Relaxing stochastic dominance constraints in portfolio optimization

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Specific measure of stochastic non-dominance in portfolio optimization

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Stochastic dominance

Definition

A random variable X dominates a random variable Y by the **n^{th}-order stochastic dominance ($X \succeq_{(n)} Y$)** *if*

$$\mathbb{E}\ u(X) \geq \mathbb{E}\ u(Y) \text{ for all } u \in U_n \text{ such that these expected values exist.}$$

- $U_1 = \{u \text{ utility function}, u' \geq 0\}$
- $U_2 = \{u \text{ utility function}, u' \geq 0, u'' \leq 0\}$
- $U_3 = \{u \text{ utility function}, u' \geq 0, u'' \leq 0, u''' \geq 0\}$

Stochastic dominance

Definition

A random variable X dominates a random variable Y by the **n^{th}-order stochastic dominance ($X \succeq_{(n)} Y$)** *if*

$\mathbb{E}\ u(X) \geq \mathbb{E}\ u(Y)$ *for all $u \in U_n$ such that these expected values exist.*

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- $U_2 = \{u \text{ utility function}, u' \geq 0, u'' \leq 0\}$
- $U_3 = \{u \text{ utility function}, u' \geq 0, u'' \leq 0, u''' \geq 0\}$

Properties:

- $X \succeq_{(1)} Y \Rightarrow X \succeq_{(2)} Y \Rightarrow X \succeq_{(3)} Y$
- $X \succeq_{(1)} Y \Leftrightarrow F_X(x) \leq F_Y(x)$
  $X \succeq_{(2)} Y \Leftrightarrow F_X^{(2)}(x) \leq F_Y^{(2)}(x) \Leftrightarrow CVaR_\alpha(X) \leq CVaR_\alpha(Y)$, $\forall \alpha$
  $X \succeq_{(3)} Y \Leftrightarrow F_X^{(3)}(x) \leq F_Y^{(3)}(x)$

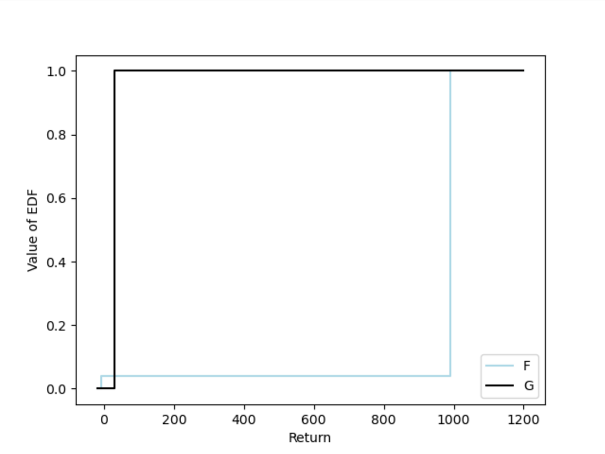
Motivation for relaxation of SD

Which investment would be preferred by most investors?

$$X = \begin{cases} 1000 \text{ EUR with probability } 0.99 \\ 0 \text{ EUR with probability } 0.01 \end{cases}$$

$Y = 1$ EUR with probability 1

Motivation for relaxation of SD



How much must X change to some Z in order to dominate Y?

General measure of stochastic non-dominance

Definition

Let X and Y be integrable random variables ($X, Y \in \mathcal{L}_1$). Denote $d_r(X, Y) = \left(\int_0^1 \mid F_X^{-1}(\alpha) - F_Y^{-1}(\alpha) \mid^r \right)^{\frac{1}{r}}$ the Wasserstein distance of order r between X and Y.

General measure of stochastic non-dominance

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Then the **general measure of stochastic non-dominance between X and Y**, $\text{GND}_n^r(X,Y)$, is defined as follows:

$$\text{GND}_n^r(X,Y) = \inf_{Z \in \mathcal{L}_1} d_r(X,Z)$$
$$\text{subject to } Z \succeq_{(n)} Y.$$

$\text{GND}_n^r(X,Y) = 0 \Longleftrightarrow X \succeq_{(n)} Y$

Specific measure of stochastic non-dominance

Definition

Let X and Y be discretely distributed random variables.
X attains values $x_1, \ldots, x_T$ with probabilities $p_1, \ldots, p_T$ and
Y attains values $y_1, \ldots, y_M$ with probabilities $q_1, \ldots, q_M$.
Denote by $\mathcal{D}$ the family of all discrete distributions with T atoms attained with probabilities $p_1, \ldots, p_T$. Then the specific measure of stochastic non-dominance between X and Y, $\mathrm{SND}_n^r(X, Y)$, is defined as follows:

Specific measure of stochastic non-dominance

Definition

Let X and Y be discretely distributed random variables.
X attains values $x_1, \ldots, x_T$ with probabilities $p_1, \ldots, p_T$ and
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Denote by $\mathcal{D}$ the family of all discrete distributions with T atoms attained with probabilities $p_1, \ldots, p_T$. Then the specific measure of stochastic non-dominance between X and Y, $\mathrm{SND}_n^r(X, Y)$, is defined as follows:

$$\mathrm{SND}_n^r(X, Y) = \min_{Z \in \mathcal{D}} d_r(X, Z)$$

$$\text{subject to } Z \succeq_{(n)} Y.$$

Wasserstein distance between discretely distributed r. v.

