Stochastic programming problems with endogenous randomness

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Standard stochastic programming model

- $\mathcal{X}_0$ is a closed set of hard constraints
- $\boldsymbol{\xi} : (\Omega, \mathcal{A}) \to (\mathbb{R}^d, \mathcal{B}^d)$ is an exogenous real random element
- uncertain formulation:

$$\text{"}\min_{x \in \mathcal{X}_0} \; f(x, \boldsymbol{\xi})\text{"}$$

$$s.t.$$

$$\text{"}g_j(x, \boldsymbol{\xi}) \leq 0\text{"}, \;\; j = 1, ..., p,$$

$$\text{"}h_k(x, \boldsymbol{\xi}) = 0\text{"}, \;\; k = 1, ..., q$$

- needs deterministic reformulations - expected value criterion, probability constraints, robust formulations etc.

Incorporating endogenous randomness

- for each feasible solution $x \in \mathcal{X}_0$ there is a random element $\boldsymbol{\xi}(x) : (\Omega, \mathcal{A}) \to (\mathbb{R}^d, \mathcal{B}^d)$ with a distribution $\mathcal{P}(x)$
- uncertain formulation:

$$"\min_{x \in \mathcal{X}_0} \; f(x, \boldsymbol{\xi}(x))"$$

$$s.t.$$

$$"g_j(x, \boldsymbol{\xi}(x)) \leq 0", \; j = 1, ..., p,$$

$$"h_k(x, \boldsymbol{\xi}(x)) = 0", \; k = 1, ..., q$$

- needs deterministic reformulation

Reformulation

- **Expected value criterion**:

$$\begin{aligned} & \min_{x \in \mathcal{X}_0} \mathbb{E}_{\mathcal{P}(x)}[f(x, \boldsymbol{\xi}(x))] \\ = & \min_{x \in \mathcal{X}_0} \int_{\mathbb{R}^d} f(x, \xi) \, d\mathcal{P}(x)(\xi) \end{aligned}$$

- robust worst-case approach, probability constraints
- specific shape of decision dependence is of high importance for reformulation

Common reference measure

- distributions $\mathcal{P}(x)$ for all $x \in \mathcal{X}_0$ have densities $\delta(x, \cdot)$ w.r.t. a common measure $\mathcal{G}$
- expected value criterion:

$$\begin{aligned} &\min_{x \in \mathcal{X}_0} \int_{\mathbb{R}^d} f(x, \xi) \cdot \delta(x, \xi) \, d\mathcal{G}(\xi) \\ = &\min_{x \in \mathcal{X}_0} \int_{\mathbb{R}^d} \tilde{f}(x, \xi) \, d\mathcal{G}(\xi) \end{aligned}$$

- the function $\tilde{f}$ need not preserve convenient properties of f
- search for special cases

Finitely many distributions

- $\boldsymbol{\xi_1}, \dots, \boldsymbol{\xi_m} : (\Omega, \mathcal{A}) \to (\mathbb{R}^d, \mathbb{B}^d)$ with distributions $\mathcal{P}_1, \dots, \mathcal{P}_m$
- disjoint partition $\mathcal{X}_0 = \cup_{i=1}^m \mathcal{X}_i$, such that $\forall i = 1, \dots, m$ and $\forall x \in \mathcal{X}_i : \boldsymbol{\xi}(x) \overset{\text{a.s.}}{=} \boldsymbol{\xi_i} \Longrightarrow$ separate into m standard subproblems
- assume $\mathbb{E}_{\mathcal{P}_i}[f(x, \boldsymbol{\xi_i})]$ to be continuous on $clo(\mathcal{X}_i)$ $\forall i$
- solve $\min_{i=1,\dots,m} \min_{x \in clo(\mathcal{X}_i)} \mathbb{E}_{\mathcal{P}_i}[f(x, \boldsymbol{\xi_i})]$
- random feasibility set $\Longrightarrow$ probability constraints:

$$\min_{i=1,\dots,m} \min_{x \in clo(\mathcal{X}_i)} \mathbb{E}_{\mathcal{P}_i}[f(x, \boldsymbol{\xi_i})]$$

$$s.t.$$

$$\mathbb{P}(g_j(x, \boldsymbol{\xi_i}) \le 0) \ge 1 - \epsilon_j^g, \; j = 1, \dots, p,$$

$$\mathbb{P}(h_k(x, \boldsymbol{\xi_i}) = 0) \ge 1 - \epsilon_k^h, \; k = 1, \dots, q$$

