Some reasons why we are interested in σ-porous sets

Martin Rmoutil

Department of Mathematics Education

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Example 0: Smallness by number of elements

Theorem (Cantor)

The set $\mathbb{R}$ is uncountable.

There are only countably many algebraic numbers.
Hence, there are uncountably many transcendental numbers.

Existence proof.

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Example 1: Baire category

Let (M, d) be a metric space.
A set $A \subseteq M$ is said to be nowhere dense if $\operatorname{Int} \overline{A} = \emptyset$.
A set $B \subseteq M$ is said to be meagre if it is the union of countably many nowhere dense sets. ($M \setminus B$ is then comeagre.)

Theorem (Baire)

Let (M, d) be complete. Then M is not meagre (in itself).

The space $(C[0,1], \|\cdot\|_\infty)$ is complete.

Theorem (Banach, Mazurkiewicz (independently), 1931)

The set $D \subseteq C[0,1]$ of all functions with a point of differentiability is meagre in $C[0,1]$.

Baire $\Rightarrow$ $C[0,1]$ is not meagre.
So $C[0,1] \setminus D \neq \emptyset$, i.e. there exist nowhere-differentiable functions. We say that the typical continuous function is n.-d.

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Example 1: Baire category & differentiability

Let X be Banach space. A function $f\colon X \to \mathbb{R}$ is said to be Fréchet differentiable at a point x_0 if there is $x^* \in X^*$ such that

$$f(x_0 + u) = f(x_0) + x^*(u) + o(\|u\|), u \to 0.$$

We also define the *directional derivative* of f at x_0 in direction u as

$$f'(x_0; u) = \lim_{t\to 0} \frac{f(x_0 + tu) - f(x_0)}{t}.$$

If $f(x_0; \cdot)$ is a bounded linear operator, f is Gâteaux diff. at x_0.

Theorem (Mazur, 1933)

X separable, $f : X \to \mathbb{R}$ cts. and convex $\Rightarrow N_G(f)$ meagre.

Theorem (Asplund, 1968)

X^* separable, $f : X \to \mathbb{R}$ cts. and convex $\Rightarrow N_F(f)$ meagre.

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Example 2: Lebesgue measure

Let $f: \mathbb{D}_f \subseteq \mathbb{R}^n \to \mathbb{R}$; set $N(f) := \{x \in \mathbb{D}_f : f'(x) \text{ does not exist}\}$.

Theorem (Rademacher)

Let $f: \mathbb{R}^n \to \mathbb{R}$ be Lipschitz. Then $|N(f)| = 0$.

This theorem is sharp for $n = 1$:

Theorem (Zahorski)

Given a $G_{\delta\sigma}$ set $A \subseteq \mathbb{R}$ with $|A| = 0$, there exists a Lipschitz function $f: \mathbb{R} \to \mathbb{R}$ such that $N(f) = A$.

For higher dimensions, Rademacher's theorem is not sharp:

Theorem (Preiss)

There exists $A \subseteq \mathbb{R}^2$ with $|A| = 0$ such that for any Lispchitz $f: \mathbb{R}^2 \to \mathbb{R}$ we have $D(f) \cap A \neq \varnothing$ (where $D(f) = \mathbb{R}^2 \setminus N(f)$).

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Example 2: Baire category & Rademacher

Rademacher's theorem states that *a Lipschitz function on* $\mathbb{R}^n$ *is differentiable up to a negligible set.* This negligibility is meant in the sense of Lebesgue measure.

Q: Is this true for another notion of negligibility? E.g. for meager?

A: NOT for meager.

Proof.

Find a dense G_δ set $G \subseteq \mathbb{R}$ with $|G| = 0$. Then $F := \mathbb{R} \setminus G$ is meagre and $\mathbb{R} = F \cup G$. Zahorski $\rightsquigarrow$ Lipschitz f with $N(f) = G$. Then f is non-differentiable on a comeagre set; so f is not differentiable up to a meagre set as $\mathbb{R}$ is not meagre by Baire. □

This can easily be generalized to any Banach space.

Hence, meagre sets are not suitable for the study of differentiability of Lipschitz functions.