X attains values $x_1, \ldots, x_T$ with probabilities $p_1, \ldots, p_T$ and
Z attains values $z_1, \ldots, z_T$ with probabilities $p_1, \ldots, p_T$.

$$
\begin{aligned}
d_r(X,Z)^r = \min_{\xi_{ts}} & \sum_{t=1}^{T} \sum_{s=1}^{T} \xi_{ts} \mid x_t - z_s \mid^r \\
\text{subject to } & \sum_{s=1}^{T} \xi_{ts} = p_t, && t = 1, \ldots, T, \\
& \sum_{t=1}^{T} \xi_{ts} = p_s, && s = 1, \ldots, T, \\
& \xi_{ts} \geq 0, && s = 1, \ldots, T, \quad t = 1, \ldots, T.
\end{aligned}
$$

Portfolio optimization with respect to SD

Search for optimal weights to maximize the expected return while dominating the benchmark by FSD or SSD.

- k assets
- $R(\boldsymbol{\lambda})$ is the return of portfolio with weights $\boldsymbol{\lambda}$.
- Y represents the return of the benchmark.

Optimal portfolio with respect to the n-th order stochastic dominance:

$$\max_{\boldsymbol{\lambda}} \mathbb{E}\, R(\boldsymbol{\lambda})$$

$$\text{s.t. } R(\boldsymbol{\lambda}) \succeq_{(n)} Y,$$

$$\boldsymbol{\lambda} \in \Lambda = \{\lambda_i \in \mathbb{R}, \sum_{i=1}^{k} \lambda_i = 1, \lambda_i \geq 0\}.$$

Larger feasible set

SD constraint

$$\max_{\boldsymbol{\lambda}} \mathbb{E}\ R(\boldsymbol{\lambda})$$
$$\text{s.t. } R(\boldsymbol{\lambda}) \succeq_{(n)} Y,$$
$$\boldsymbol{\lambda} \in \Lambda$$

Larger feasible set

SD constraint

$$\max_{\boldsymbol{\lambda}} \mathbb{E}\ R(\boldsymbol{\lambda})$$
$$\text{s.t. } R(\boldsymbol{\lambda}) \succeq_{(n)} Y,$$
$$\boldsymbol{\lambda} \in \Lambda$$

Weaker constraint

$$\max_{\boldsymbol{\lambda}} \mathbb{E}\ R(\boldsymbol{\lambda})$$
$$\text{s.t. } SND_n^r(R(\boldsymbol{\lambda}), Y) \leq \varepsilon$$
$$\boldsymbol{\lambda} \in \Lambda$$

- For $\varepsilon = 0$ the constraints are the same.
- The higher the ε, the more relaxed the SD condition is.

Portfolio optimization with SND_n^r constraints

$$
\begin{aligned}
&\max_{\boldsymbol{\lambda}, Z} \mathbb{E}\ R(\boldsymbol{\lambda}) \\
&\text{s.t.} \quad \min_{Z} d_r(R(\boldsymbol{\lambda}), Z) \le \varepsilon, \\
&\qquad Z \succeq_{(n)} Y \\
&\qquad \boldsymbol{\lambda} \in \Lambda.
\end{aligned}
$$

Portfolio optimization with SND_n^r constraints

$$\begin{aligned} \max_{\boldsymbol{\lambda}, Z} \; & \mathbb{E}\; R(\boldsymbol{\lambda}) \\ \text{s.t.} \; & \min_{Z} d_r(R(\boldsymbol{\lambda}), Z) \leq \varepsilon, \\ & Z \succeq_{(n)} Y \\ & \boldsymbol{\lambda} \in \Lambda. \end{aligned}$$

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Portfolio optimization with SND_n^r constraints

$$
\begin{aligned}
\max_{\boldsymbol{\lambda},\boldsymbol{z},\boldsymbol{x},\boldsymbol{\xi}} \quad & \sum_{t=1}^{T} p_t x_t \\
\text{subject to} \quad & \sum_{t=1}^{T}\sum_{s=1}^{T} \xi_{ts} \mid x_t - z_s \mid^r \le \varepsilon^r \\
& x_t = \sum_{i=1}^{k} \lambda_i \cdot r_{it} && t = 1,\dots,T, \\
& \sum_{s=1}^{T} \xi_{ts} = p_t, && t = 1,\dots,T, \\
& \sum_{t=1}^{T} \xi_{ts} = p_s, && s = 1,\dots,T, \\
& \xi_{ts} \ge 0, && s = 1,\dots,T, \quad t = 1,\dots,T, \\
& Z \succeq_{(n)} Y, \\
& \boldsymbol{\lambda} \in \Lambda.
\end{aligned}
$$

Portfolio optimization with SND_n^r constraints

$$\boldsymbol{Z} \succeq_{(1)} \boldsymbol{Y}$$

becomes

$$
\begin{aligned}
& z_s \geq \sum_{m=1}^{M} \pi_{ms} y_m, && \forall s \\
& \sum_{m=1}^{M} \pi_{ms} = 1, && \forall s \\
& \sum_{s=1}^{S} \pi_{ms} p_s = v_m, && \forall m \\
& \sum_{m=1}^{j-1} v_m \leq \sum_{m=1}^{j-1} q_m && j = 2, \dots, M, \\
& \pi_{ms} \in \{0, 1\}, && \forall m, \ \forall s.
\end{aligned}
$$

$$\boldsymbol{Z} \succeq_{(2)} \boldsymbol{Y}$$

becomes

$$
\begin{aligned}
& \sum_{s=1}^{T} p_s s_{ts} \leq F_Y^{(2)}(y_t), && \forall t, \\
& z_s + s_{ts} \geq y_t, && \forall s, t, \\
& s_{ts} \geq 0, && \forall s, t.
\end{aligned}
$$

Simplification for empirical distributions

Wasserstein distance between empirically distributed r. v.

