3) $y''' + y' - 10y = 0$

$$\lambda^3 + \lambda - 10 = 0$$

$$(\lambda^3 + \lambda - 10) : (\lambda - 2) = \lambda^2 + 2\lambda + 5$$

$$-(\lambda^3 - 2\lambda^2)$$

$$2\lambda^2 + \lambda - 10$$

$$-(2\lambda^2 - 4\lambda)$$

$$5\lambda - 10$$

$$\lambda_1 = 2$$

$$\lambda_{2,3} = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

$$F.S. = \{e^{2t}, e^{-t} \cdot \cos 2t, e^{-t} \cdot \sin 2t\}$$

$$y(x) = Ae^{2t} + Be^{-t}\cos 2t + Ce^{-t}\sin 2t$$

$$y'(x) = 2Ae^{2t} + B(-e^{-t}\cos 2t - 2e^{-t}\sin 2t) + C(-e^{-t}\sin 2t + 2e^{-t}\cos 2t) =$$

$$= 2Ae^{2t} + e^{-t}\cos 2t \cdot (2C - B) + e^{-t}\sin 2t(-2B - C)$$

$$y''(x) = 4Ae^{2t} + (2C - B)(-e^{-t}\cos 2t - 2e^{-t}\sin 2t) - (2B + C)(-e^{-t}\sin 2t + 2e^{-t}\cos 2t)$$

$$= 4Ae^{2t} + e^{-t}\cos 2t(B - 2C - 4B - 2C) + e^{-t}\sin 2t(2B - 4C + 2B + C)$$

$$1 = y(0) = A + B$$

$$1 = y'(0) = 2A + 2C - B$$

$$0 = y''(0) = 4A - 3B - 4C$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & -1 & 2 & 1 \\ 4 & -3 & -4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -3 & 2 & -1 \\ 0 & -7 & -4 & -4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 7 & 4 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & \frac{26}{3} & \frac{5}{3} \end{array}\right)$$

$$\boxed{C = \frac{5}{26}}$$

$$B = \frac{1}{3} + \frac{2}{3} \cdot \frac{5}{26} = \frac{1}{3}\left(1 + \frac{5}{13}\right) = \frac{18}{39}$$

$$\boxed{B = \frac{6}{13}} \quad \boxed{A = 1 - \frac{6}{13} = \frac{7}{13}}$$

Řešení splňující P.P.:

$$\boxed{y(x) = \frac{7}{13}e^{2t} + \frac{6}{13}e^{-t}\cos 2t + \frac{5}{26}e^{-t}\sin 2t}$$

(1.) $y' = x^2 e^x \cdot y$, $y(2) = 1$

STACIONÁRNÍ ŘEŠENÍ $y \equiv 0$. nyní $[y \neq 0]$:

$$\frac{y'}{y} = x^2 e^x \quad \dots$$

$$\ln|y| = e^x(x^2-2x+2) + C$$

$$\left[\int x^2 e^x \,dx = x^2 e^x - \int 2x e^x \,dx = \right.$$
$$= x^2 e^x - \left(2x e^x - \int 2 e^x \,dx\right) =$$
$$\left. = x^2 e^x - 2x e^x + 2 e^x + C \right]$$

(přičleníme i S.Ř. $y \equiv 0$)

$$|y| = e^C \cdot \exp\left(e^x(x^2-2x+2)\right)$$

$$y(x) = K \cdot \exp\left(e^x(x^2-2x+2)\right) , \quad K \in \mathbb{R}$$

$\boxed{\text{P.P.}}$ $1 = y(2) = K \cdot \exp\left(e^2 \cdot (2^2 - 2\cdot 2 + 2)\right) =$

$$= K \cdot \exp(e^2 \cdot 2) \implies K = \exp(-2e^2).$$

Tedy $y(x) = \exp(-2e^2) \cdot \exp\left(e^x(x^2-2x+2)\right)$

$\underline{x \in \mathbb{R}}$ $= \exp\left(e^x(x^2-2x+2) - 2e^2\right).$

(2.) $y' = (1-\cos y)\sqrt[5]{y^3-y^2+y-1} =: g(y)$

$y^3-y^2+y-1 = (y-1)(y^2+1)$

$(1-\cos y) = 0 \iff \cos y = 1 \iff y = 2k\pi,\ k\in\mathbb{Z}$

STAC. ŘEŠENÍ:
- $y \equiv 1$
- $y \equiv 2k\pi,\quad k\in\mathbb{Z}.$

Lepení na $2k\pi$: $\int_{2k\pi}^{2k\pi+\frac{1}{2}} \frac{1}{g}$ K. ?

