1. $2x - e^{\sin y^2} - e^{\sin xy} = -2$

na okolí bodu $(0,0) \in \mathbb{R}^2$:

$F(x,y) := e^{\sin y^2} + e^{\sin xy} - 2x - 2$

Předpoklady VIF: (i) $F \in C^\infty(\mathbb{R}^2)$, protože je poskládána z C^∞ - fcí.

(ii) $F(0,0) = e^{\sin 0} + e^{\sin 0} - 2\cdot 0 - 2 = 0$ ✓

(iii) $\frac{\partial F}{\partial y}(x,y) = e^{\sin y^2} \cdot \cos y^2 \cdot 2y + e^{\sin xy} \cdot \cos xy \cdot x$

$$\frac{\partial F}{\partial y}(0,0) = e^0 \cdot \cos 0 \cdot 0 + e^0 \cdot \cos 0 \cdot 0 = 0$$

Zkusíme vyjádřit x jako fci proměnné y:

$$\frac{\partial F}{\partial x}(x,y) = 0 + e^{\sin xy} \cdot \cos xy \cdot y - 2$$

$$\frac{\partial F}{\partial x}(0,0) = e^0 \cdot \cos 0 \cdot 0 - 2 = -2 \neq 0 \checkmark.$$

Tedy křivka je na nějakém okolí bodu $(0,0)$ grafem funkce $\varphi(y)$. Přesněji:

$$\left[\ \exists\, \delta, \Delta > 0 \quad \forall (x,y) \in (-\Delta, \Delta) \times (-\delta, \delta): \quad F(x,y) = 0 \iff x = \varphi(y).\ \right]$$

Navíc: VIF $\Rightarrow$ $\varphi \in C^\infty(-\delta, \delta)$.

Speciálně víme, že $F(0,0) = 0$, a tedy $\varphi(0) = 0$.

Spočtěme $\psi'(0)$, $\psi''(0)$:

Protože $\forall y \in (-\delta, \delta)$ je $F(\psi(y), y) = 0$, je

$$0 = \frac{d}{dy}\left(F(\psi(y), y)\right) = \frac{\partial F}{\partial x}(\psi(y), y)\cdot\psi'(y) + \frac{\partial F}{\partial y}(\psi(y), y),$$

a tedy $\psi'(y) = -\dfrac{\frac{\partial F}{\partial y}(\psi(y), y)}{\frac{\partial F}{\partial x}(\psi(y), y)}$, odkud

$$\psi'(0) = -\frac{\frac{\partial F}{\partial y}(0,0)}{\frac{\partial F}{\partial x}(0,0)} \overset{\text{viz výše}}{=} -\frac{0}{-2} = 0.$$

POZOR: Následující výpočet $\psi''(0)$ není nejjednodušší možný. Pomocí řetízkového pr. vlastně odvozuje obecný vzorec. Jednodušší je opakovaně derivovat zadanou rovnici.

Dále: $\psi''(y) = \dfrac{d}{dy}(\psi'(y)) = \dfrac{d}{dy}\left(-\dfrac{\frac{\partial F}{\partial y}(\psi(y), y)}{\frac{\partial F}{\partial x}(\psi(y), y)}\right)$

$$= -\frac{(F_{xy}\cdot\psi'(y) + F_{yy})F_x - F_y(F_{xx}\cdot\psi'(y) + F_{yx})}{(F_x)^2},$$

kde $F_{xy} := \dfrac{\partial^2 F}{\partial x \partial y}(\psi(y), y)$, $F_x := \dfrac{\partial F}{\partial x}(\psi(y), y)$ apod.