Theorem

X attains values $x_1 \leq \cdots \leq x_T$ with probabilities $1/T$.
Z attains values $z_1 \leq \cdots \leq z_T$ with probabilities $1/T$.
Then their Wasserstein distance of integer order r is

$$d_r(X, Z)^r = \frac{1}{T} \sum_{t=1}^{T} \mid x_t - z_t \mid^r .$$

Portfolio optimization with SND_n^r constraints

X attains values $x_1 \leq \cdots \leq x_T$ with probabilities $1/T$.

$$
\begin{aligned}
&\max_{\boldsymbol{\lambda},\boldsymbol{z},\boldsymbol{x}} \frac{1}{T}\sum_{t=1}^{T} x_t \\
&\text{s.t. } \frac{1}{T}\sum_{t=1}^{T} \mid x_t - z_t \mid^r \leq \varepsilon^r \\
&\quad x_t = \sum_{i=1}^{k} \lambda_i \cdot r_{it} \qquad \forall t \\
&\quad Z \succeq_{(n)} Y \\
&\quad \boldsymbol{\lambda} \in \Lambda.
\end{aligned}
$$

Portfolio optimization with SND_n^r constraints

Y attains values $y_1 \leq \cdots \leq y_T$ with probabilities $1/T$.

$$\boldsymbol{Z} \succeq_{(1)} \boldsymbol{Y}$$

becomes

$$z_t \geq \sum_{s=1}^{T} \pi_{ts} y_s, \qquad \forall t,$$

$$\sum_{t=1}^{T} \pi_{ts} = 1, \qquad \forall s,$$

$$\sum_{s=1}^{T} \pi_{ts} = 1, \qquad \forall t,$$

$$\pi_{ts} \in \{0, 1\}, \qquad \forall s, t.$$

$$\boldsymbol{Z} \succeq_{(2)} \boldsymbol{Y}$$

becomes

$$z_t \geq \sum_{s=1}^{T} w_{ts} y_s, \quad t = 1, \ldots, T,$$

$$\sum_{t=1}^{T} w_{ts} = 1, \qquad s = 1, \ldots, T,$$

$$\sum_{s=1}^{T} w_{ts} = 1, \qquad t = 1, \ldots, T,$$

Further useful properties under empirical distribution

X and Y are empirically distributed with atoms $x_1, \ldots, x_T$ and $y_1, \ldots, y_T$.

- $\mathrm{SND}_1^r(X, Y) = \mathrm{GND}_1^r(X, Y)$ and $\mathrm{SND}_2^1(X, Y) = \mathrm{GND}_2^1(X, Y)$
- When $\mathrm{SND}_n^r(X, Y)$ is computed, it holds for the optimal closest dominating Z that $z_t \geq x_t$. As a result,

$$\frac{1}{T} \sum_{t=1}^{T} \mid x_t - z_t \mid^r \leq \varepsilon^r$$

can be replaced by

$$\frac{1}{T} \sum_{t=1}^{T} (z_t - x_t)^r \leq \varepsilon^r,$$

$$z_t \geq x_t.$$

Almost stochastic dominance

Almost first-order stochastic dominance

Definition (Leshno, Levy(2002)):
Let X and Y be two random variables defined on a support $[a, b]$ with distribution functions F_X and F_Y. For $\varepsilon \in (0, 0.5)$, we say that X dominates Y by ε-Almost FSD ($X \succeq_{(1)}^{\varepsilon} Y$) if

$$\int_{S_1} F_X(t) - F_Y(t)dt \leq \varepsilon \cdot \int_a^b \mid F_X(t) - F_Y(t) \mid dt,$$

where

$$S_1 = \{t \in [a, b] : F_X(t) > F_Y(t)\}.$$

Relationships to d_1 and GND_1^1

Theorem

Suppose the support of X and Y is in $[a, b]$, and F_X and F_Y are their distribution functions. Then

$$d_1(X, Y) = \int_a^b \mid F_X(t) - F_Y(t) \mid dx.$$

Theorem:

Suppose the support of X and Y is in $[a, b]$, and F_X and F_Y are their distribution functions. Then

$$\int_{S_1} F_X(t) - F_Y(t) dt = GND_1^1(X, Y)$$

Theorem:

Suppose X and Y are empirically distributed. Then

$$GND_1^1(X, Y) = SND_1^1(X, Y)$$

Relationship of AFSD and SND$_1^1$

Theorem:

Let X and Y be two random variables defined on a support $[a, b]$, $\varepsilon \in (0, 0.5)$. Then X dominates Y by ε-Almost FSD if and only if

$$SND_1^1(X, Y) \leq \varepsilon \cdot d_1(X, Y)$$

Portfolio optimization with SND_1^1 vs. AFSD constraints

SND_1^1 constraints

$$\max_{\boldsymbol{\lambda}} \mathbb{E}\, R(\boldsymbol{\lambda})$$
$$\text{s.t. } SND_1^1(R(\boldsymbol{\lambda}), Y) \leq \varepsilon$$
$$\boldsymbol{\lambda} \in \Lambda$$

AFSD constraints

$$\max_{\boldsymbol{\lambda}} \mathbb{E}\, R(\boldsymbol{\lambda})$$
$$\text{s.t. } SND_1^1(R(\boldsymbol{\lambda}), Y) \leq \tilde{\varepsilon} \cdot d_1(R(\boldsymbol{\lambda}), Y)$$
$$\boldsymbol{\lambda} \in \Lambda$$

SND_1^1 and AFSD constraints lead to different feasible sets

X_1 attains values -1 and 20 with probabilities 0.1 and 0.9, respectively.
X_2 attains values -0.5 and 5 with probabilities 0.1 and 0.9, respectively.
$Y = 0$ almost surely.

