ELEMENTARY CONSTRUCTION OF HÖLDER FUNCTIONS S.T. THE KURZWEIL-STIELTJES INTEGRAL DOES NOT EXIST

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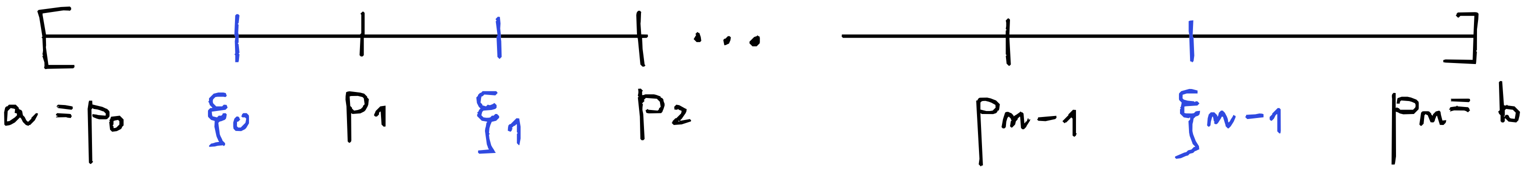
Definition: • Tagged partition of $[a,b] \subseteq \mathbb{R}$:

A pair (D,ξ), $D = (p_i)_{i=0}^{n}$, $\xi = (\xi_i)_{i=0}^{n-1}$;

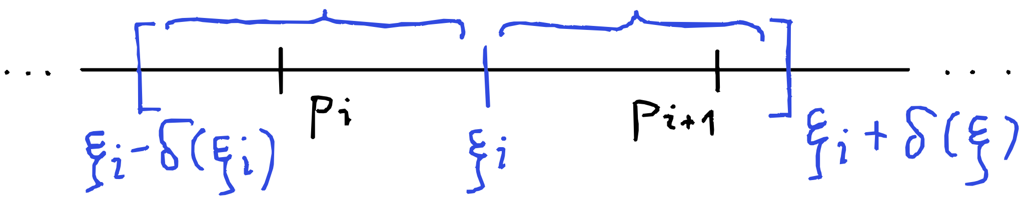
$$a = p_0 \leq \xi_0 \leq p_1 \leq \xi_1 \leq \dots \leq \xi_{n-1} \leq p_n = b$$



• gauge $\equiv$ any positive function $\delta : [a,b] \to (0,\infty)$

• (D,ξ) is δ-fine if, for all i,

$$[p_i, p_{i+1}] \subseteq [\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)]$$



<u>Definition</u>: Let $f, g : [a,b] \longrightarrow \mathbb{R}$,

and let (D, ξ) be a tagged partition of $[a,b]$.

- We define: (The K.-S. sum ...)

$$S(f, dg, D, \xi) = \sum_{i=0}^{m-1} f(\xi_i) \cdot (g(p_{i+1}) - g(p_i)).$$

- $I \in \mathbb{R}$ is the Kurzweil-Stieltjes integral of f w.r.t. g, if:

$$\forall \varepsilon > 0 \quad \exists \delta : [a,b] \longrightarrow (0, \infty) \quad \forall \delta\text{-fine } (D, \xi):$$

$$|S(f, dg, D, \xi) - I| < \varepsilon. \qquad \left[I = (KS)\int_a^b f\, dg \right]$$

g ... integrator

<u>Thm</u>: (Riesz) $\Phi \in \left(C[a,b]\right)^* \iff$

$\exists\, p \in BV[a,b]$: $\Phi(f) = \int_a^b f\, dp$, $f \in C[a,b]$.

<u>Proof</u>: „$\Leftarrow$" Easy.

„$\Rightarrow$" $\Phi \in C[a,b]^*$ given.

H.-B. $\rightsquigarrow$ $\tilde{\Phi} \in \left(l_\infty[a,b]\right)^*$ extension of Φ

Define: $p(t) := \tilde{\Phi}\left(\chi_{[a,t]}\right)$, $t \in (a,b]$

$p(a) := 0$.