Srovnáme s funkcí $h(y) = (y-2k\pi)^{-2}$:

LSK: $\lim\limits_{y\to 2k\pi} \frac{\frac{1}{g(y)}}{(y-2k\pi)^{-2}} = \lim\limits_{y\to 2k\pi} \frac{(y-2k\pi)^2}{(1-\cos y)\sqrt[5]{\cdots}} =$

$= \underbrace{\left((2k\pi)^3-(2k\pi)^2+2k\pi-1\right)^{-\frac{1}{5}}}_{C\in\mathbb{R}\setminus\{0\}} \cdot \lim\limits_{y\to 2k\pi}\left(\frac{1-\cos y}{(y-2k\pi)^2}\right)^{-1} =$

$= C\cdot \lim\limits_{y\to 0}\left(\frac{1-\cos(y+2k\pi)}{y^2}\right)^{-1} \overset{\text{periodicita}}{=} C\cdot\lim\limits_{y\to 0}\left(\frac{1-\cos y}{y^2}\right)^{-1}$

$\overset{\text{VOAL}}{=} C\cdot\left(\lim\limits_{y\to 0}\frac{1-\cos y}{y^2}\right)^{-1} = C\cdot\left(\frac{1}{2}\right)^{-1} = 2C\in\mathbb{R}\setminus\{0\}.$

Tedy $\left[\int_{2k\pi}^{2k\pi+\frac{1}{2}} \frac{1}{g} \text{ K.} \iff \int_{2k\pi}^{2k\pi+\frac{1}{2}} h \text{ K.}\right].$

Ale $\int_{2k\pi}^{2k\pi+\frac{1}{2}} h = \int_{2k\pi}^{2k\pi+\frac{1}{2}} \frac{1}{(y-2k\pi)^2}\,dy = \int_0^{\frac{1}{2}} \frac{1}{z^2}\,dz = \infty.$

SUBSTITUCE: $z = y-2k\pi$, $dz = dy$

y	$2k\pi$	$2k\pi+\frac{1}{2}$
z	0	$\frac{1}{2}$

Tedy $\int_{\cdots}^{\cdots}\frac{1}{g}$ D., a tedy nelze lepit na S.Ř. shora (analogicky: ani zdola).

Lepení na $y\equiv 1$ shora: $\int_1^2 \frac{1}{g}$ K. ?

Srovnáme s funkcí $h(y) = (y-1)^{-\frac{1}{5}}$

LSK: $\lim\limits_{y\to 1}\frac{\frac{1}{g(y)}}{h(y)} = \lim\limits_{y\to 1}\frac{\sqrt[5]{y-1}}{\sqrt[5]{y-1}\cdot\sqrt[5]{y^2+1}\cdot(1-\cos y)} =$

