1. $x^y + y^x - 2y = 0$ v bodě $(1,1)$.

Ověřím předpoklady VIF:

$$F(x,y) = x^y + y^x - 2y = e^{y\ln x} + e^{x\ln y} - 2y$$

(i) $F \in C^\infty(G)$, kde $G = \{(x,y)\in\mathbb{R}^2 : x, y \neq 0\}$

G je otevřená, $(1,1) \in G$.

(ii) Bod $(1,1)$ leží na křivce: skutečně,

$F(1,1) = 1^1 + 1^1 - 2 = 0$ ✓

(iii) $$\frac{\partial F}{\partial y}(x,y) = e^{y\ln x}\cdot \ln x + e^{x\ln y}\cdot \frac{x}{y} - 2 =$$
$$= \ln x \cdot x^y + \frac{x}{y} y^x - 2 =$$
$$= \ln x \cdot x^y + x\cdot y^{x-1} - 2$$

$$\frac{\partial F}{\partial y}(1,1) = 0\cdot 1 + 1\cdot 1^0 - 2 = -1 \neq 0$$ ✓

Podle VIF tedy existují otevřené $U, V \subseteq \mathbb{R}$, $1 \in U$, $1 \in V$ a funkce $\varphi: U \to V$, že $\forall (x,y) \in U\times V$: $F(x,y) = 0 \iff y = \varphi(x)$.

navíc $\varphi \in C^\infty(U)$. Speciálně $\varphi(1) = 1$, tj. bod $(1,1)$ leží na grafu φ.

$\varphi'(1) = ?$ Víme, že body tvaru $(x, \varphi(x))$ (tj. body grafu φ) splňují rovnici $F(x,\varphi(x)) = 0$,

tj. $x^{\varphi(x)} + (\varphi(x))^x - 2\varphi(x) = 0$, $x \in U$.

Rovnici zderivuji podle x (po úpravě):

$$0 = e^{\varphi(x)\ln x} + e^{x\ln\varphi(x)} - 2\varphi(x) \quad /(\ldots)'$$

$$0 = e^{\varphi(x)\ln x}\left(\varphi'(x)\ln x + \frac{\varphi(x)}{x}\right) + e^{x\ln\varphi(x)}\cdot\left(\ln\varphi(x) + x\,\frac{\varphi'(x)}{\varphi(x)}\right) - 2\varphi'(x).$$

Dosadím $[x = 1]$ (tedy $[\varphi(x) = \varphi(1) = 1]$):

$$0 = e^{1\cdot 0}\left(\varphi'(1)\cdot 0 + \frac{1}{1}\right) + e^{1\cdot 0}\cdot\left(0 + 1\cdot\frac{\varphi'(1)}{1}\right) - 2\varphi'(1)$$

$$= 1 + \varphi'(1) - 2\varphi'(1).$$ Tedy $\varphi'(1) = 1$.

(2.) $f(x,y) = \exp\left(\sin x + \arcsin y\right)$,

$a = \left(\pi, \frac{1}{2}\right)$, $u = (1,0)$, $v = (0,1)$, $w = (-2,5)$

(a) • $D_f = ?$ $\quad D_{\exp} = D_{\sin} = \mathbb{R}$. $D_{\arcsin} = [-1,1]$.

Proto $D_f = \mathbb{R} \times [1,1]$.

• PD: $\frac{\partial f}{\partial y}(x,y) = \exp\left(\sin x + \arcsin y\right) \cdot \frac{1}{\sqrt{1-y^2}}$,

$(x,y) \in \mathbb{R} \times (-1,1)$. (musíme vyloučit $y \in \{-1,1\}$, protože tam by limita z def $\frac{\partial f}{\partial y}$ byla pouze jednostranná.)

$\frac{\partial f}{\partial x}(x,y) = \exp(\dots) \cdot \cos x$, $\quad (x,y) \in \mathbb{R} \times [-1,1]$

(Tentokrát není důvod vylučovat $y = \pm 1$, protože derivujeme ve směru osy x.)

(b) $\nabla f(a)$ je def. $\Longleftarrow$ existuje totální dif. $df(a)$. Pro to stačí spojitost $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ v bodě a. Ta je ovšem zřejmá, protože obě PD jsou složeny ze spojitých funkcí. Proto existuje

$$\nabla f(a) = \nabla f\left(\pi, \tfrac{1}{2}\right) = \left(\frac{\partial f}{\partial x}\left(\pi, \tfrac{1}{2}\right), \frac{\partial f}{\partial y}\left(\pi, \tfrac{1}{2}\right)\right) =$$

$$= \left(e^{0+\frac{\pi}{6}} \cdot (-1),\ e^{\frac{\pi}{6}} \cdot \frac{1}{\sqrt{1-\frac{1}{4}}}\right) =$$

$$= \left(-e^{\frac{\pi}{6}},\ \frac{2}{\sqrt{3}} e^{\frac{\pi}{6}}\right)$$

(c) $\frac{\partial f}{\partial u}(a) = \left\langle \nabla f(a), u \right\rangle = -e^{\frac{\pi}{6}}$