Fixed parametric family

- assume some parametric family of distributions $\{\mathcal{F}(\boldsymbol{\theta}) \mid \boldsymbol{\theta} \in \boldsymbol{\Theta}\}$ where $\boldsymbol{\theta} \in \mathbb{R}^p$
- parametric mapping $\theta : \mathcal{X}_0 \to \boldsymbol{\Theta} \implies$ the distribution of $\boldsymbol{\xi}(x)$ is

$$\mathcal{P}(x) = \mathcal{F}(\theta(x))$$

  with a density $\delta(\theta(x), \cdot)$ w.r.t. $\mathcal{G}$
- solve

$$\int_{\mathbb{R}^d} f(x, \xi) \cdot \delta(\theta(x), \xi)\, d\mathcal{G}(\xi)$$

- example for exponential distribution:

$$\min_{x \in \mathcal{X}_0} \int_0^\infty f(x, \xi) \cdot \lambda(x) \cdot e^{-\lambda(x) \cdot \xi}\, d\xi.$$

Example - separated random element

- say we have a probability constraint of the form

$$\mathbb{P}(g(x) \geq \boldsymbol{\xi}(x)) \geq 1 - \epsilon$$

- assume for each $x \in \mathcal{X}_0$ that $\boldsymbol{\xi}(x) \sim N(\mu(x), \sigma^2(x))$
- the constraint can be rewritten as

$$\mathbb{P}\left(\frac{g(x) - \mu(x)}{\sigma(x)} \geq \frac{\boldsymbol{\xi}(x) - \mu(x)}{\sigma(x)}\right) \geq 1 - \epsilon$$

- and therefore

$$g(x) \geq \mu(x) + q(1 - \epsilon) \cdot \sigma(x)$$

Finitely many scenarios

- **Decision dependent probabilities**: fixed set of scenarios $\{\xi^1, ..., \xi^S\}$, decisions only influence probabilities

$$\min_{x \in \mathcal{X}_0} \sum_{s=1}^{S} p^s(x) \cdot f(x, \xi^s)$$

- **Decision dependent scenarios**: fixed probability structure, decisions shape scenarios

$$\min_{x \in \mathcal{X}_0} \sum_{s=1}^{S} p^s \cdot f(x, \xi^s(x))$$

- arguably the most important case in practice

Example - big M reformulation

- say we have a probability constraint of the form

$$\mathbb{P}(g(x, \boldsymbol{\xi}(x)) \leq 0) \geq 1 - \epsilon$$

- we can reformulate using $S + 1$ constraints of the shape

$$g(x, \xi^s) \leq M \cdot (1 - y^s), \; s = 1, ..., S,$$
$$\sum_{s=1}^{S} p^s(x) \cdot y^s \geq 1 - \epsilon$$

Effect on dependence structure

- use of copulas to separate the dependence structure and marginal distributions of the multivariate random element $\xi(x)$
- **Fixed copula**: decisions influence marginal distributions, dependence remains unchanged
- **Fixed marginals**: decisions influence dependence, marginals remain unchanged
- utilizing the Sklar theorem, we can rewrite the expected value criterion

Set-valued dependence

- omit the assumption that the distribution is assigned uniquely to $x \in \mathcal{X}_0$
- can happen when the decisions only partially affect the underlying random element
- proposed general form (fixed feasibility set):

$$\min_{x \in \mathcal{X}_0} \int_{\mathbb{R}^d} f(x,\xi)\, d\mathcal{P}(\xi)$$
$$s.t.\ (\mathcal{P}, x) \in \mathcal{K}$$

- now assume that there is a whole set of possible distributions $\mathcal{U}(x)$ for each feasible x
- search for tractable cases

Fixed and finite cardinality of $\mathcal{U}(x)$

- for each $x \in \mathcal{X}_0$ we have $\mathcal{U}(x) = \{\mathcal{P}_1(x), \ldots, \mathcal{P}_k(x)\}$
- **Robust approach**:

$$\min_{x \in \mathcal{X}_0} \max_{\mathcal{P} \in \mathcal{U}(x)} \mathbb{E}_{\mathcal{P}}[f(x, \boldsymbol{\xi}(x))] = \min_{x \in \mathcal{X}_0} \max_{\mathcal{P} \in \mathcal{U}(x)} \int_{\mathbb{R}^d} f(x, \xi) d\mathcal{P}(\xi)$$