Note that we used a decomposition result on $\mathbb{R}$, namely, $\mathbb{R}$ can be expressed as the union of two sets "negligible" in different senses.

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Example 3: Haar null sets

Let G be a locally compact Polish group. Then there exists a ("unique") Haar measure μ on G, i.e. a non-trivial (left-)translation invariant Radon (σ-additive) measure. Lebesgue measure is Haar.

The sets $A \subseteq G$ with $\mu(A) = 0$ are small in the sense of measure, (but not necessarily e.g. in the sense of Baire category).

But: In non-locally compact groups there is no Haar measure. (Easy to see.) Nonetheless, is it possible to define a corresponding notion of small sets? Yes!

Definition (Christensen: Haar null sets (HN))

Let G be an Abelian Polish group.

$A \subseteq G$ Borel is HN $\overset{\text{def.}}{\Longleftrightarrow}$ $\exists$ Borel prob. μ $\forall x \in G : \mu(x + A) = 0$.

(i) G loc. cpt. $\Rightarrow$ HN $\equiv$ Haar measure zero;

(ii) G non-loc. cpt. & $A \subseteq G$ cpt. $\Rightarrow$ A is HN;

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Example 3: HN & Rademacher

Theorem (Christensen, 1972)

X separable BS, $f : X \to \mathbb{R}$ Lipschitz $\Rightarrow N_G(f)$ is HN.

This was followed by Mankiewicz, Aronszajn and Phelps who proved the same result for different notions of smallness, namely *cube null*, *Aronszajn null* and *Gauss null*.

Remark

X separable BS; consider Lipschitz functions $f_n : X \to \mathbb{R}$. Then, by Christensen, $B = \bigcup_{n=1}^{\infty} N_G(f_n)$ is HN. Since X is not HN, there is a large set (namely $X \setminus B$) where all the functions are Gâteaux d.

Compare to the following (difficult) result:

Theorem (Preiss, 1990)

X^* separable, f Lipschitz $\Rightarrow f$ is Fréchet diff. on a dense set.

Example 3: HN & Rademacher

Theorem (Christensen, 1972)

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This was followed by Mankiewicz, Aronszajn and Phelps who proved the same result for different notions of smallness, namely *cube null*, *Aronszajn null* and *Gauss null*.

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σ-ideals

Common property of these notions:

Definition

Let X be a set and $\mathcal{S} \subseteq \mathcal{P}(X)$. We say $\mathcal{S}$ is a σ-ideal if:

(i) $A \in \mathcal{S}$ and $B \subseteq A \Rightarrow B \in \mathcal{S}$;

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$\mathcal{S}$ is nontrivial if $X \notin \mathcal{S}$.

All the aforementioned notions of smallness correspond to σ-ideals.

- Countable: well-known;
- Meagre: trivial;
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The letter σ corresponds to countable unions, sums, limits etc. Its importance is clear to anyone who can appreciate the difference between Riemann and Lebesgue integrals.

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Rademacher type theorem: *On a certain space X there exists a nontrivial σ-ideal $\mathcal{S} \subseteq \mathcal{P}(X)$ such that every Lipschitz $f : X \to \mathbb{R}$ is differentiable up to a set from $\mathcal{S}$.*

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Lebesgue: *Every Lipschitz $f : \mathbb{R} \to \mathbb{R}$ is diff. a.e.*

Hence, given a Lip. $f : X \to \mathbb{R}$, the set $B(u)$ of points where f is non-diff. in direction u is from $\mathcal{A}(u)$.

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A set $A \subseteq X$ is Aronszajn null if for every sequence $\{u_i\}_{i=1}^{\infty} \subseteq X$ with $\overline{\text{span}}\{u_i\} = X$ we have $A = \bigcup_{i=1}^{\infty} A_i$ where $A_i \in \mathcal{A}(u_i)$.

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Theorem (Lindenstrauss, Preiss, Tišer, 2008)

Any two Lipschitz functions on a separable Hilbert space have a common point of Fréchet differentiability.

More generally, this works on any spaces with sufficient (asymptotic) smoothness.

Open problem: Is this true for 3 functions?

The main Rademacher-type thm of Lindenstrauss and Preiss:

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Let X be a Banach space with separable dual. If σ-porous sets in X are Γ-null, then every Lipschitz $f : X \to \mathbb{R}$ satisfies $N_F(f) \in \Gamma$.

