + 2F_{2n} = F_{2n+1} + F_{2n} = F_{2n+2}$. ■

An alternative method of proving the first part of Lemma 4 is to use Remark 2, which provides a recurrence relation for the number of arborescences of $T_n = H_{n,1,1}$.

5. PATH GRAPHS. We are finally ready to calculate the winning probabilities in *Pass the Buck* on various graphs. We begin with the path graph P_n with vertices numbered consecutively by 1, ..., n, and calculate the probability that player j wins, provided that player i starts. Without loss of generality, we restrict ourselves to the case $j \geq i$ (otherwise, one can simply reverse the labelling of the vertices). Our result generalizes [**9**, Theorem 1], which deals with the case when the first (or last) player starts.

Theorem 5. *Consider Pass the Buck on the graph P_n. If player $i \in \{1, \dots, n\}$ starts, the probability that player $j \in \{i, \dots, n\}$ wins is*

$$p_{ij} = \frac{F_{2i-1} F_{2n-2j+1}}{F_{2n}}.$$

Proof. We apply the Markov chain tree theorem, and calculate the winning probabilities by counting arborescences in the graph obtained from P_n by replacing all undirected edges by pairs of directed edges, and joining each vertex $1, \dots, n$ to the absorbing vertex S. This is exactly the graph $P_{n,1,1}$ introduced in Section 4. From Corollary 3, we know that it has F_{2n} arborescences.

It remains to count arborescences containing the path from i to S via the edge $j \to S$. Suppose first that $i \geq 2$ and $j \leq n-1$. We claim that the required number is the same as the number of arborescences of the graph obtained from $P_{n,1,1}$ by deleting the vertices $i, \dots, j$, joining $i-1$ to S by an additional edge e, and joining $j+1$ to S by an additional edge f. Indeed, each arborescence of the new graph is easily transformed to an arborescence of the old graph. We add the path $i \to \cdots \to j \to S$; moreover, the edge e in the new graph corresponds to the edge $i-1 \to i$ in the old graph, and f in the new graph corresponds to $j+1 \to j$ in the old graph. This process can be always reversed, i.e., we have a bijection between the arborescences of the two graphs.

Now, each arborescence of the new graph is obtained by taking an arborescence of the subgraph on vertices $1, \dots, i-1, S$, and an arborescence of the subgraph on the vertices $j+1, \dots, n, S$. These two subgraphs are isomorphic to T_{i-1} and T_{n-j}, respectively. Thus, by Lemma 4, the total number of arborescences is $F_{2(i-1)+1}F_{2(n-j)+1}$.

It is not difficult to check that the formula $F_{2(i-1)+1}F_{2(n-j)+1}$ is correct even if $i = 1$ or $j = n$, respectively. If $i = 1$, the first subgraph is empty, and $F_{2(i-1)+1} = 1$. If $j = n$, then the second subgraph is empty, and $F_{2(n-j)+1} = 1$. ■

The technique employed in the proof will be used repeatedly throughout this paper. In short, the number of arborescences containing a certain path P can be obtained by redirecting all edges leading to the vertices of P into S, deleting P, and counting the arborescences of the new graph.

6. CYCLE GRAPHS. We now calculate the winning probabilities for *Pass the Buck* on the cycle graph C_n. We can assume that the vertices are numbered consecutively by $1, \dots, n$, and player 1 starts.

Theorem 6. *Consider Pass the Buck on the graph C_n. If player 1 starts, the probability that player $j \in \{1, \dots, n\}$ wins is*

$$p_{1j} = \frac{F_{2(n-j+1)} + F_{2(j-1)}}{F_{2n+1} + F_{2n-1} - 2}. \tag{6}$$

Proof. By the Markov chain tree theorem, the calculation can be reduced to counting arborescences in the graph G obtained from C_n by replacing undirected edges by pairs of directed ones, and joining each vertex to the absorbing state S. Recalling that

$$p_{1j} = \frac{|\mathcal{A}_{1j}(G)|}{|\mathcal{A}(G)|},$$

we need to calculate $|\mathcal{A}_{1j}(G)|$, i.e., the number of arborescences containing a path from 1 to S via the edge $j \to S$. If $j = 1$, we are interested in arborescences containing the edge $1 \to S$. We will use the method described in Section 5: By deleting the vertex 1 and adding edges from 2 and n to S, we obtain a graph isomorphic to D_{n-1}, which has exactly F_{2n} arborescences.

If $j \geq 2$, the arborescences containing a path from 1 to S via the edge $j \to S$ can be divided into two types: either they contain the path $1 \to 2 \to 3 \to \cdots \to j \to S$, or the path $1 \to n \to n-1 \to \cdots \to j \to S$.