This p works (tedious). $\square$

General Q: For which pairs (f, g) does the integral (KS) $\int_a^b f\,dg$ exist?

Theorem: $f, g : [a,b] \to \mathbb{R}$ are regulated and one of them is BV $\Longrightarrow$ YES

Definition: $f : [a,b] \to \mathbb{R}$ is regulated $\overset{\text{def.}}{\Longleftrightarrow}$

$$f(a+),\ f(b-) \in \mathbb{R} \quad \& \quad \forall t \in (a,b) : f(t_-), f(t_+) \in \mathbb{R}.$$

In particular: $f \in C[a,b]$, $g \in BV[a,b] \Longrightarrow$

$\int_a^b f\,dg$, $\int_a^b g\,df$ exist (in (KS)-sense).

<u>Theorem</u>: (Young, 1936) Let $\alpha, \beta \in (0,1)$, $\alpha + \beta > 1$, $f \in C^{0,\alpha}[a,b]$, $g \in C^{0,\beta}[a,b]$.

THEN $(RS)\int_a^b f\,dg$ exists. [„constant gauge"]

<u>Recall</u>: f is <u>α-Hölder</u> on $[a,b]$ $(\alpha \in (0,1])$

$\overset{\text{def.}}{\Longleftrightarrow}$ There is a constant C s.t.:

$$\forall x, y \in [a,b] : \quad |f(y) - f(x)| \leq C \cdot |y - x|^{\alpha}.$$

<u>Theorem</u>: (Dudley, Norvaiša [1999] + Hardy [1916] + Leśniewicz, Orlicz [1973])

Let $\alpha, \beta \in (0,1)$, $\boxed{\alpha + \beta \leq 1}$. THEN there exist $f \in C^{0,\alpha}[0,1]$, $g \in C^{0,\beta}[0,1]$: $\nexists$ $(KS)\int_0^1 f\,dg$.

Proof for $\alpha+\beta<1$: We define $f \in C^{0,\alpha}[0,1]$ as

$$f := \lim_{n\to\infty} f_n \quad \text{where:}$$

f_n: piece-wise affine w.r.t. $Z_n := \left\{ \frac{k}{2^n} : k = 0,1,\dots,2^n \right\}$

$f_0 :\equiv 0$. Now: induction step:

- $f_{n+1}\big|_{Z_n} := f_n\big|_{Z_n}$
- $x \in Z_{n+1} \setminus Z_n \iff x = \frac{\overset{\in\{0,\dots,2^{n+1}\}}{k}}{2^{n+1}}$, k odd:

Set $x_- = \frac{k-1}{2^{n+1}}$, $x_+ = \frac{k+1}{2^n+1}$. $(x_\pm \in Z_n)$.

- If $|f_n(x_+) - f_n(x_-)| \geq 2\cdot\left(\frac{1}{2^{n+1}}\right)^\alpha$, SET $f_{n+1}(x) := f_n(x)$.
- Otherwise, SET $f_{n+1}(x) := \max\{f_n(x_+), f_n(x_-)\} + \left(\frac{1}{2^{n+1}}\right)^\alpha$.

THEN: (i) $f_n \rightrightarrows f$ on $[0,1]$.

(ii) $\forall \gamma > \alpha \quad \forall I \subseteq [0,1] : f \in C^{0,\alpha}(I) \setminus C^{0,\gamma}(I)$.

(iii) $f(x) = f_n(x)$ whenever $x \in Z_n$, SO: $(n \geq 1)$

for any adjacent $x, y \in Z_n$ (i.e. $|x-y| = 2^{-n}$):

$$\left(\frac{1}{2^n}\right)^\alpha \leq |f(x) - f(y)| \leq 3 \cdot \left(\frac{1}{2^n}\right)^\alpha .$$

Same method $\rightsquigarrow g \in C^{\beta,0}(I) \setminus C^{\gamma,0}(I)$, $\gamma > \beta$

ETC. (analogous to f).