Víme: $F_y(0,0) = 0$, $F_x(0,0) = -2$, tedy dosazením $y = 0$ (a tedy také $\psi(0) = 0$, $\psi'(0) = 0$)

$$= -\frac{(F_{xy}\cdot 0 + F_{yy})\cdot(-2) - 0\cdot(\dots)}{(-2)^2} = \frac{2\cdot F_{yy}}{4}.$$

Zbývá tedy dopočítat $F_{yy} = \dfrac{\partial^2 F}{\partial y^2}(0,0)$:

$$\frac{\partial^2 F}{\partial y^2}(x,y) =$$

$$\frac{\partial}{\partial y}\left(e^{\sin y^2}\cdot\cos y^2\cdot 2y + e^{\sin xy}\cdot\cos xy\cdot x\right)$$

$$= e^{\sin y^2}\cdot\cos^2 y^2\cdot 4y^2 - e^{\sin y^2}\sin y^2\cdot 4y^2 + e^{\sin y^2}\cos y^2\cdot 2$$

$$+ x\cdot(\dots\dots),$$ a tedy

$$\frac{\partial^2 F}{\partial y^2}(0,0) = e^0\cos^2 0\cdot 4\cdot 0^2 - e^0\cdot\sin 0^2\cdot 4\cdot 0^2 + e^0\cdot\cos 0^2\cdot 2 + 0\cdot(\dots)$$

$$= 0 - 0 + 1\cdot 1\cdot 2 + 0 = 2.$$

Celkem: $\psi''(0) = \dfrac{2\cdot 2}{4} = 1.$

(2.) $z = \operatorname{arctg}\left(\frac{x}{y}\right)$, $\varphi(r,\theta) = (r\cos\theta, r\sin\theta)$

(a) $\tilde{z} = z\circ\varphi$, tj. $\tilde{z}(r,\theta) = \operatorname{arctg}\frac{r\cos\theta}{r\sin\theta} =$

$$= \operatorname{arctg}\frac{\cos\theta}{\sin\theta}.$$

$$\frac{\partial\tilde{z}}{\partial r} = 0$$

$$\frac{\partial\tilde{z}}{\partial\theta}(r,\theta) = \frac{1}{1+\frac{\cos^2\theta}{\sin^2\theta}}\cdot\frac{-\sin^2\theta-\cos^2\theta}{\sin^2\theta} =$$

$$= \frac{1}{\frac{1}{\sin^2\theta}}\cdot\frac{-1}{\sin^2\theta} = -1$$

(b) $(x_1,y_1) = \left(x(r_1,\theta_1), y(r_1,\theta_1)\right) =$

$$= \left(x\left(2,\frac{\pi}{6}\right), y\left(2,\frac{\pi}{6}\right)\right) = \left(2\cos\frac{\pi}{6}, 2\cdot\sin\frac{\pi}{6}\right) =$$

$$= \left(2\cdot\frac{\sqrt{3}}{2}, 2\cdot\frac{1}{2}\right) = (\sqrt{3}, 1)$$

(c) $\mathbb{D}_z = \{(x,y)\in\mathbb{R}^2 : y\neq 0\} = \mathbb{R}^2 \setminus [\text{„osa } x\text{"}]$

$\mathbb{D}_{\tilde{z}} = \{(r,\theta)\in[0,\infty)\times\underbrace{(-\pi,\pi]}_{\text{stačila stačila}} : \theta\neq 0, \theta\neq\pi\}$

Protože $\frac{\partial\tilde{z}}{\partial r}\equiv 0$, $\frac{\partial\tilde{z}}{\partial\theta}\equiv -1$, jsou obě PD spoj. na okolí bodu $\left(2,\frac{\pi}{6}\right)$, a tedy existuje TD $d\tilde{z}\left(2,\frac{\pi}{6}\right)$. Proto existuje i

$$\nabla\tilde{z}\left(2,\frac{\pi}{6}\right) = (0,-1).$$

(c) ∇z:

$$\frac{\partial z}{\partial x} = \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{-x}{y^2} = \frac{-x}{x^2+y^2}$$

Obě P.D. jsou spoj. na $\mathbb{R}^2 \setminus$ [„osa x“]

(Pozor: po úpravě se zdá, že dokonce jsou spoj. na $\mathbb{R}^2 \setminus \{(0,0)\}$, což ale není pravda!)