The number of arborescences of the first type is the same as the total number of arborescences of the graph obtained from G by deleting the vertices $1, 2, 3 \dots, j$, and joining $j+1$ and n to S by additional edges. The new graph has $n - j$ vertices and is isomorphic to D_{n-j}. By Corollary 4, the number of its arborescences is $F_{2(n-j+1)}$. The previous argument does not work if $j = n$ or $j = n-1$, but one can easily check that the result remains correct even in these cases.

The number of arborescences of the second type is the same as the total number of arborescences of the graph obtained from G by deleting the vertices $j, j+1, \dots, n, 1$, and joining 2 and $j-1$ to S by additional edges. This graph has $j-2$ vertices, is isomorphic to D_{j-2}, and has $F_{2(j-1)}$ arborescences. The argument does not work if $j = 2$, but the result remains correct.

In total, we have $|\mathcal{A}_{1j}(G)| = F_{2(n-j+1)} + F_{2(j-1)}$ for all $j \in \{1, \dots, n\}$. To determine $|\mathcal{A}(G)|$, we observe that $p_{i1} + \cdots + p_{in} = 1$ (the probability that the game will never end is zero). Therefore,

$$\begin{aligned}|\mathcal{A}(G)| &= \sum_{j=1}^{n} |\mathcal{A}_{1j}(G)| = \sum_{j=1}^{n} F_{2(n-j+1)} + \sum_{j=1}^{n} F_{2(j-1)} = \sum_{i=1}^{n} F_{2i} + \sum_{j=1}^{n} F_{2(j-1)} \\ &= \sum_{i=1}^{n} (F_{2i+1} - F_{2i-1}) + \sum_{j=1}^{n} (F_{2j-1} - F_{2j-3}) = F_{2n+1} + F_{2n-1} - 2,\end{aligned}$$

which completes the proof.[2] ■

Pass the Buck on cycles was already analyzed in the paper [**8**], whose authors arrived at the following results:

$$p_{1j} = \begin{cases} \frac{F_{n-2j+2}}{L_n} & \text{for } n \text{ odd}, \\ \frac{L_{n-2j+2}}{5F_n} & \text{for } n \text{ even}, \end{cases} \tag{7}$$

where L_n are the Lucas numbers. (Actually, the results in [**8**, Theorem 2] have a shifted index j, since the players there are labelled by the numbers $0, \ldots, n-1$.) The advantage of our formula (6) is that there is no need to distinguish between even and odd values of n. To verify that (6) and (7) coincide, one can use standard Lucas number identities to rewrite the denominator in (6) as follows:

$$F_{2n+1} + F_{2n-1} - 2 = L_{2n} - 2 = \begin{cases} L_n^2 & \text{for } n \text{ odd}, \\ 5F_n^2 & \text{for } n \text{ even}. \end{cases}$$

To conclude that (6) and (7) are equivalent, it remains to check that

$$F_{2(n-j+1)} + F_{2(j-1)} = \begin{cases} L_n F_{n-2j+2} & \text{for } n \text{ odd}, \\ F_n L_{n-2j+2} & \text{for } n \text{ even}. \end{cases}$$

One way to accomplish this is to use the explicit formulas $F_n = (\varphi^n - \psi^n)/\sqrt{5}$ and $L_n = \varphi^n + \psi^n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = -1/\varphi$ are the roots of $x^2 - x - 1$.

7. COMPLETE GRAPHS. What are the winning probabilites in *Pass the Buck* on the complete graph K_n? These can be calculated fairly easily by elementary considerations as in [**9**, p. 389]. However, we will use this opportunity to demonstrate a technique known in Markov chain theory as "lumping".

Without loss of generality, assume that the starting player has label 1, and the remaining players are 2, …, n. As before, denote by p_{1j} the probability that player $j \in \{1, \ldots, n\}$ wins.

Theorem 7. *Consider Pass the Buck on the graph K_n. If player 1 starts, the probability p_{1j} that player $j \in \{1, \ldots, n\}$ wins is*

$$p_{11} = \frac{2}{n+1}, \quad p_{12} = p_{13} = \cdots = p_{1n} = \frac{1}{n+1}. \tag{8}$$

[2] Alternatively, the relation $|\mathcal{A}(G)| = \sum_{j=1}^{n} |\mathcal{A}_{1j}(G)|$ follows from the fact that each arborescence contains a unique path from 1 to S.

Proof. Because of symmetry, it is clear that the chances of players 2, ..., n are identical, i.e., $p_{12} = p_{13} = \cdots = p_{1n}$. This suggests that we could merge these players into a single group B. Thus, instead of considering the original Markov chain, we will deal with a lumped chain having transient states 1, B, and absorbing states S_1, S_B. Being in state B (or S_B) in the lumped chain corresponds to being in one of the states $2, \ldots, n$ (or $S_2, \ldots, S_n$) in the original chain.