- **Aggregation approach**: create one candidate distribution $\mathcal{P}(x)$ as a mixture and solve the classical way
  - no additional information $\Longrightarrow$ use $\mathcal{P}(x) = \frac{1}{k} \sum_{i=1}^{k} \mathcal{P}_i(x)$
  - an external influence with finitely many scenarios and known (or estimated) probabilities $p^1, \ldots, p^k \Longrightarrow$ use $\mathcal{P}(x) = \sum_{i=1}^{k} p^i \cdot \mathcal{P}_i(x)$

Ambiguity sets

- decision maker wants to account for uncertainty in estimated quantities about the underlying distribution
- widely used in practical applications for finding robust solutions
- for each feasible $x \in \mathcal{X}_0$ there is a reference distribution $\mathcal{P}(x)$ and the ambiguity set $\mathcal{U}(x)$ is a sort of "neighborhood"
- solve using distributionally robust optimization

$$\min_{x \in \mathcal{X}_0} \max_{\mathcal{P} \in \mathcal{U}(x)} \int_{\mathbb{R}^d} f(x, \xi) \, d\mathcal{P}(\xi)$$

Example - Ambiguity sets

Ambiguity sets induced by the Wasserstein metric $\mathcal{W}$:

$$\mathcal{U}^{\mathcal{W}}(x) = \{\mathcal{P} \in \mathcal{P}(\mathbb{R}^d, \mathcal{B}^d) \mid \mathcal{W}(\mathcal{P}, \mathcal{P}(x)) \leq \epsilon(x)\}$$

Ambiguity sets induced by the ϕ-divergence metric $\mathcal{D}_\phi$:

$$\mathcal{U}^{\phi}(x) = \{\mathcal{P} \in \mathcal{P}(\mathbb{R}^d, \mathcal{B}^d) \mid \mathcal{D}_\phi(\mathcal{P}||\mathcal{P}(x)) \leq \epsilon(x)\}$$

Scenario-based ambiguity sets:

$$\mathcal{U}^{S}(x) = \{\mathcal{P} \text{ with prob. } \{p^s\}_{s=1}^{S} \mid |p^s - p^s(x)| \leq \epsilon(x) \forall s\}$$

Contamination

- for each decision $x \in \mathcal{X}$ we have a reference distribution $\mathcal{P}(x)$ and a stressing distribution $\tilde{\mathcal{P}}(x)$
- the set of distributions is taken as the line segment

$$\mathcal{U}(x) = \{(1 - \lambda) \cdot \mathcal{P}(x) + \lambda \cdot \tilde{\mathcal{P}}(x) \mid \lambda \in [0, 1]\}$$

- measuring the sensitivity of the optimal solution and optimal value to distributional changes in a particular direction

Multistage optimization

- assuming a fixed stage structure the causal path of the decision process is

$$\text{decide } x_1 \rightsquigarrow \text{observe } \xi_1(x_1) \rightsquigarrow \text{decide } x_2 \rightsquigarrow \dots \\ \rightsquigarrow \text{observe } \xi_{T-1}(x_{[T-1]}) \rightsquigarrow \text{decide } x_T$$

- can generally be highly intractable due to changes in tree structure

Single-stage newsvendor problem

- unit price $p \in [a, b]$, unit cost $c < p$, order quantity $x \geq 0$, random demand $\boldsymbol{\xi}(p)$
- the decision dependent distribution is chosen as

$$\mathcal{P}(p) = \mathcal{U}\left[D(p) - \frac{\sigma(p)}{2}, D(p) + \frac{\sigma(p)}{2}\right]$$

Formulation

- solve the problem

$$\max_{p\in[a,b],\ x\in[D(p)-\frac{\sigma(p)}{2},D(p)+\frac{\sigma(p)}{2}]} \ p\cdot\mathbb{E}[\min(x,\xi(p))]-c\cdot x$$

- or

$$\max_{p\in[a,b],\ x\in\left[D(p)-\frac{\sigma(p)}{2},D(p)+\frac{\sigma(p)}{2}\right]} \ \frac{p}{\sigma(p)}\cdot\int_{D(p)-\frac{\sigma(p)}{2}}^{D(p)+\frac{\sigma(p)}{2}}\min(x,z)\,dz-c\cdot x,$$