$\frac{\partial f}{\partial v}(a) = \left\langle \nabla f(a), v \right\rangle = \frac{2}{\sqrt{3}} e^{\frac{\pi}{6}}$

$\frac{\partial f}{\partial w}(a) = \left\langle \nabla f(a), w \right\rangle = e^{\frac{\pi}{6}}\left(\frac{10}{\sqrt{3}} + 2\right)$

(d) $z := \nabla f(a)$.

$$\frac{\partial f}{\partial z}(a) = \left\langle \nabla f(a), z \right\rangle = \underbrace{\left\langle \nabla f(a), \nabla f(a) \right\rangle}_{\|\nabla f(a)\|^2} =$$

$$= e^{\frac{\pi}{3}}\left(1 + \frac{4}{3}\right)$$

Derivace ve směru vektoru z:

$$\hookrightarrow = \frac{\frac{\partial f}{\partial z}(a)}{\|z\|} = \frac{\|\nabla f(a)\|^2}{\|\nabla f(a)\|} = \|\nabla f(a)\| = \sqrt{e^{\frac{\pi}{3}}\left(1 + \frac{4}{3}\right)}$$

$$= \sqrt{\frac{7}{3}} \cdot e^{\frac{\pi}{6}}.$$

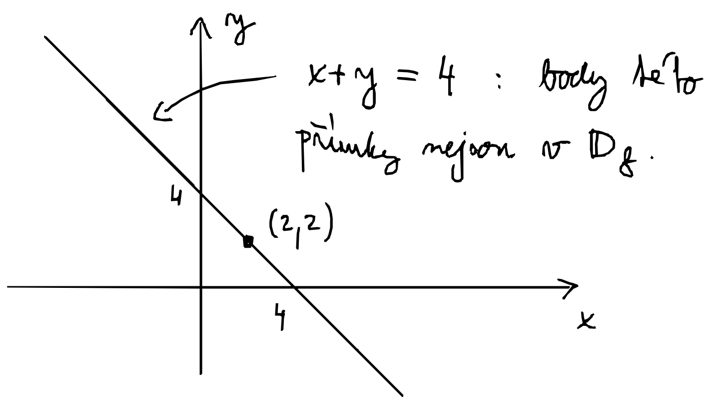
3. $$\lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{xy-4y+y^2}{\sqrt{x+y}-2} =$$

$$= \lim \frac{y(x+y-4)\cdot(\sqrt{x+y}+2)}{(\sqrt{x+y}-2)(\sqrt{x+y}+2)} =$$

$$= \lim \frac{x+y-4}{x+y-4}\cdot \lim y(\sqrt{x+y}+2) =$$

$$= 1\cdot 2\cdot(\sqrt{2+2}+2) = \underline{\underline{8}}$$

Na množinu $\{(x,y)\in\mathbb{R}^2 : x+y\neq 4\}$ se omezujeme, protože funkce není def. na žádném prstencovém okolí bodu $(2,2)$, tudíž limita vzhledem k celému $\mathbb{R}^2$ není definována.



4. $f(x,y) = x^2 + y^2 - 4x + 4y + 8$

$M = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$

$f_x(x,y) = 2x - 4$, $\quad f_y(x,y) = 2y + 4$

1) Vnitřek M: Hledáme stac. body f:

$2x - 4 = 0 \qquad x = 2$

$2y + 4 = 0 \qquad y = -2$

ovšem $[2,-2] \notin M$.

2) Hranice M: položme $g(x,y) = x^2 + y^2$

$g_x(x,y) = 2x$, $\quad g_y(x,y) = 2y$

a) P.B., kde $\nabla g = 0$: $\quad [0,0] \notin H(M)$. $\Rightarrow$ není PB

b) $\nabla f(x,y) = \lambda \nabla g(x,y)$ dá soustavu

$$\begin{bmatrix} 2x - 4 = \lambda 2x & (1) \\ 2y + 4 = \lambda 2y & (2) \\ x^2 + y^2 = 4 & (3) \end{bmatrix} \Rightarrow 1 - \lambda \neq 0$$

$x(1-\lambda) = 2 \qquad x = \frac{2}{1-\lambda} \qquad (1)$

$y(1-\lambda) = -2 \qquad y = -\frac{2}{1-\lambda} \qquad (2)$

(3): $\quad \frac{4}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} = 4$

$(1-\lambda)^2 = 2 \qquad 1 - \lambda = \pm\sqrt{2}$

P.B.: $[\sqrt{2}, -\sqrt{2}]$, $[-\sqrt{2}, \sqrt{2}]$

$f(x,y) = x^2 + y^2 - 4x + 4y + 8$

$f(\sqrt{2}, -\sqrt{2}) = 2 + 2 - 8\sqrt{2} + 8$ ... minimum f na M

$f(-\sqrt{2}, \sqrt{2}) = 2 + 2 + 8\sqrt{2} + 8$ ... maximum f na M