The matrices of transition probabilities in the lumped chain are

$$Q = \begin{pmatrix} \pi_{11} & \pi_{1B} \\ \pi_{B1} & \pi_{BB} \end{pmatrix} = \begin{pmatrix} 0 & \frac{n-1}{n} \\ \frac{1}{n} & \frac{n-2}{n} \end{pmatrix}, \quad R = \begin{pmatrix} \pi_{1S_1} & \pi_{1S_B} \\ \pi_{BS_1} & \pi_{BS_B} \end{pmatrix} = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{n} \end{pmatrix}.$$

We could now calculate the absorbing probabilities by looking at the first row of the matrix $(I - Q)^{-1}R$, but let us stick with the Markov chain tree theorem, and count arborescences in the graph shown in Figure 6.

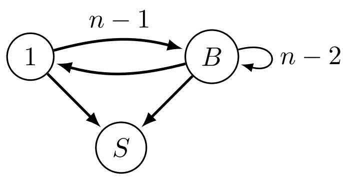

Figure 6. Lumped graph corresponding to *Pass the Buck* on K_n.

This graph has $n + 1$ arborescences; two of them contain the path $1 \to S$, and $n - 1$ of them contain the path $1 \to B \to S$. Hence, the probability of absorption from state 1 is $\frac{2}{n+1}$, while the probability of absorption from state B is $\frac{n-1}{n+1}$. We now recall that B was obtained by lumping $n - 1$ states, and therefore divide the latter probability by $n - 1$. This gives the probabilities in (8). ■

The mean duration of the game can be calculated from the fundamental matrix

$$N = (I - Q)^{-1} = \begin{pmatrix} \frac{2n}{n+1} & \frac{(n-1)n}{n+1} \\ \frac{n}{n+1} & \frac{n^2}{n+1} \end{pmatrix}$$

by adding the values in the first row; it turns out that the mean duration is precisely n.

Here is a different proof of Theorem 7, which avoids the lumping technique: Starting with K_n, replace all undirected edges by pairs of directed ones, and join the vertices $1, \ldots, n$ to an absorbing vertex $S = n + 1$. In this way, we obtain a directed graph G, which has the same number of arborescences as the number of undirected spanning trees of K_{n+1}; by Cayley's formula, this number is $(n + 1)^{n-1}$. Indeed, there is a bijection between undirected spanning trees of K_{n+1} and arborescences of G with root $n + 1$, where each undirected edge is oriented along the unique path towards the root.

This time, it will be more convenient to denote the starting player by n. To calculate the winning probability of this player, we need the number of arborescences containing the edge $n \to n + 1$. However, it is easier to count arborescences that do not contain this edge; this is simply the number of undirected spanning trees of K_{n+1} minus one edge, which is known to be $(n - 1)(n + 1)^{n-2}$; see the next paragraph. Hence, the winning probability of the starting player is

$$\frac{(n + 1)^{n-1} - (n - 1)(n + 1)^{n-2}}{(n + 1)^{n-1}} = \frac{2}{n + 1}.$$

Because of symmetry, all remaining players have winning probabilities $\frac{1}{n+1}$.

One way[3] to prove the formula $(n-1)(n+1)^{n-2}$ is to use [**3**, Lemma 1]—a corollary of the matrix tree theorem, which implies that the number of spanning trees of an undirected graph with $n+1$ vertices and Laplacian matrix L is

$$\frac{1}{(n+1)^2}\det(L+J),$$

where J is a square matrix of order $n+1$ whose entries are all equal to one. For K_{n+1} minus the edge $n \to n+1$, we get the matrix

$$\begin{aligned} L+J &= \begin{pmatrix} n & -1 & \cdots & -1 & -1 \\ -1 & n & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & -1 & -1 \\ -1 & -1 & \cdots & n-1 & 0 \\ -1 & -1 & \cdots & 0 & n-1 \end{pmatrix} + \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \\ &= \begin{pmatrix} n+1 & 0 & \cdots & 0 & 0 \\ 0 & n+1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & n & 1 \\ 0 & 0 & \cdots & 1 & n \end{pmatrix}, \end{aligned}$$

whose determinant is $(n+1)^n(n-1)$.

8. COMPLETE BIPARTITE GRAPHS. We now consider *Pass the Buck* on the complete bipartite graph $K_{a,b}$ with part A having a vertices, and part B having b vertices. The results in the present section are new.

Without loss of generality, suppose that the starting player in *Pass the Buck* has label 1 and is located in part A; denote his/her winning probability by p_{11}. By symmetry, the winning probabilities of all remaining players in part A are equal to a certain number p_{1A}, and the winning probabilities of all players from part B are all equal to p_{1B}.