- arrive at the final form

$$\max_{p\in[a,b],\ x\in\left[D(p)-\frac{\sigma(p)}{2},D(p)+\frac{\sigma(p)}{2}\right]} \alpha(p)\cdot x^2+\beta(p)\cdot x+\gamma(p),$$

where

$$\alpha(p) = -\frac{p}{2\cdot\sigma(p)},$$

$$\beta(p) = \frac{p}{\sigma(p)}\cdot(D(p)+\frac{\sigma(p)}{2})-c,$$

$$\gamma(p) = \frac{p}{\sigma(p)}\cdot\Big(-\frac{1}{2}\cdot D(p)^2+\frac{1}{2}\cdot\sigma(p)\cdot D(p)-\frac{\sigma(p)^2}{8}\Big)$$

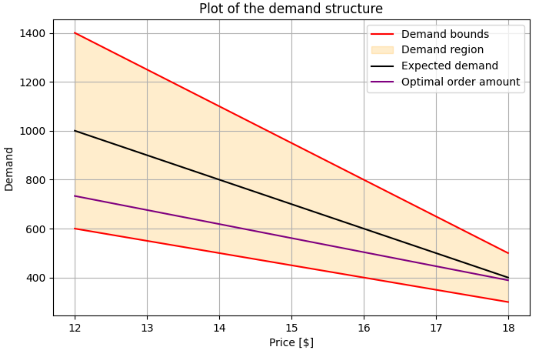
- the quadratic function can be solved to obtain

$$x = D(p)+(\frac{1}{2}-\frac{c}{p})\cdot\sigma(p)$$

Numerical example

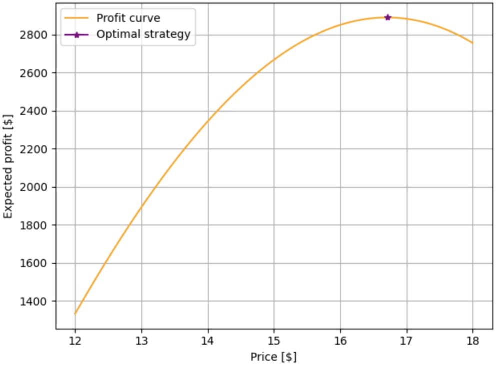
- set the parameters to

$$D(p) = 1000 - 100 \cdot (p - 12),$$
$$\sigma(p) = 800 - 100 \cdot (p - 12),$$
$$c = 10, \;\; p \in [12, 18]$$



Results

Optimal Expected Profit	2889.03 $
Optimal Order Quantity	496.68
Optimal Unit Selling Price	16.71 $



Three-stage newsvendor model

- adds an option for a recourse decision based on the first observed demand demand
- this time set a uniform distribution on a finite number of scenarios

$$\mathcal{P}(p) = \mathcal{U}\left\{ D(p) - \frac{\sigma(p)}{2} + i \cdot \frac{\sigma(p)}{S-1} \;:\; i \in \{0, \dots, S-1\} \right\}$$

1. stage variables	Constraints	Meaning
p_1	$\in [a, b]$	Selling price for first period
x_1	≥ 0	Order amount for first period
2. stage variables after s_1	**Constraints**	**Meaning**
$\mathrm{h}_1^{s_1}$	$\in \left[0, \min\left(x_1, \xi_1^{s_1}(p_1)\right)\right]$	Amount sold in the first period
s^{s_1}	$= \mathrm{x}_1 - h_1^{s_1}$	Amount kept after the first period
$\mathrm{p}_2^{s_1}$	$\in [a, b]$	Selling price for second period
$\mathrm{x}_2^{s_1}$	≥ 0	Order amount for second period
3. stage variable after s_1, s_2	**Constraints**	**Meaning**
$\mathrm{h}_2^{s_1, s_2}$	$\in \left[0, \min\left(x_2^{s_1} + s^{s_1}, \xi_2^{s_2}(p_2^{s_1})\right)\right]$	Amount sold in the second period

- formulate the full model as

$$\max_{p,x,s,h\geq 0} -c\cdot x_1 + \sum_{s_1=1}^{S} \pi^{s_1}\cdot\left[p_1\cdot h_1^{s_1} - c\cdot x_2^{s_1} + \sum_{s_2=1}^{S}\pi^{s_1,s_2}\cdot p_2^{s_1}\cdot h_2^{s_1,s_2}\right]$$

s.t.