WTP: $(KS)\int_a^b f \, dg \; \nexists$. BY CONTRADICTION.

$(KS)\int_a^b f\,dg$ exists $\Longrightarrow$ for $\varepsilon = \frac{1}{2}$ $\exists$ gauge δ :

$\forall\ (D_1, \xi_1), (D_2, \xi_2)$ T.P. of $[0,1]$:

$$\left| S(f, dg, D_1, \xi_1) - S(f, dg, D_2, \xi_2) \right| < 1.$$

WTF: $(D_1, \xi_1), (D_2, \xi_2)$ s.t. : $\left| S(\dots) - S(\dots) \right| \geq 1.$

$$\delta : [0,1] \longrightarrow (0,\infty) \Longrightarrow [0,1] = \bigcup_{n=1}^{\infty} H_n ,$$

where $H_n := \{ x \in [0,1] : \delta(x) > 2^{-n} \}$.

non-deg.

Baire THM $\rightsquigarrow$ $\underline{n_0}$: $\overline{H_{n_0}} \supseteq I \subseteq [0,1]$.

$$\delta : [0,1] \longrightarrow (0,\infty) \implies [0,1] = \bigcup_{n=1}^{\infty} H_n,$$

where $H_n := \{ x \in [0,1] : \delta(x) > 2^{-n} \}$.

Baire THM $\rightsquigarrow$ $\underline{n_0}$: $\overline{H_{n_0}} \supseteq I \subseteq [0,1]$ non-deg.

$I := [a,b]$ ... we shall work only here. NOW:

- fix $\underline{n > n_0}$: $\left(2^n(b-a)-2\right)\left(2^{-(n+1)}\right)^{\alpha+\beta} > 1$ and ... $\alpha+\beta<1$!
- fix $\underline{\varepsilon > 0}$: $\left(\frac{1}{2^n}\right)^\alpha - 2\varepsilon > \left(\frac{1}{2^{n+1}}\right)^\alpha$ & same for β.
- fix $\eta \in (0, 2^{-(n+1)})$: $\forall x,y \in [0,1]$:

$$|x-y| < \eta \implies |f(x)-f(y)| < \varepsilon \quad \& \text{ same for } g.$$

• fix $\eta \in (0, 2^{-(m+1)})$: $\forall x, y \in [0,1]$:

$|x - y| < \eta \Rightarrow |f(x) - f(y)| < \varepsilon$ & same for g.

• Set $\tilde{D} := \mathbb{Z}_m \cap I$ ... dyadic pts in $I = [a, b]$.

• PERTURB $\tilde{D}$ (by $\leq \eta$) $\rightsquigarrow$ $\tilde{P} = \{p_0, \dots, p_N\}$

so that $p_i \in H_{m_0} \cap I \cap (d_i - \eta, d_i + \eta)$.

By uniform continuity,

[choice of m]

$|f(p_{i+1}) - f(p_i)| \geq$ Δ-ineq. ... $> \left(\frac{1}{2^m}\right)^\alpha - 2\varepsilon > \left(\frac{1}{2^{m+1}}\right)^\alpha$

TAGS: $\{t_i, T_i\} = \{p_i, p_{i+1}\}$. depends on $\mathrm{sgn}((\square - \square)\cdot(\square - \square))$

Then the T.P. $(\tilde{P}, t)$, $(\tilde{P}, T)$ are δ-fine!

$$\left| S(f, dg, D_1, \xi_1) - S(f, dg, D_2, \xi_2) \right| = \text{[restr. to } I\text{]}$$

$$= \left| \sum_{i=0}^{N-1} f(T_i)\left(g(p_{i+1}) - g(p_i)\right) - \sum_{i=0}^{N-} f(t_i)\left(g(p_{i+1}) - g(p_i)\right) \right|$$

$$= \left| \sum \left(f(T_i) - f(t_i)\right) \cdot \left(g(p_{i+1}) - g(p_i)\right) \right|$$

$$\left| S(f, dg, D_1, \xi_1) - S(f, dg, D_2, \xi_2) \right| = \text{[restr. to I]}$$