Theorem 8. *Consider Pass the Buck on the graph $K_{a,b}$. If player 1 starts, the winning probabilities are*

$$p_{11} = \frac{a+2b+1}{(a+b+1)(b+1)}, \quad p_{1A} = \frac{b}{(a+b+1)(b+1)}, \quad p_{1B} = \frac{1}{a+b+1}.$$

Proof. We consider a lumped Markov chain with three transient states corresponding to three groups of players: 1, $\overline{A} = A \setminus \{1\}$, and B. It is not difficult to write down the transition probabilities for the lumped chain:

$$Q = \begin{pmatrix} \pi_{11} & \pi_{1\overline{A}} & \pi_{1B} \\ \pi_{\overline{A}1} & \pi_{\overline{A}\overline{A}} & \pi_{\overline{A}B} \\ \pi_{B1} & \pi_{B\overline{A}} & \pi_{BB} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{b}{b+1} \\ 0 & 0 & \frac{b}{b+1} \\ \frac{1}{a+1} & \frac{a-1}{a+1} & 0 \end{pmatrix},$$

[3]For a combinatorial proof, see e.g., `https://math.stackexchange.com/questions/575163/spanning-trees-of-the-complete-graph-minus-an-edge`.

$$R = \begin{pmatrix} \pi_{1S_1} & \pi_{1S_{\overline{A}}} & \pi_{1S_B} \\ \pi_{\overline{A}S_1} & \pi_{\overline{A}S_{\overline{A}}} & \pi_{\overline{A}S_B} \\ \pi_{BS_1} & \pi_{BS_{\overline{A}}} & \pi_{BS_B} \end{pmatrix} = \begin{pmatrix} \frac{1}{b+1} & 0 & 0 \\ 0 & \frac{1}{b+1} & 0 \\ 0 & 0 & \frac{1}{a+1} \end{pmatrix}.$$

Hence, we need to count arborescences in the graph shown in Figure 7.

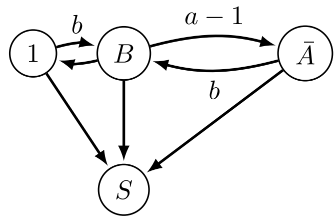

Figure 7. Lumped graph corresponding to *Pass the Buck* on $K_{a,b}$.

The number of arborescences containing the path $1 \to S$ is

$$(a+1)(b+1) - (a-1)b = a + 2b + 1$$

(multiply the outdegrees of B and $\overline{A}$, and subtract the number of cycles created in this way). Moreover, there are $b(b+1)$ arborescences containing the path $1 \to B \to S$, and $b(a-1)$ arborescences containing the path $1 \to B \to \overline{A} \to S$. The sum of all these numbers (which coincides with the total number of arborescences) is

$$(a+1)(b+1) - (a-1)b + b(b+1) + b(a-1) = (a+b+1)(b+1).$$

Thus, the absorbing probabilities for states 1, $\overline{A}$, and B are

$$p_1 = \frac{a+2b+1}{(a+b+1)(b+1)}, \quad p_{\overline{A}} = \frac{b(a-1)}{(a+b+1)(b+1)}, \quad p_B = \frac{b}{a+b+1}.$$

Dividing these values by the sizes of the groups, we obtain the winning probabilities for individual players: $p_{11} = p_1, p_{1A} = p_{\overline{A}}/(a-1), p_{1B} = p_B/b$. ■

To calculate the mean duration of the game, we write down the fundamental matrix

$$N = (I-Q)^{-1} = \begin{pmatrix} \frac{1+a+2b}{1+a+b} & \frac{(-1+a)b}{1+a+b} & \frac{(1+a)b}{1+a+b} \\ \frac{b}{1+a+b} & \frac{1+a+ab}{1+a+b} & \frac{(1+a)b}{1+a+b} \\ \frac{1+b}{1+a+b} & \frac{(-1+a)(1+b)}{1+a+b} & \frac{(1+a)(1+b)}{1+a+b} \end{pmatrix}$$

and add the values in its first row, which yields $(1+a)(1+2b)/(a+b+1)$.

Note that the previous results apply to star graphs, which are special cases of complete bipartite graphs with $a = 1$.

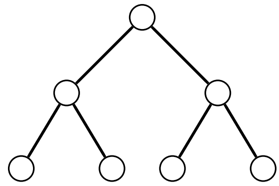

Figure 8. The graph $T_{2,3}$, i.e., a binary tree with three layers.

9. COMPLETE k-ARY TREES.

Using the lumping technique, we can solve *Pass the Buck* on complete k-ary trees, provided that the starting player is in the root; this case was analyzed in [**6**], and we present a short alternative solution based on counting arborescences.

Denote by $T_{k,n}$ the complete k-ary tree of depth $n-1$. The i-th layer of the tree (when counted from the top) contains k^{i-1} vertices; an example is shown in Figure 8.

Since the starting player corresponds to the root, it is clear that all players in the same layer have the same winning probability; denote it by p_{1i}.