$$\begin{aligned}
p_1 &\in [a,b]\\
p_2^{s_1} &\in [a,b],\ \forall s_1\in\{1,...,S\}\\
s^{s_1} &= x_1 - h_1^{s_1},\ \forall s_1\in\{1,...,S\}\\
h_1^{s_1} &\leq x_1,\ \forall s_1\in\{1,...,S\}\\
h_1^{s_1} &\leq \xi_1^{s_1}(p_1),\ \forall s_1\in\{1,...,S\}\\
h_2^{s_1,s_2} &\leq s^{s_1} + x_2^{s_1},\ \forall s_1,s_2\in\{1,...,S\}\\
h_2^{s_1,s_2} &\leq \xi_2^{s_2}(p_2^{s_1})\ \forall s_1,s_2\in\{1,...,S\}.
\end{aligned}$$

- add an advertisement option $y\in\{0,1\}$ with a cost $c_A>0$, effect factor $(1+\rho)$ and a decay parameter $\beta\in[0,1]$

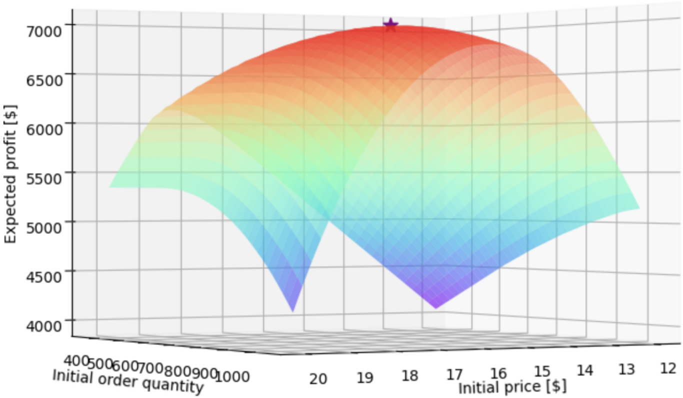
Numerical example

- set the parameters to

$$D(p) = 1000 - 100 \cdot (p - 12),$$
$$\sigma(p) = 800 - 100 \cdot (p - 12),$$
$$c = 10,\ \ p_1, p_2 \in [12, 18],\ \ S = 9,$$

$$\Pi = \begin{pmatrix}
\frac{7}{10} & \frac{2}{10} & \frac{1}{10} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3}{10} & \frac{4}{10} & \frac{2}{10} & \frac{1}{10} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{10} & \frac{2}{10} & \frac{4}{10} & \frac{2}{10} & \frac{1}{10} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{10} & \frac{2}{10} & \frac{4}{10} & \frac{2}{10} & \frac{1}{10} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{10} & \frac{2}{10} & \frac{4}{10} & \frac{2}{10} & \frac{1}{10} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{10} & \frac{2}{10} & \frac{4}{10} & \frac{2}{10} & \frac{1}{10} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{10} & \frac{2}{10} & \frac{4}{10} & \frac{2}{10} & \frac{1}{10} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{10} & \frac{2}{10} & \frac{4}{10} & \frac{3}{10} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10} & \frac{2}{10} & \frac{7}{10}
\end{pmatrix}$$

Results



Expected profit [$]
7000
6500
6000
5500
5000
4500
4000
Initial order quantity
400 500 600 700 800 900 1000
Initial price [$]
20 19 18 17 16 15 14 13 12

Results

p_1	x_1	$\xi_1^{s_1}(p_1)$	$h_1^{s_1}$	s^{s_1}	$p_2^{s_1}$	$x_2^{s_1}$
		395.45	395.45	390.91	16.18 $	0.00
		444.32	444.32	342.05	17.02 $	44.12
		493.18	493.18	293.18	16.79 $	147.89
		542.05	542.05	244.32	16.52 $	260.61
16.09 $	786.36	590.91	590.91	195.45	16.32 $	372.97
		639.77	639.77	146.59	16.16$	485.10
		688.64	688.64	97.73	16.04 $	597.06
		737.50	737.50	48.86	15.94 $	708.92
		786.36	786.36	0.00	15.86 $	820.69