$$= \left| \sum_{i=0}^{N-1} f(T_i)\big(g(p_{i+1}) - g(p_i)\big) - \sum_{i=0}^{N-} f(t_i)\big(g(p_{i+1}) - g(p_i)\big) \right|$$

$$= \left| \sum \big(f(T_i) - f(t_i)\big) \cdot \big(g(p_{i+1}) - g(p_i)\big) \right|$$

$$= \Bigg| \sum_{i \in E} \big(f(p_{i+1}) - f(p_i)\big) \cdot \big(g(p_{i+1}) - g(p_i)\big) +$$

$$+ \sum_{i \in F} \big(f(p_i) - f(p_{i+1})\big) \cdot \big(g(p_{i+1}) - g(p_i)\big) \Bigg|$$

$$
\begin{aligned}
&\left| S(f, dg, D_1, \xi_1) - S(f, dg, D_2, \xi_2) \right| \quad = \quad \text{[restr. to } I\text{]} \\
&= \left| \sum_{i=0}^{N-1} f(T_i)\big(g(p_{i+1}) - g(p_i)\big) - \sum_{i=0}^{N-} f(t_i)\big(g(p_{i+1}) - g(p_i)\big) \right| \\
&= \left| \sum \big(f(T_i) - f(t_i)\big) \cdot \big(g(p_{i+1}) - g(p_i)\big) \right| \\
&= \left| \sum_{i \in E} \big(f(p_{i+1}) - f(p_i)\big) \cdot \big(g(p_{i+1}) - g(p_i)\big) \right. + \\
&\left. + \sum_{i \in F} \big(f(p_i) - f(p_{i+1})\big) \cdot \big(g(p_{i+1}) - g(p_i)\big) \right| \\
&= \sum_{i=0}^{N-1} \left| f(p_{i+1}) - f(p_i) \right| \cdot \left| g(p_{i+1}) - g(p_i) \right| \\
&> \sum_{i=0}^{N-1} \left(2^{-(m+1)}\right)^{\alpha} \cdot \left(2^{-(m+1)}\right)^{\beta} = N \cdot \left(2^{-(m+1)}\right)^{\alpha+\beta} \\
&\geq \left(2^m (b-a) - 2\right) \cdot \left(2^{-(m+1)}\right)^{\alpha+\beta} > 1. \qquad \square
\end{aligned}
$$

$$\left| S(f, dg, D_1, \xi_1) - S(f, dg, D_2, \xi_2) \right| = \text{[restr. to } I\text{]}$$

$$= \left| \sum_{i=0}^{N-1} f(T_i)\left(g(p_{i+1}) - g(p_i)\right) - \sum_{i=0}^{N-1} f(t_i)\left(g(p_{i+1}) - g(p_i)\right) \right|$$

$$= \left| \sum \left(f(T_i) - f(t_i) \right) \cdot \left(g(p_{i+1}) - g(p_i) \right) \right|$$

$$= \left| \sum_{i \in E} \left(f(p_{i+1}) - f(p_i) \right) \cdot \left(g(p_{i+1}) - g(p_i) \right) + \right.$$

$$\left. + \sum_{i \in F} \left(f(p_i) - f(p_{i+1}) \right) \cdot \left(g(p_{i+1}) - g(p_i) \right) \right|$$

$$= \sum_{i=0}^{N-1} \left| f(p_{i+1}) - f(p_i) \right| \cdot \left| g(p_{i+1}) - g(p_i) \right|$$

$$> \sum_{i=0}^{N-1} \left(2^{-(m+1)}\right)^{\alpha} \cdot \left(2^{-(m+1)}\right)^{\beta} = N \cdot \left(2^{-(m+1)}\right)^{\alpha+\beta}$$

$$\geq \left(2^m (b-a) - 2\right) \cdot \left(2^{-(m+1)}\right)^{\alpha+\beta} > 1. \qquad \square$$

<u>PROBLEM</u>: Similar proof for $\alpha + \beta = 1$?