Theorem 9. *Consider Pass the Buck on the graph $T_{k,n}$. If the player corresponding to the root starts, the winning probability for all players in layer $j \in \{1, \ldots, n\}$ is*

$$p_{1j} = \frac{a_{n-j}}{b_n}, \tag{9}$$

where the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are given by

$$a_n = (k+2)a_{n-1} - ka_{n-2}, \quad n \geq 2, \quad a_0 = 1, \quad a_1 = 2, \tag{10}$$

$$b_n = (k+2)b_{n-1} - kb_{n-2}, \quad n \geq 3, \quad b_1 = 1, \quad b_2 = k+2. \tag{11}$$

Proof. Lumping all players in the same layer together, we see that our problem reduces to counting arborescences in the graph $P_{n,k,1}$ introduced in Section 4. By Lemma 1, the arborescences of this graph are counted by the sequence $(b_n)_{n=1}^{\infty}$ described in (11).

To calculate the winning probability of player j, we need the number of arborescences containing a path $1 \to 2 \to \cdots \to j \to S$. There are k^{j-1} such paths.

Suppose first that $j < n$. When choosing the remaining edges of the arborescence, we can remove the vertices $1, \ldots, j$, and join $j+1$ by an additional edge to S (this edge will correspond to the edge $j+1 \to j$ in the old graph). This results in a graph that is isomorphic to the graph $H_{n-j,k,1}$, where $H_{n,a,b}$ was introduced in Remark 2. According to that remark, the number of arborescences of $H_{n-j,k,1}$ is a_{n-j}, where the numbers a_n satisfy $a_n = (k+2)a_{n-1} - ka_{n-2}$ for $n \geq 3$, and $a_1 = 2$, $a_2 = k+4$. If we let $a_0 = 1$, the recurrence holds for all $n \geq 2$.

To sum up, if $j < n$, we have shown that the number of arborescences containing a path $1 \to 2 \to \cdots \to j \to S$ is $k^{j-1}a_{n-j}$. However, since $a_0 = 1$, this result is also correct for $j = n$.

These considerations show that the absorption probability for the group of vertices in layer j is $k^{j-1}a_{n-j}/b_n$. Dividing by the number of vertices in layer j, we see that the winning probability for each player in layer j is indeed given by (9). ■

The recurrence relation (10) is the same as in [**6**, p. 353]; in that paper, b_n is instead calculated using the formula $b_n = \sum_{i=1}^{n} k^{i-1}a_{n-i}$ (which holds because the sum of all probabilities must be 1). Note that one can solve the recurrences (10) and (11) to obtain explicit formulas for a_n and b_n, but the results are not too enlightening:

$$b_n = \frac{(k+2+s)^n - (k+2-s)^n}{2^n s},$$
$$a_n = \frac{(k-2+s)(k+2-s)^n - (k-2-s)(k+2+s)^n}{2^{n+1} s},$$

where $s = \sqrt{k^2+4}$.

The previous examples (complete graphs, complete bipartite graphs, and complete k-ary trees) demonstrate the usefulness of the lumping technique. A word of caution: Given a general Markov chain with states $1, \ldots, n$, the process of lumping the states into groups $A_1, \ldots, A_r$ yields a Markov chain if and only if the following lumpability condition holds. For all i, j and $s \in A_i$, the probability of transition from s to A_j, i.e., $\pi_{s,A_j} = \sum_{t \in A_j} \pi_{st}$, depends only on i and j, and not on the particular choice of $s \in A_i$. If this is the case, we denote this value by π_{A_i,A_j}; repeating this process for all i, j, we obtain the transition probabilities for the lumped Markov chain (see [**2**, Section 6.3]). In the three examples we have presented, we believe it is intuitively clear that the lumpability condition is satisfied, but it is not difficult to check it explicitly.

10. CONNECTED CYCLES. The next example is *Pass the Buck* on the graph $C_{m,n}$ formed by joining two cycles, C_m and C_n, at a single vertex. Denote the vertices of C_m by $a_1, a_2, \ldots, a_m$, and the vertices of C_n by $b_1, b_2, \ldots, b_n$, with the common vertex being $a_1 = b_1 = 1$. An example is provided in Figure 9.

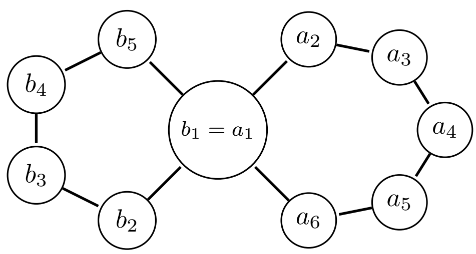

Figure 9. The graph $C_{m,n}$ with cycles of lengths $m = 6$ and $n = 5$.

The lumping technique is not appropriate here, but we can proceed as before: Replace all undirected edges by pairs of directed edges, and connect all vertices to the absorbing state S, obtaining a graph G. No matter which player starts, we need the total number of arborescences of G.