The spaces c_0, even $C(K)$ for K countable compactum, and the Tsirelson space are known to have the property. By separable reduction this was shown by Cúth even for $C(K)$ with K scattered.

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Any two Lipschitz functions on a separable Hilbert space have a common point of Fréchet differentiability.

More generally, this works on any spaces with sufficient (asymptotic) smoothness.
Open problem: Is this true for 3 functions?
The main Rademacher-type thm of Lindenstrauss and Preiss:

Theorem

Let X be a Banach space with separable dual. If σ-porous sets in X are Γ-null, then every Lipschitz $f : X \to \mathbb{R}$ satisfies $N_F(f) \in \Gamma$.

The spaces c_0, even $C(K)$ for K countable compactum, and the Tsirelson space are known to have the property. By separable reduction this was shown by Cúth even for $C(K)$ with K scattered.

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Definition

Let (M, d) be a metric space, $A \subseteq M$, $x \in M$. We say that A is porous at x, if there are $x_n \to x$ and $r_n \to 0$ with:

- $B(x_n, r_n) \cap A = \emptyset$;
- $\lim_{n\to\infty} \frac{r_n}{d(x_n,x)} > 0$.

Definition (equivalent)

$\gamma(x, R, A) = \sup\{r > 0 : \text{for some } z \in M,\ B(z, r) \subseteq B(x, R) \setminus A\},$

$$\overline{p}(A, x) = \limsup_{R\to 0_+} \frac{2 \cdot \gamma(x, R, A)}{R}, \qquad \underline{p}(A, x) = \liminf_{R\to 0_+} \frac{2 \cdot \gamma(x, R, A)}{R}.$$

A is porous at x if $\overline{p}(A, x) > 0$ and lower porous at x if $\underline{p}(A, x) > 0$.
A set B is σ-porous if $B = \bigcup_{n=1}^{\infty} A_n$ with A_n porous for all n.

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Facts.

(i) The definition makes sense in any metric space.

(ii) M uncountable and complete $\Rightarrow$ σ-porous sets form a nontriv. σ-ideal.

(iii) σ-porous sets are always meagre (trivial).

(iv) In $\mathbb{R}^n$, each σ-porous set is Lebesgue null (easy – density).

(v) (Zajíček:) There is a closed $F \subseteq \mathbb{R}^n$ which is both n.-d. ($\Rightarrow$ meagre) and Lebesgue null, but is not σ-porous.

(vi) (Foran:) Graph of a cts. function. (Zelený:) AC function.

Observation.

Let $A \subseteq \mathbb{R}$. Then

M is porous $\Leftrightarrow$ $x \mapsto \mathrm{dist}(x, A)$ is differentiable at no point of A.

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σ-porosity & Rademacher (for cts. convex)

Recall Asplund, 1968: X BS with X^* separable $\Rightarrow$ for any cts convex $f\colon X \to \mathbb{R}$, $N_F(f)$ is meagre.

Preiss, Zajíček (1980s): Same for σ-porous, even cone small sets.

These results can be generalized to non-separable Asplund spaces by a separable-reduction technique.

We first prove that the relevant notions, such as σ-porosity and Fréchet differentiability, are separably determined. For example:

Theorem (Marek Cúth, M.R.)

Let X be a Banach space, $A \subset X$ be a Souslin set. Then for every separable subspace $V_0 \subset X$ there exists a closed separable space $V \subset X$ such that $V_0 \subset V$ and

(i) *A is σ-upper porous $\iff$ $A \cap V$ is σ-upper porous in V;*

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A separable reduction theorem

Using more refined methods we were able to prove the separable determination of cone smallness, a smaller σ-ideal than that of σ-porous sets.

As a consequence we obtained the following result, which is a generalization of Zajíček's result to the non-separable setting.

Theorem (Marek Cúth, M.R., Miroslav Zelený)

Let X be an Asplund space and $G \subset X$ be open. Let $f : G \to \mathbb{R}$ be a continuous and approximately convex function. Then the set of all points of G at which f is not Fréchet differentiable is cone small.

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Thank you for your attention.