	X_1	X_2
$\mathrm{SND}_1^1(X, Y)$	0.1	0.05
$d_1(X, Y)$	18.1	4.55
$\tilde{\varepsilon} = \mathrm{SND}_1^1(X, Y)/\mathrm{d}_1(X, Y)$	0.0055	0.011

Special case when the optimal solution is the same

For any $\tilde{\varepsilon} \in [0, 0.5]$ and the corresponding optimal solution $\boldsymbol{\lambda}$ of

$$\begin{aligned} &\max_{\boldsymbol{\lambda}} \mathbb{E}\ R(\boldsymbol{\lambda}) \\ &\text{s.t. } SND_1^1(R(\boldsymbol{\lambda}), Y) \leq \tilde{\varepsilon} \cdot d_1(R(\boldsymbol{\lambda}), Y) \\ &\qquad \boldsymbol{\lambda} \in \Lambda \end{aligned}$$

there exists an $\varepsilon = \tilde{\varepsilon} \cdot d_1(R(\boldsymbol{\lambda}), Y)$ such that the optimal solution of

$$\begin{aligned} &\max_{\boldsymbol{\lambda}} \mathbb{E}\ R(\boldsymbol{\lambda}) \\ &\text{s.t. } SND_1^1(R(\boldsymbol{\lambda}), Y) \leq \varepsilon \\ &\qquad \boldsymbol{\lambda} \in \Lambda \end{aligned}$$

is the same.

Idea of proof

$$
\begin{aligned}
&\mathbb{E}\ R(\boldsymbol{\lambda}) - \mathbb{E}\ Y \\
=&\int_{-\infty}^{\infty} F_Y(t) - F_X(t) dt \\
=&\int_{-\infty}^{\infty} \mid F_Y(t) - F_X(t) \mid dt - 2\int_{-\infty}^{\infty} [F_Y(t) - F_X(t)]^{-} dt \\
=&d_1(R(\boldsymbol{\lambda}), Y) - 2\int_{S_1} F_Y(t) - F_X(t) dt \\
=&d_1(R(\boldsymbol{\lambda}), Y) - 2 \cdot \mathrm{SND}_1^1(R(\boldsymbol{\lambda}), Y).
\end{aligned}
$$

For $\forall\ \tilde{\varepsilon}$-AFSD solution, SND_1^1 solution exists. Not vice versa.

X_1 attains values -1 and 20 with probabilities 0.1 and 0.9, respectively.

X_2 attains values -0.5 and 5 with probabilities 0.1 and 0.9, respectively.

$Y = 0$ almost surely.

	X_1	X_2
$\mathrm{SND}_1^1(X, Y)$	0.1	0.05
$d_1(X, Y)$	18.1	4.55
$\tilde{\varepsilon} = \mathrm{SND}_1^1(X, Y)/\mathrm{d}_1(X, Y)$	0.0055	0.011
$\mathbb{E}\, X$	17.9	4.45

Portfolio optimization AFSD constraints

$$
\begin{aligned}
\max_{\boldsymbol{\lambda},\boldsymbol{z},\boldsymbol{x},\boldsymbol{\pi}} \; & \frac{1}{T}\sum_{t=1}^{T} x_t \\
\text{subject to } & 1/T \cdot \sum_{t=1}^{T} (z_t - x_t) \leq \varepsilon \cdot d_1(R(\boldsymbol{\lambda}), Y) \\
& z_t \geq x_t && t = 1, \ldots, T. \\
& x_t = \sum_{i=1}^{k} \lambda_i \cdot r_{it} && t = 1, \ldots, T \\
& Z \succeq_{(1)} Y \\
& \boldsymbol{\lambda} \in \Lambda.
\end{aligned}
$$

Dual formulation of Wasserstein distance

$$
\begin{aligned}
d_r(X, Y)^r = \max_{\rho_t, \mu_t} & \frac{1}{T}\sum_{t=1}^{T}\rho_t + \frac{1}{T}\sum_{t=1}^{T}\mu_t \\
& \rho_t + \mu_s \leq \mid x_t - z_s \mid^r, \quad && t = 1, \dots, T, s = 1, \dots, T, \\
& \rho_t \in \mathbb{R}, && t = 1, \dots, T, \\
& \mu_t \in \mathbb{R}, && t = 1, \dots, T.
\end{aligned}
$$

Portfolio optimization AFSD constraints

$$\max_{\boldsymbol{\lambda}, \boldsymbol{z}, \boldsymbol{x}, \boldsymbol{\pi}, \rho, \mu} \frac{1}{T} \sum_{t=1}^{T} x_t$$

$$\begin{aligned}
\text{subject to } & 1/T \cdot \sum_{t=1}^{T} (z_t - x_t) \leq \varepsilon \cdot \sum_{t=1}^{T} (\rho_t + \mu_t) & \\
& \rho_t + \mu_s \leq \mid x_t - y_s \mid, & t = 1, \dots, T, s = 1, \dots, T, \\
& \rho_t, \mu_t \in \mathbb{R}, & t = 1, \dots, T, \\
& z_t \geq x_t & t = 1, \dots, T. \\
& x_t = \sum_{i=1}^{k} \lambda_i \cdot r_{it} & t = 1, \dots, T \\
& Z \succeq_{(1)} Y & \\
& \boldsymbol{\lambda} \in \Lambda. &
\end{aligned}$$