One way to create an arborescence of G is to begin with an arborescence of the subgraph consisting of C_n and S, which can be chosen in $F_{2n+1} + F_{2n-1} - 2$ ways (see Section 6). Then it remains to choose outgoing edges for the vertices $a_2, \ldots, a_m$. Replacing $a_2 \to a_1$ by a second edge $a_2 \to S$, and $a_m \to a_1$ by a second edge $a_m \to S$, we see that the task is equivalent to choosing an arborescence of a graph isomorphic to D_{m-1}, which can be done in F_{2m} ways (see Lemma 4). In this way, we have constructed all arborescences of G where 1 is linked to b_2, b_n, or S; their number is $(F_{2n+1} + F_{2n-1} - 2)F_{2m}$.

Symmetrically, beginning with an arborescence of the subgraph consisting of C_m and S and then choosing outgoing edges for the vertices $b_2, \ldots, b_n$, we construct all arborescences of G where 1 is linked to a_2, a_m, or S; their number is $(F_{2m+1} + F_{2m-1} - 2)F_{2n}$.

At this moment, we have counted twice all arborescences containing the edge $1 \to S$. How many such arborescences are there? Given an arborescence containing $1 \to S$, we can transform all edges leading to 1 into edges leading to S. This creates arborescences of two graphs that are isomorphic to D_{m-1} and D_{n-1}, respectively. Since the process is reversible, we see that there are $F_{2m}F_{2n}$ arborescences of G containing the edge $1 \to S$.

Consequently, the total number of arborescences of G is

$$\begin{aligned}&(F_{2n+1} + F_{2n-1} - 2)F_{2m} + (F_{2m+1} + F_{2m-1} - 2)F_{2n} - F_{2m}F_{2n} \\ &= F_{2n+1}F_{2m} + F_{2n}F_{2m-1} + F_{2n}F_{2m+1} + F_{2n-1}F_{2m} - 2F_{2m} - 2F_{2n} - F_{2m}F_{2n} \\ &= 2F_{2m+2n} - 2F_{2m} - 2F_{2n} - F_{2m}F_{2n},\end{aligned}$$

where the second equality follows from the identity $F_{i+j+1} = F_{i+1}F_{j+1} + F_iF_j$.

Theorem 10. *Consider Pass the Buck on the graph $C_{m,n}$. If player 1 starts, the winning probability for player a_i, where $i \in \{1, \ldots, m\}$, is*

$$p_{1,a_i} = \frac{(F_{2(m-i)+2} + F_{2(i-1)})F_{2n}}{2F_{2m+2n} - 2F_{2m} - 2F_{2n} - F_{2m}F_{2n}}. \tag{12}$$

Proof. It suffices to calculate the number of arborescences containing the path from 1 to S via the edge $a_i \to S$. Suppose first that $i \geq 2$. Vertex a_i can be reached in two different ways: via the path $a_1 \to a_2 \to \cdots \to a_i$, or via the path $a_1 \to a_m \to a_{m-1} \to \cdots \to a_i$.

The number of arborescences containing $a_1 \to a_2 \to \cdots \to a_i$ can be obtained similarly as in the proof of Theorem 6: We can redirect the edges $a_{i+1} \to a_i$, $a_m \to a_1$, $b_2 \to a_1$, and $b_n \to a_1$ into the absorbing vertex S, and delete the vertices $a_1, \ldots, a_i$. This operation preserves the number of arborescences, and splits the graph into two parts, which are isomorphic to D_{n-1} and D_{m-i}, respectively. Hence, by Lemma 4, the number of arborescences is $F_{2n}F_{2m-2i+2}$ (this number is correct even for $i = m$, when there is no edge $a_{i+1} \to a_i$). Similarly, calculation of arborescences containing the path $a_1 \to a_m \to a_{m-1} \to \cdots \to a_i$ leads to a pair of graphs that are isomorphic to D_{n-1} and D_{i-2}, respectively; there are $F_{2n}F_{2i-2}$ of them.

Consequently, the winning probability for player a_i with $i \neq 1$ is given by (12). The calculation for $i = 1$ has already been performed at the beginning of this section, where we investigated the number of arborescences containing the edge $1 \to S$. ∎

Let us consider a scenario involving a different starting player.

Theorem 11. *Consider Pass the Buck on the graph $C_{m,n}$. If player a_i with $i \neq 1$ starts, the winning probability for player b_j with $j \neq 1$ is*

$$p_{a_i,b_j} = \frac{F_{2(m-i)+2}(F_{2(n-j)+2} + F_{2(j-2)+2}) + F_{2(i-2)+2}(F_{2(n-j)+2} + F_{2(j-2)+2})}{2F_{2m+2n} - 2F_{2m} - 2F_{2n} - F_{2m}F_{2n}}. \tag{13}$$

Proof. We need to consider four different paths from a_i to b_j listed below. Each of them is handled as in the proof of Theorem 10, i.e., by removing the path and counting arborescences in a pair of graphs isomorphic to D_k for appropriate values of k:

- $a_i \to a_{i-1} \to \cdots \to a_1 = b_1 \to b_2 \to \cdots \to b_j$, leads to D_{m-i} and D_{n-j}.