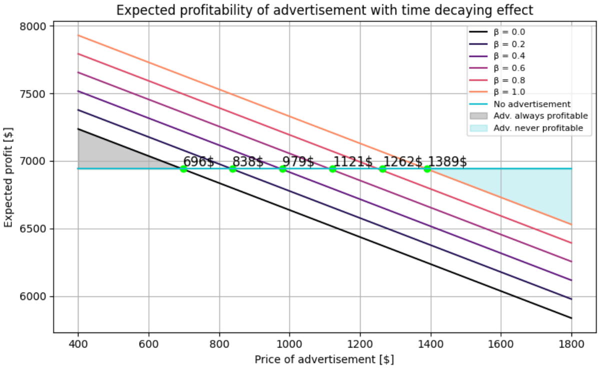
Results

1. stage scenario s_1	Most likely 2. stage demand	Available stock $x_2^{s_1} + s^{s_1}$
Lowest	391.00	390.91
Very low	386.25	386.17
Low	440.75	441.07
Below average	504.50	504.93
Average	568.00	568.36
Above average	632.00	631.69
High	695.00	694.79
Very high	744.5	757.78
Highest	821.0	820.69

Table: Comparison of available stock and the likeliest future demand

Advertisement

- set $\rho = 0.2$, treat c_A and β as a parameter



β	0	0.2	0.4	0.6	0.8	1
Turnover price	696$	838$	979$	1121$	1262$	1389$

CVaR risk measure and setting

- for a given confidence level $\alpha \in (0,1)$ the Conditional Value-at-risk is

$$CVaR_\alpha(\boldsymbol{\xi}) = \frac{1}{1-\alpha} \cdot \int_\alpha^1 q(p)\, dp = \min_{a \in \mathbb{R}} \left(a + \frac{1}{1-\alpha} \cdot \mathbb{E}[\boldsymbol{\xi} - a]^+ \right)$$

- we have $d \in \mathbb{N}$ assets and for each portfolio weights $\lambda \in \Lambda$ the asset losses are $\boldsymbol{\xi}(\lambda)$

CVaR portfolio optimization

- basic model formulation:

$$\min_{\lambda\in\Lambda, a\in\mathbb{R}} \left(a + \frac{1}{1-\alpha}\cdot \mathbb{E}[\lambda^T \boldsymbol{\xi}(\lambda) - a]^+\right)$$
$$\text{s.t.} \; -\mathbb{E}[\lambda^T \boldsymbol{\xi}(\lambda)] \geq r_{min}.$$

- reformulate using finitely many past scenarios

$$\min_{\lambda\in\Lambda, a\in\mathbb{R}, z_1,\ldots,z_m\in\mathbb{R}} \left(a + \frac{1}{1-\alpha}\cdot \sum_{i=1}^{S} \frac{1}{S}\cdot z_i\right)$$
$$\text{s.t.} \; \lambda^T r(\lambda) \geq r_{min}$$
$$z_i \geq \lambda^T \xi^i(\lambda) - a, \; i = 1, \ldots, S$$
$$z_i \geq 0, \; i = 1, \ldots, S.$$

Endogenous randomness

- assume that a large enough investment can alter the loss distribution
- we set two ownership tresholds: 10% and 50%
- crossing an ownership treshold for a stock lowers the loss scenarios by some predefined margin
- compute tresholds for portfolio weights using the investors budget and stocks market capitalization
- for an asset with market capitalization M and an investor with budget B, the weight needed to cross the treshold $\gamma \in [0, 1]$ is $\lambda = \gamma \cdot \frac{M}{B}$
- denote by T_i^t the portfolio weight needed to cross the t-th treshold on stock i ($t \in \{0, 1, 2, 3\}$)

- we have three ownership levels for each asset
  - $< 20\%$
  - $20 - 50\%$
  - $> 50\%$
- each ownership combination can be indexed by $k \in K = \{0, 1, 2\}^d$ and corresponds to a set of feasible weights

  $$\Lambda(k) = \Lambda(k_1, ..., k_d) = \{\lambda \in \Lambda : \lambda_i \in [T_i^{k_i}, T_i^{k_i+1}) \ \forall i \in \{1, ..., d\}\}$$