Absolute value constrained from below

$\rho_t + \mu_s \leq \mid x_t - y_s \mid$ means that

$$\text{either } \rho_t + \mu_s \leq x_t - y_s,$$
$$\text{or } \rho_t + \mu_s \leq -x_t + y_s.$$

Using a big constant M and integer variable b_{ts}, we can reformulate it as

$$\rho_t + \mu_s \leq x_t - y_s + b_{ts} \cdot M$$
$$\rho_t + \mu_s \leq -x_t + y_s + (1 - b_{ts}) \cdot M$$
$$b_{ts} \in \{0, 1\}.$$

Tractable ASD

Definition (Lizyayev, Ruszczynski (2012)):
X dominates Y by **Tractable ε-Almost n$^{\text{th}}$ order stochastic dominance** if there exists a non-negative random variable V such that $\mathbb{E}\ V \leq \varepsilon$ and $X + V \succeq_{(n)} Y$.

Relationship between tractable AnSD and GND_n^1

Suppose that Z is the optimal solution of the problem that searches for $\text{GND}_n^1(X, Y)$. Set $V = Z - X$. Then

- $V \geq 0$,
- $X + V \succeq_{(n)} Y$,
- $\mathbb{E}\ V = d_1(Z, X) = \text{GND}_n^1(X, Y)$.

$$\Downarrow$$

X dominates Y by Tractable ε-Almost n$^{\text{th}}$-order stochastic dominance

$$\Leftrightarrow$$

$$\text{GND}_n^1(X, Y) \leq \varepsilon.$$

Further, $\text{GND}_n^1(X, Y) = \text{SND}_n^1$ for $n = 1$ and $n = 2$
for empirically distributed X and Y.

Empirical study

Empirical study

- 49 industry representative portfolios
- CRSP index as benchmark
- We compare the optimal portfolios with respect to SSD, SND_2^1, SND_2^2 and TSD constraints.
- FSD portfolio and the corresponding SND_1^1 portfolios are outperformed by the SSD portfolio in terms of returns as well as risks.
- Considered ε: $0.000025, 0.0001, 0.00025, 0.0005, 0.00075, \ldots, 0.06$

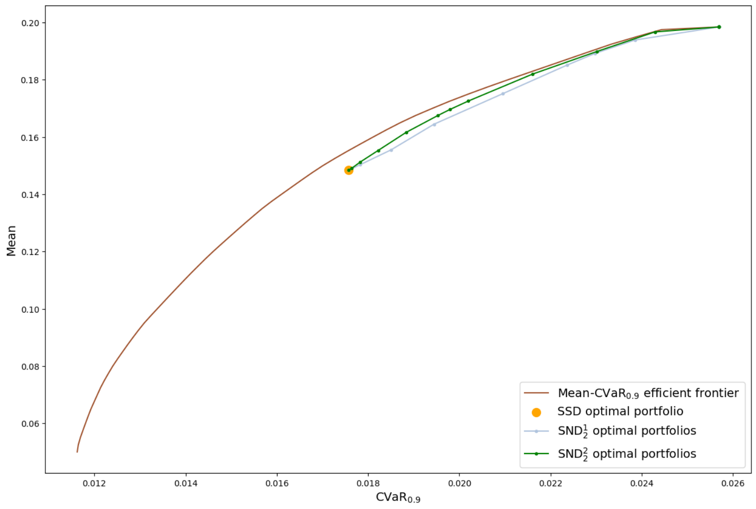
Single period results

- Daily returns from Oct 2022 to Sept 2023

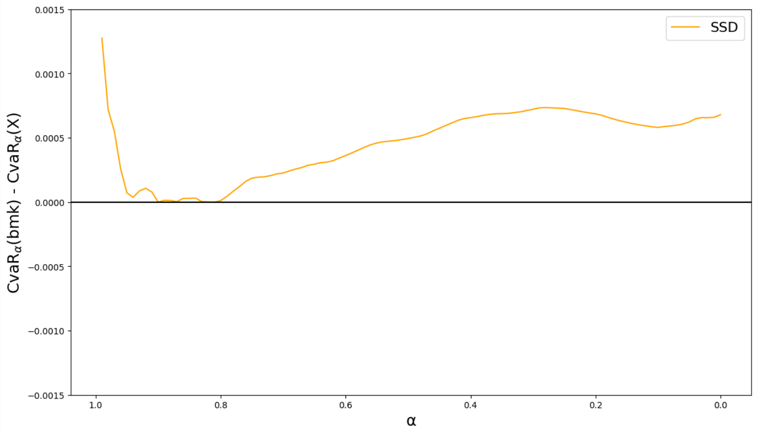
Mean daily return and CVaR $\cdot 10^4$

ε	SND_2^1 mean	SND_2^2 mean	SND_2^1 $\mathsf{CVaR}_{0.9}$	SND_2^2 $\mathsf{CVaR}_{0.9}$
0	14.84	14.84	1.76	1.76
0.000025	15.03	14.91	1.78	1.76
0.0001	15.55	15.13	1.85	1.78
0.00025	16.45	15.53	1.94	1.82
0.0005	17.51	16.16	2.09	1.88
0.00075	18.52	16.75	2.24	1.95
0.001	19.39	17.26	2.39	2.02
0.0025	19.84	19.67	2.57	2.43
0.06	19.84	19.84	2.57	2.57
TSD	15.08		1.75	