Tedy existuje TD, a tedy i gradient, v bodě $(x_1, y_1) = (\sqrt{3}, 1)$, přičemž

$$\nabla z(\sqrt{3}, 1) = \left(\frac{1}{4}, \frac{-\sqrt{3}}{4}\right).$$

(d) $u = (0,1)$, $v = \left(\frac{1}{2}, \frac{\sqrt{3}}{4}\right)$

$$\frac{\partial \tilde{z}}{\partial v}\left(2, \frac{\pi}{6}\right) = \left\langle \nabla \tilde{z}\left(2, \frac{\pi}{6}\right), v \right\rangle$$

$$= \left\langle (0,-1), \left(\frac{1}{2}, \frac{\sqrt{3}}{4}\right) \right\rangle = -\frac{\sqrt{3}}{4}$$

$$\frac{\partial z}{\partial u}(\sqrt{3}, 1) = \left\langle \nabla z(\sqrt{3}, 1), u \right\rangle$$

$$= \left\langle \left(\frac{1}{4}, -\frac{\sqrt{3}}{4}\right), (0,1) \right\rangle = -\frac{\sqrt{3}}{4}$$

3. $$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2+4}-2}{x^2+y^2} =$$

Použijeme vzorec $(a-b)\cdot(a+b) = a^2-b^2$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2+y^2+4 - 4}{(x^2+y^2)\cdot\left(\sqrt{x^2+y^2+4}+2\right)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{1}{\sqrt{x^2+y^2+4}+2} \overset{(*)}{=} \frac{1}{\sqrt{4}+2} = \frac{1}{4}.$$

(*): Funkce je spojitá na $\mathbb{R}^2$:

- polynom pod odm. je spoj. a nabývá hodnot z $[4,\infty)$
- $\sqrt{\dots}$ je tedy spoj. na $\mathbb{R}^2$
- v žádném bodě nedělíme 0.

4. $f(x,y) = 27y^2 + 15x^2 + 9xy^2 + 2x^3$

$$\frac{\partial f}{\partial x} = 30x + 9y^2 + 6x^2$$

$$\frac{\partial f}{\partial y} = 54y + 18xy$$

STACIONÁRNÍ BODY: $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$:

$$6x^2 + 30x + 9y^2 = 0 \qquad (1)$$

$$18y(x+3) = 0 \qquad (2)$$

$(2) \iff y = 0 \quad \vee \quad x = -3$

(a) $\underline{y=0}$: (1): $6x(x+5) = 0 \quad \begin{cases} x = 0 \\ x = -5 \end{cases}$

$\underline{\text{P.B.}}$: $[0,0]$, $[-5,0]$

(b) $\underline{x=-3}$: (1): $54 - 90 + 9y^2 = 0$

$9y^2 = 36$, $y^2 = 4$ $\quad y = \pm 2$

$\underline{\text{P.B.}}$: $[-3,-2]$, $[-3,2]$

$\frac{\partial^2 f}{\partial x^2} = 30 + 12x$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 18y$, $\frac{\partial^2 f}{\partial y^2} = 54 + 18x$

$\underline{[0,0]}$: $d^2f(0,0) = \begin{pmatrix} 30 & 0 \\ 0 & 54 \end{pmatrix}$ PD $\Rightarrow$ min.

$[-5,0]$: $d^2f(-5,0) = \begin{pmatrix} -30 & 0 \\ 0 & -36 \end{pmatrix}$ ND $\Rightarrow$ max.

$[-3,-2]$: $d^2f(-3,-2) = \begin{pmatrix} -6 & -36 \\ -36 & 0 \end{pmatrix}$ ... det $< 0 \Rightarrow$ sedlo

$[-3,2]$: $d^2f(-3,2) = \begin{pmatrix} -6 & 36 \\ 36 & 0 \end{pmatrix}$ ... det $< 0 \Rightarrow$ sedlo