  (which provide a disjoint partition of Λ) and a scenario matrix $\mathbb{L}(k)$

Scenario matrix

- we have a basic scenario matrix $\mathbb{L}$ estimated from past losses
- for each $k \in K$ the scenario matrix is defined as

$$\mathbb{L}(k) = \begin{bmatrix} \mathbb{L}_{1,1} - \kappa_{k_1} \cdot \tau_1 & \cdots & \mathbb{L}_{1,d} - \kappa_{k_d} \cdot \tau_d \\ \mathbb{L}_{2,1} - \kappa_{k_1} \cdot \tau_1 & \cdots & \mathbb{L}_{2,d} - \kappa_{k_d} \cdot \tau_d \\ \vdots & \ddots & \vdots \\ \mathbb{L}_{S-1,1} - \kappa_{k_1} \cdot \tau_1 & \cdots & \mathbb{L}_{S-1,d} - \kappa_{k_d} \cdot \tau_d \\ \mathbb{L}_{S,1} - \kappa_{k_1} \cdot \tau_1 & \cdots & \mathbb{L}_{S,d} - \kappa_{k_d} \cdot \tau_d \end{bmatrix}$$

- denote $\xi^i(k)$ the i-th loss vector scenario for combination k

Cut infeasible combinations

- likely a majority of the sets $\Lambda(k)$ are empty
- in order for k to be feasible, both these conditions must hold

$$\sum_{i=1}^{d} T_i^{k_i} \leq 1,$$

$$\sum_{i=1}^{d} T_i^{k_i+1} > 1.$$

- denote $K^* = \{k \in K : \Lambda(k) \neq \emptyset\}$

Final model

$$\begin{aligned}
\min_{k \in K^*} \min_{\lambda \in \mathbb{R}^d, a, z_1, \dots, z_S \in \mathbb{R}} & \left(a + \frac{1}{1-\alpha} \cdot \sum_{i=1}^{S} \frac{1}{S} \cdot z_i \right) \\
\text{s.t. } & \lambda^T r(k) \geq r_{min} \\
& \sum_{i=1}^{d} \lambda_i = 1 \\
& \lambda_i \geq 0, \ i = 1, \dots, d \\
& \lambda_i \in [T_i^{k_i}, T_i^{k_i+1}), \ i = 1, \dots, d \\
& z_i \geq \lambda^T \xi^i(k) - a, \ i = 1, \dots, S \\
& z_i \geq 0, \ i = 1, \dots, S.
\end{aligned}$$

Numerical example

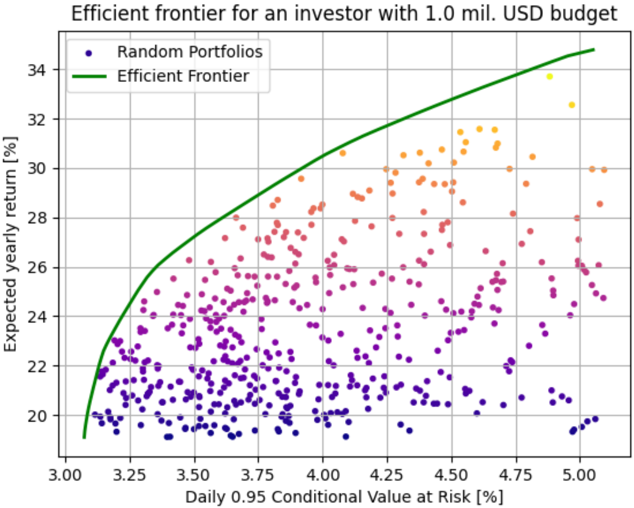
- daily adjusted closing prices from January 1st 2022 to June 1st 2024
- confidence level $\alpha = 0.95$

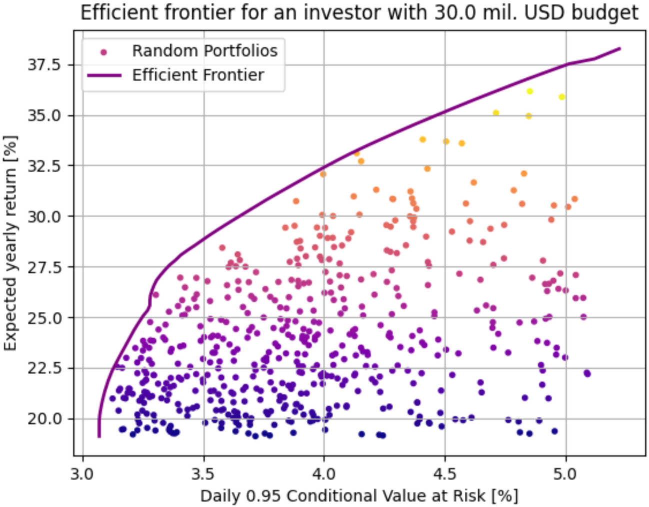
Stock ticker	Sector of interest	Market cap. [mil. USD]
ALOT	Computer Hardware / Technology	115.78
CAAS	Auto Parts / Consumer Cyclical	114.67
HMENF	Oil & Gas / Energy	122.13
INTT	Semiconductor Equipment / Technology	119.53
ISSC	Aerospace & Defense / Industrials	111.25