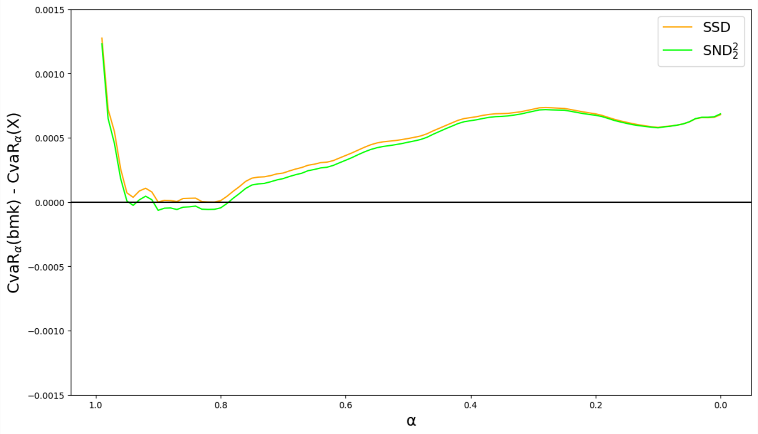
Relationship to mean-CVaR efficient frontier



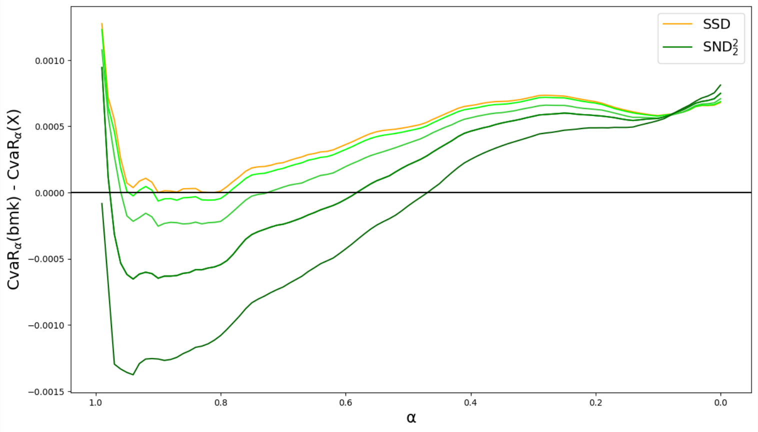
Risk reduction compared to benchmark



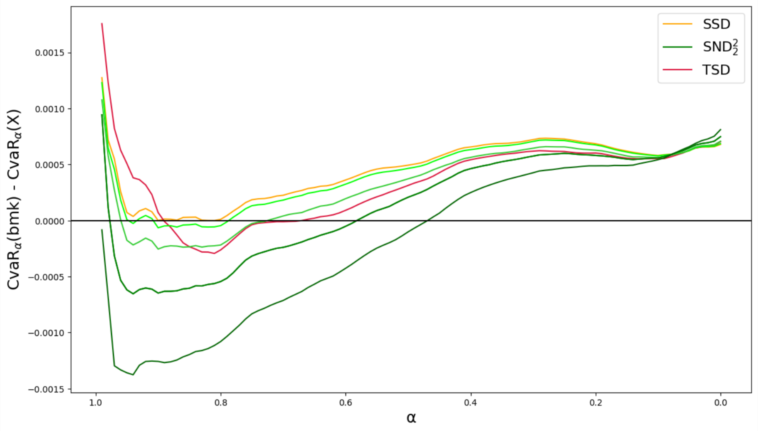
Risk reduction compared to benchmark



Risk reduction compared to benchmark



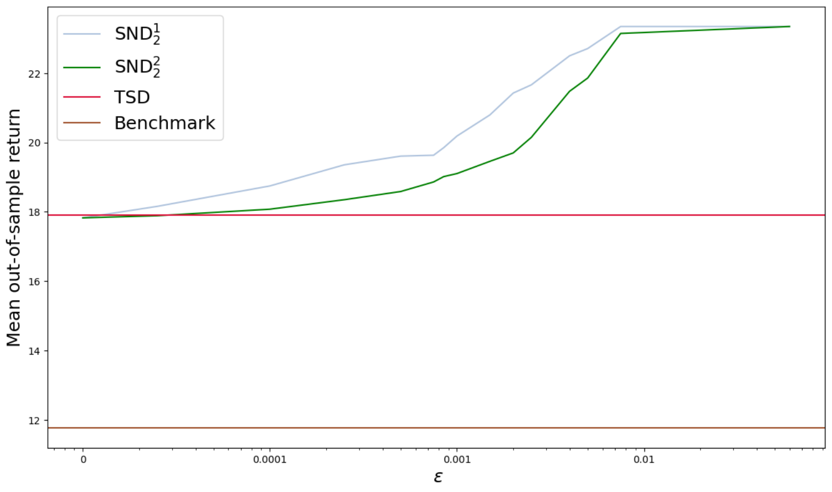
Risk reduction compared to benchmark



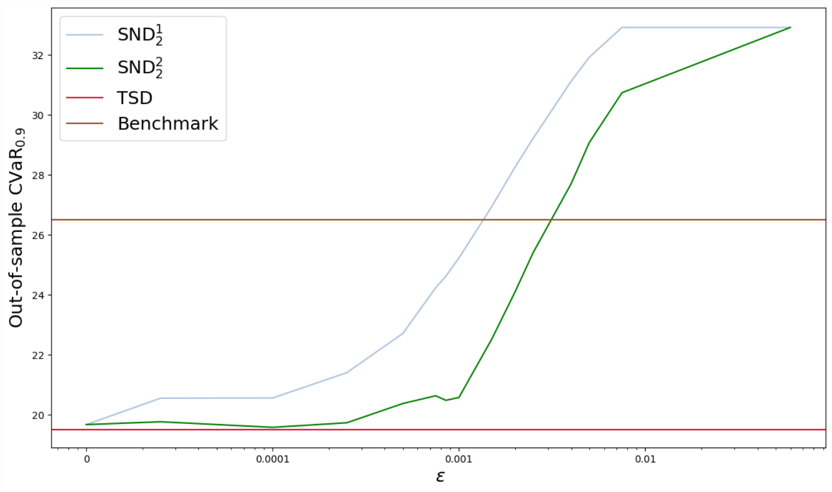
Empirical study - moving window out-of-sample analysis

- Portfolios are formed at the beginning of every quarter based on the preceding 12 months of daily returns.
- Their out-of-sample performance is evaluated in 96 non-overlapping periods from January 1 through December 31 in every year from 1928 through 2023.

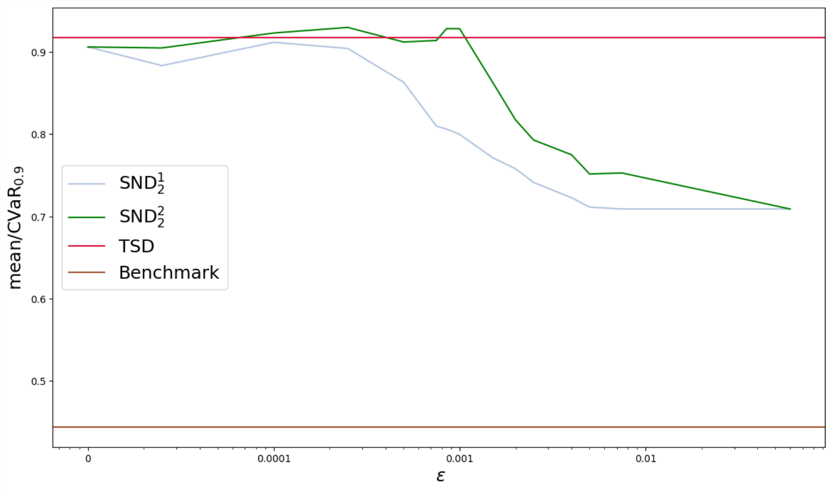
Average annual out-of-sample returns depending on allowed ε