- $a_i \to a_{i-1} \to \cdots \to a_1 = b_1 \to b_n \to b_{n-1} \to \cdots \to b_j$, leads to D_{m-i} and D_{j-2}.
- $a_i \to a_{i+1} \to \cdots \to a_n \to a_1 = b_1 \to b_2 \to \cdots \to b_j$, leads to D_{i-2} and D_{n-j}.
- $a_i \to a_{i+1} \to \cdots \to a_n \to a_1 = b_1 \to b_n \to b_{n-1} \to \cdots \to b_j$, leads to D_{i-2} and D_{j-2}.

By Lemma 4, the winning probability for player b_j is given by (13). ■

The final unsolved case involves two players in a single cycle; we leave it as an exercise.

11. SPOKE GRAPHS. The spoke graph $S_{m,n}$ consists of m paths (referred to as the spokes) of length n, which start at a common vertex. We will denote this common vertex by 1, and the remaining vertices along each spoke will be numbered by consecutive integers as in Figure 10. (An alert reader might be worried about using the same labels for different vertices. This will greatly simplify the notation, and if necessary, we can always distinguish between such vertices by labelling the spokes.) The fact that all spokes have the same length is not crucial; we focus on this case only for the purpose of clarity and simplicity, but similar methods work for spokes of unequal lengths (although the resulting formulas become messy, cf., the end of this section).

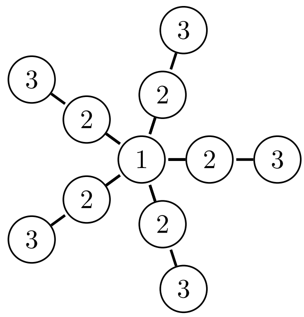

Figure 10. The spoke graph $S_{m,n}$ with $m = 5$ spokes of length $n = 3$.

Pass the Buck on $S_{m,n}$ was already analyzed in the paper [**9**] (where it is referred to as the star graph), but only in the case when player 1 starts. We will provide a short alternative derivation of the same result, and then investigate the case of different starting players.

Theorem 12. *Consider Pass the Buck on the graph $S_{m,n}$. If player 1 starts, the winning probability for each player having number $j \in \{1, \ldots, m\}$ is*

$$p_{1j} = \frac{F_{2(n-j)+1}}{mF_{2n} - (m-1)F_{2n-1}}. \tag{14}$$

Proof. Consider the graph G obtained from $S_{m,n}$ by introducing directed edges and adding an absorbing vertex S. We choose one of the vertices with label j, and calculate the number of arborescences containing the path from 1 to S via the edge $j \to S$.

Suppose first that $j = 1$. Given an arborescence containing $1 \to S$, we can redirect all edges leading to 1 into S. This creates arborescences of m graphs isomorphic to T_{n-1}, and the process is reversible. Hence, the number of arborescences containing $1 \to S$ is F^m_{2n-1}.

Suppose next that $j > 1$, and that the selected vertex j lies on spoke k. Given an arborescence containing the path $1 \to \cdots \to j \to S$, we can redirect all edges leading to 1, as well as edge $j+1 \to j$ on spoke k, into the absorbing vertex S. This process is reversible, and gives rise to $m-1$ arborescences of graphs isomorphic to T_{n-1}, plus an arborescence of a graph isomorphic to T_{n-j}. The total number of possibilities is $F_{2n-1}^{m-1} F_{2(n-j)+1}$; this result is correct even if $j = n$, when there is no edge $j+1 \to j$.

The total number of arborescences of G is obtained by summing the previous results (note that the cases $j > 1$ have to be counted m times). We get

$$m \sum_{j=2}^{n} F_{2n-1}^{m-1} F_{2(n-j)+1} + F_{2n-1}^{m} = m \sum_{j=1}^{n} F_{2n-1}^{m-1} F_{2(n-j)+1} - (m-1) F_{2n-1}^{m}$$

$$= F_{2n-1}^{m-1} \left(m \sum_{j=1}^{n} F_{2(n-j)+1} - (m-1) F_{2n-1} \right) = F_{2n-1}^{m-1} \left(m F_{2n} - (m-1) F_{2n-1} \right).$$

The proof is now finished by applying the Markov chain tree theorem. ■

The previous result is in agreement with [**9**, Theorem 2] (the vertex numbers are reversed there). Next, we examine the case when we start and end on different spokes.

Theorem 13. *Consider Pass the Buck on the graph $S_{m,n}$. If player $i > 1$ starts, the winning probability of player $j > 1$ on a different spoke is*

$$p_{ij} = \frac{F_{2(n-i)+1} F_{2(n-j)+1}}{F_{2n-1}(m F_{2n} - (m-1) F_{2n-1})}. \tag{15}$$

Proof. We consider the same graph G as in the proof of Theorem 12, where we found that the number of its arborescences is $F_{2n-1}^{m-1} \left(m F_{2n} - (m-1) F_{2n-1} \right)$.