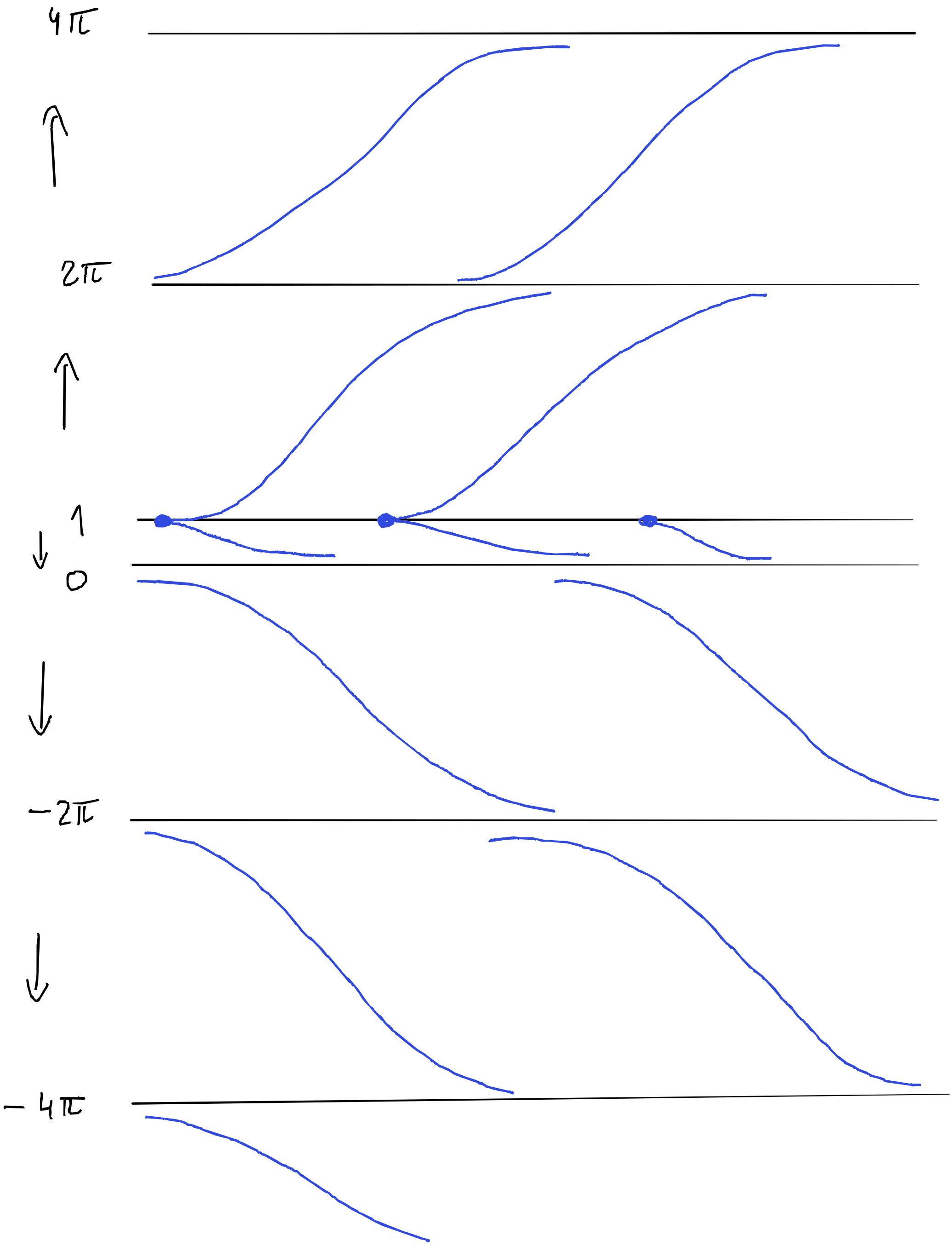
$= \frac{1}{\sqrt[5]{2}}\cdot\frac{1}{1-\cos 1}\in(0,\infty).$ Tedy opět:

$\Rightarrow \left[\int_1^2 \frac{1}{g} \text{ K.} \iff \int_1^2 h \text{ K.}\right]$, ovšem

$\int_1^2 h = \int_1^2 (y-1)^{-\frac{1}{5}}\,dy \overset{\text{SUBST.}}{=} \int_0^1 y^{-\frac{1}{5}}\,dy \ \ldots$ K.

(protože $= \left[\frac{5}{4}y^{\frac{4}{5}}\right]_0^1 = \frac{5}{4}1^{\frac{4}{5}} - \frac{5}{4}0^{\frac{4}{5}} = \frac{5}{4}$).

Tedy $\int_1^2\frac{1}{g}$ K., a lze lepit na $y\equiv 1$ shora (a analogicky i zdola).



<u>Poznámka</u>: nebylo nutné řešit lepení na $2k\pi$ v plné obecnosti. Stačí napsat úplné řešení lepení na $y \equiv 0$ a pak konstatovat, že díky periodicitě cos je situace pro další S.Ř. tvaru $y \equiv 2k\pi$ $(k \in \mathbb{Z})$ analogická.