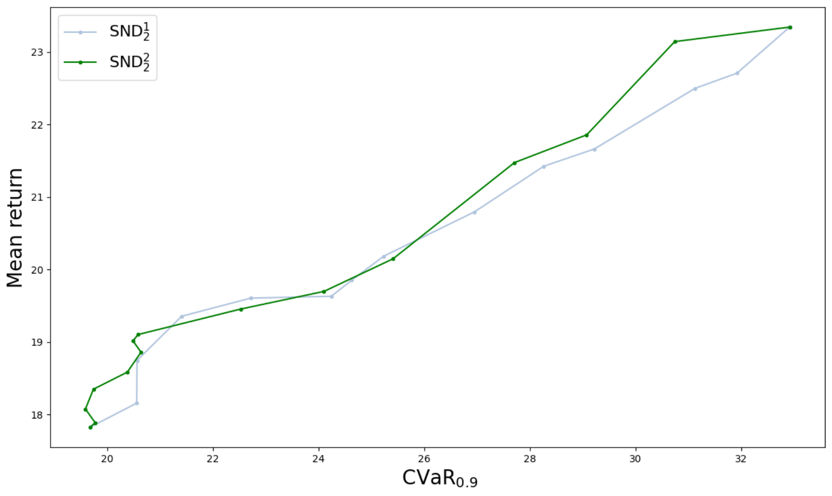
$\text{CVaR}_{0.9}$ of annual losses



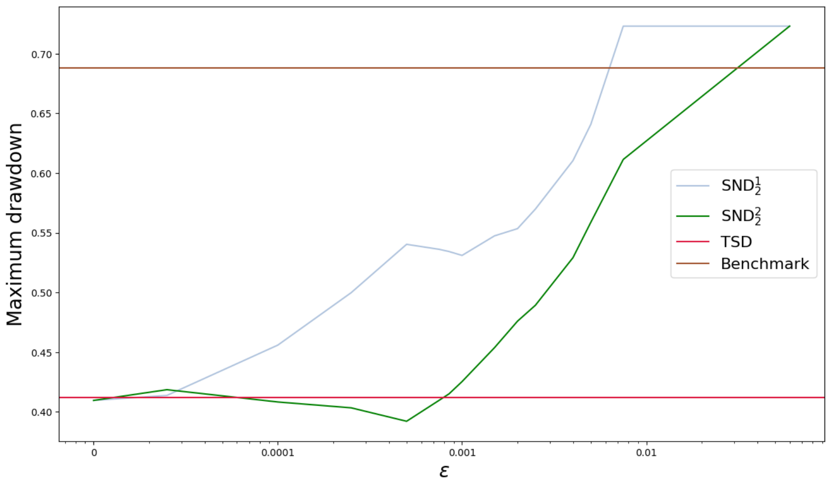
Mean-CVaR ratio



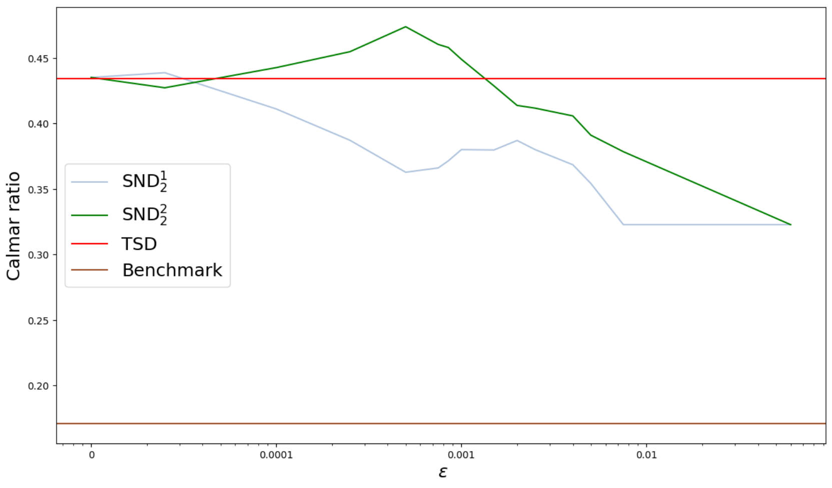
Out-of-sample performance with respect to mean and $\text{CVaR}_{0.9}$



Maximum drawdown



Calmar ratio



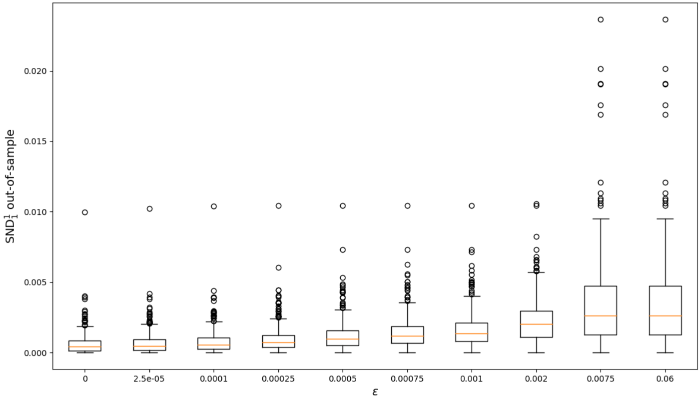
Out-of-sample measure of stochastic non-dominance

How often is the in-sample constraint satisfied out-of-sample?

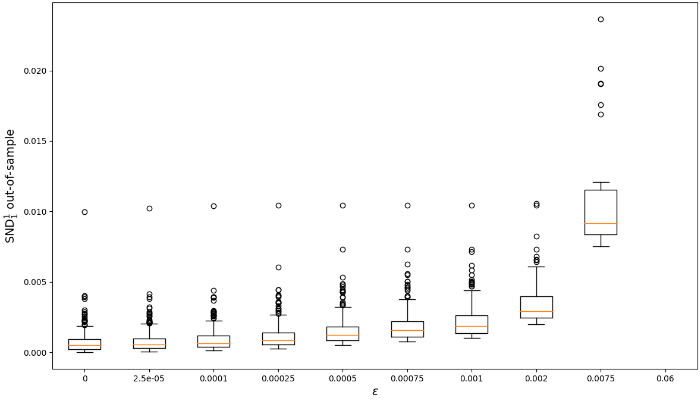
What is the average measure of non-dominance out-of-sample?

ε	SND_2^1 % satisfied	SND_2^2 % satisfied	SND_2^1 avg SND_2^1	SND_2^2 avg SND_2^2
0	10	10	0.0006	0.0013
0.0001	13	10	0.0008	0.0014
0.00025	17	13	0.001	0.0015
0.0005	23	16	0.0012	0.0017
0.00075	27	19	0.0015	0.002
0.001	33	23	0.0017	0.0022
0.002	49	34	0.0023	0.003
0.0075	93	78	0.0034	0.0052
0.06	100	100	0.0034	0.0064

Boxplots of the oos measures of non-dominance



Boxplots of the excessive measures of non-dominance



% of periods when optimal portfolio consisted of 1 asset

ε	SND_2^1 % in 1 asset	SND_2^2 % in 1 asset
0	6	6
0.000025	7	6
0.0001	9	7
0.00025	13	8
0.0005	19	11
0.00075	22	12
0.001	29	15
0.002	55	26
0.0075	95	76
0.06	100	100

Conclusions

- Formulation of portfolio optimization with SND_n^r constraints
- Formulation of portfolio optimization with AFSD constraints and relations to SND_1^1 constraints
- Improved out-of-sample performance by allowing certain SND_n^r.
- Measuring the violation of SD by the Wasserstein distance of order 2 leads to more risk-averse results

Thank you for your attention

J. Junová, M. Kopa (2024): Measures of Stochastic Non-Dominance in Portfolio Optimization, EJOR.

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Increases in annual returns are statistically significant

Increase in average out-of-sample annual returns in percent compared with SSD optimal portfolio (and p-value)

	SND_2^1	SND_2^1	SND_2^2	SND_2^2
ε	0.0001	0.001	0.0001	0.001
average return spread	0.94	2.37	0.25	1.28
p-value	(0.04)	(0.03)	(0.05)	(0.04)