It remains to calculate the number of arborescences containing the path from i to S via the edge $j \to S$. There is a unique path from i to j, and we need to count arborescences containing this path. We delete the path, and redirect edges leading to its endpoints to S.

The spokes containing neither i nor j will lose only the central vertex, and will become isomorphic to T_{n-1}. The two spokes containing i or j will be transformed into graphs isomorphic to T_{n-i} and T_{n-j}, respectively. Hence, the number of arborescences containing the path from i to j is $F_{2n-1}^{m-2} F_{2(n-i)+1} F_{2(n-j)+1}$. Dividing this by the total number of arborescences yields (15). ■

Note that if we formally substitute $i = 1$ into (15), the formula reduces to (14), i.e., the formula (15) actually holds even if $i = 1$.

What happens if the starting player i and the player j, whose winning probability is to be calculated, are located on the same spoke? This case is more complicated, because after removing the path from i to j, we get a graph having only two components. The rest of the spoke where i and j are located is again isomorphic to a graph T_k for a suitable k, but the second component is isomorphic to a graph obtained from G by shortening one spoke, whose terminal vertex has a double edge to S. Hence, we need to solve the following auxiliary problem: Find the number of arborescences of the graph G_l obtained from G by shortening one spoke to length l, whose terminal vertex has a double edge to S.

To create an arborescence of G_l, we can begin with an arborescence of the subgraph containing S and one of the $m-1$ long spokes, which is isomorphic to $P_{n,1,1}$. Choosing outgoing edges in the rest of the graph is then equivalent to finding arborescences of $m-2$ graphs isomorphic to T_{n-1}, and one graph isomorphic to D_{l-1}. Alternatively, we can begin with the subgraph containing S and the short spoke, which is isomorphic to T_l. Choosing outgoing edges in the rest of the graph is then equivalent to finding arborescences of $m-1$ graphs isomorphic to T_{n-1}.

The process we have described creates all arborescences of G_l (the step in which a particular arborescence was created depends on the edge leaving the vertex 1). However, all arborescences containing $1 \to S$ were counted m times, so we need to subtract their number $m-1$ times. Finding such arborescences is equivalent to finding arborescences of $m-1$ graphs isomorphic to T_{n-1}, and one graph isomorphic to D_{l-1}.

Using the formulas for arborescences of $P_{n,1,1}$, T_n and D_n, we conclude that the total number of arborescences of G_l is

$$(m-1)F_{2n}F_{2n-1}^{m-2}F_{2l} + F_{2l+1}F_{2n-1}^{m-1} - (m-1)F_{2n-1}^{m-1}F_{2l}.$$

With this result, it is not difficult to solve *Pass the Buck* on $S_{m,n}$ in the case when players i and j are on the same spoke; details are left to the reader.

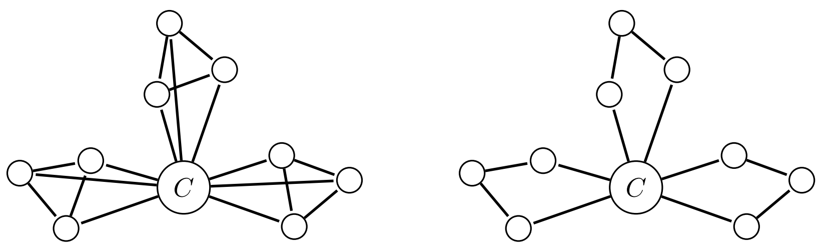

Figure 11. Windmill graph $Wd_{4,3}$ and Dutch windmill graph $Dw_{4,3}$.

12. FINAL REMARKS.

The methods discussed in earlier sections make it possible to solve *Pass the Buck* on a wide range of graphs. They are particularly suitable for graphs which, after the removal of a path, decompose into components for which it is possible to count their arborescences. For example, the reader might enjoy investigating the following two classes of graphs illustrated in Figure 11:

- The windmill graph $Wd_{k,n}$ consists of n copies of the complete graph K_k sharing a common vertex C. Using the lumping technique, the reader will have no trouble analyzing the case when player C starts. Here is a hint for a different starting vertex (because of symmetry, it does not matter which one). Consider a lumped Markov chain obtained by dividing the vertices into four classes: the starting vertex 1, the central vertex C, the set B of all vertices different from 1 contained in the same copy of K_k as 1, and the set D of all other vertices.
- The Dutch windmill graph $Dw_{m,n}$ is formed by joining n copies of the cycle graph C_m. The case $n = 2$ corresponds to the connected cycles graph $C_{m,m}$, which was solved in Section 10. The general case with arbitrary n can be solved in a similar manner. In fact, one can be even more general and consider n connected cycles of unequal lengths.

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