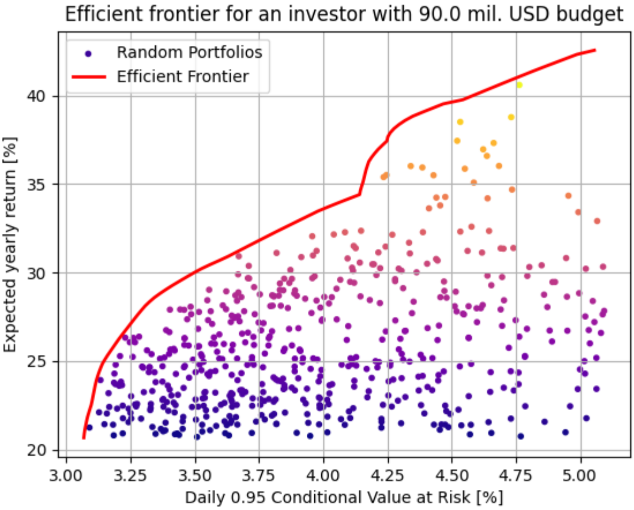
Optimal portfolios

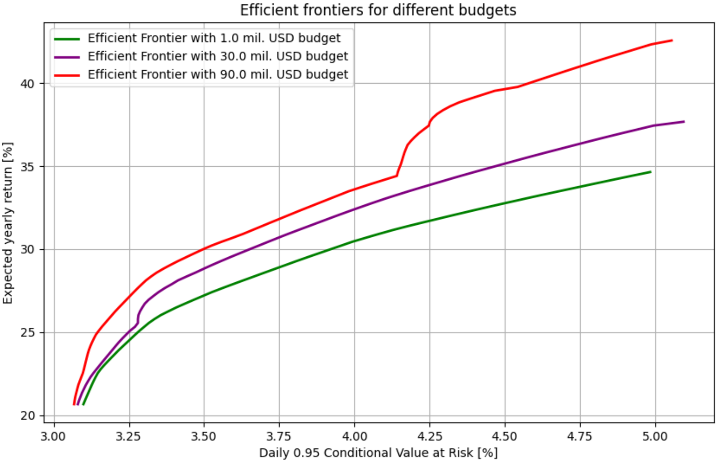
Budget	r_{min}	**ALOT**	**CAAS**	**HMENF**	**INTT**	**ISSC**
1 mil.	0.25	0.320	0.116	0.387	0.004	0.173
30 mil.	0.25	0.386	0.102	0.348	0.000	0.164
90 mil.	0.25	0.281	0.161	0.276	0.050	0.232
1 mil.	0.3	0.258	0.199	0.520	0.000	0.023
30 mil.	0.3	0.237	0.169	0.491	0.000	0.102
90 mil.	0.3	0.267	0.168	0.442	0.000	0.124

Efficient frontiers




Efficient frontier for an investor with 30.0 mil. USD budget
Random Portfolios
Efficient Frontier
Expected yearly return [%]
37.5
35.0
32.5
30.0
27.5
25.0
22.5
20.0
3.0
3.5
4.0
4.5
5.0
Daily 0.95 Conditional Value at Risk [%]


Efficient frontier for an investor with 90.0 mil. USD budget
Random Portfolios
Efficient Frontier
Expected yearly return [%]
Daily 0.95 Conditional Value at Risk [%]
20
25
30
35
40
3.00
3.25
3.50
3.75
4.00
4.25
4.50
4.75
5.00


Efficient frontiers for different budgets
Efficient Frontier with 1.0 mil. USD budget
Efficient Frontier with 30.0 mil. USD budget
Efficient Frontier with 90.0 mil. USD budget
Expected yearly return [%]
20
25
30
35
40
3.00
3.25
3.50
3.75
4.00
4.25
4.50
4.75
5.00
Daily 0.95 Conditional Value at Risk [%]

Thank you for your attention!