Emergence of equilibria in games with random payoffs

NMEK615

Lukáš Račko

5.12.2024

Contents

Game with random payoff

Definition

By a game with random payoff on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ we consider a triplet $G = (I, \{X_i\}_{i\in I}, \{u_i\}_{i\in I})$, where $I = \{1, \dots, N\}$ is a fixed finite set of players, X_i is a set of (mixed) strategies of the player i and $u_i(\boldsymbol{x})$ is $\mathcal{A}$-measurable random variable, for each $\boldsymbol{x} \in X$.

On the probability space

- Let $X = \bigtimes_{i \in I} X_i$ be the set of mixed strategy profiles. It follows that $X \simeq S \subseteq \mathbb{R}^K$, where $S = \bigtimes_{i \in I} S_i$ and $S_i = \{x_i \in \mathbb{R}^{K_i} \mid \sum_{p \in P_i} x_i(p) = 1,\ x_i(q) \geq 0,\ q \in P_i\}$, where $K = \sum_{i \in I} K_i$ is the number of different pure strategy profiles.
- Natural choice is $\Omega = \{\boldsymbol{u} \in C^{N+}(X) \mid ||\boldsymbol{u}||_{C^{N+}(X)} \leq 1\}$, where $||\boldsymbol{u}||_{C^{N+}(X)} = \sum_{i \in I} \sup_{\boldsymbol{x} \in X} |u_i(\boldsymbol{x})|$. Since u_i is linear on X it follows that $u_i(\boldsymbol{x}) =< \boldsymbol{\omega}_i, \boldsymbol{x} > +\theta_i$ for some $\boldsymbol{\omega}_i \in \mathbb{R}^{K_i}$, $\theta_i \in \mathbb{R}$.
- If we set $\theta_i = 0, i \in I$ we can construct a homeomorphism $H : C^{N+} \to \mathbb{R}^{K+}$ so that $H(\boldsymbol{u}) = (u_i(p_i))_{i \in I, p \in P_i} \in \mathbb{R}^{K+}$. From which we can construct Ω equivalently as $\Omega = \{\omega \in \mathbb{R}^{K+} \mid ||\omega||_{\mathbb{R}^{K+}} \leq 1\}$.

α-Best response

Definition

Let $G = (I, \{X_i\}_{i \in I}, \{u_i\}_{i \in I})$ be a game with random payoff on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $\alpha_i \in [0, 1]$. We define the set of best response strategies on a confidence level of α_i for the player i with individual constraints, given strategy profile of other players $\boldsymbol{x}_{-i} \in X_{-i}$ as

$$BRI_i^{\alpha_i}(\boldsymbol{x}_{-i}) = \{x_i \in X_i \,|\, \forall p \in P_i : \mathbb{P}\left(u_i(x_i, \boldsymbol{x}_{-i}) \geq u_i(p, \boldsymbol{x}_{-i})\right) \geq \alpha_i\}.$$

Similarly we define the set of best response strategies on a confidence level of α_i for the player i with joint constraints, given strategy profile of other players $\boldsymbol{x}_{-i} \in X_{-i}$ as

$$BRJ_i^{\alpha_i}(\boldsymbol{x}_{-i}) = \{x_i \in X_i \,|\, \mathbb{P}\left(\forall p \in P_i : u_i(x_i, \boldsymbol{x}_{-i}) \geq u_i(p, \boldsymbol{x}_{-i})\right) \geq \alpha_i\}.$$

$\boldsymbol{\alpha}$-Nash equilibria

Definition

Let $G = (I, \{X_i\}_{i\in I}, \{u_i\}_{i\in I})$ be a game with random payoff on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $\boldsymbol{\alpha} = (\alpha_i)_{i\in I} \in [0,1]^I$ be a vector of confidence levels. We define the $\boldsymbol{\alpha}$-Nash equilibrium of the game G with individual constraints as $\boldsymbol{x}^* \in X$ such that $\forall i \in I : x_i^* \in BRI_i^{\alpha_i}(\boldsymbol{x}_{-i}^*)$ and $\boldsymbol{\alpha}$-Nash equilibrium of the game G with joint constraints as $\boldsymbol{x}^* \in X$ such that $\forall i \in I : x_i^* \in BRJ_i^{\alpha_i}(\boldsymbol{x}_{-i}^*)$. The set of $\boldsymbol{\alpha}$-Nash equilibria of the game G with individual constraints will be denoted as $\boldsymbol{\alpha}$-$INE(G)$ and similarly the set of $\boldsymbol{\alpha}$-Nash equilibria of the game G with the joint constraints will be denoted as $\boldsymbol{\alpha}$-$JNE(G)$.

Existence and other nice properties

Theorem

Let $G = (I, \{X_i\}_{i \in I}, \{u_i\}_{i \in I})$ be a game with random payoff on an arbitrary probability space $(\Omega, \mathcal{A}, \mathbb{P})$ then:

1. *(Existence)* $\exists \boldsymbol{\alpha} > \mathbf{0}$ *and* $\boldsymbol{x}^* \in X$ *such that* $\boldsymbol{x}^* \in \boldsymbol{\alpha}\text{-}INE(G)$.
2. *(Confidence monotonicity)* $\forall \boldsymbol{\alpha} \geq \hat{\boldsymbol{\alpha}}$ *it holds that* $\boldsymbol{\alpha}\text{-}INE(G) \subseteq \hat{\boldsymbol{\alpha}}\text{-}INE(G)$ *and* $\boldsymbol{\alpha}\text{-}JNE(G) \subseteq \hat{\boldsymbol{\alpha}}\text{-}JNE(G)$.
3. *(Almost sure Nash equilibrium) If* $\boldsymbol{\alpha} = \mathbf{1}$ *then* $\boldsymbol{\alpha}\text{-}INE(G) = \boldsymbol{\alpha}\text{-}JNE(G) = NE(G)$, *where* $NE(G)$ *is the set of strategy profiles that are Nash equilibria up to some* $\mathbb{P}$*-null set.*
4. *(No uncertainty equilibrium) If* $\mathbb{P} = \delta_\omega$ *for some* $\omega \in \Omega$ *(that is scenario* ω *occurs with probability* 1*) then* $\forall \boldsymbol{\alpha} > \mathbf{0}$ *it holds that* $\boldsymbol{\alpha}\text{-}INE(G) = \boldsymbol{\alpha}\text{-}JNE(G) = NE(G(\omega))$, *where by* $NE(G(\omega))$ *we denote the set of Nash equilibria of the deterministic game with payoff given by the scenario* $\omega \in \Omega$.

Unique confidence levels

Theorem

Let $G = (I, \{X_i\}_{i \in I}, \{u_i\}_{i \in I})$ be a game with random payoff on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then $\forall \boldsymbol{x} \in X$ there exists a unique $\boldsymbol{\alpha} \in [0,1]^I$ so that $\boldsymbol{x} \in \boldsymbol{\alpha}\text{-}INE(G)$ and $\forall \hat{\boldsymbol{\alpha}} \in [0,1]^I$ with $\boldsymbol{x} \in \hat{\boldsymbol{\alpha}}\text{-}INE(G)$ it holds that $\hat{\boldsymbol{\alpha}} \leq \boldsymbol{\alpha}$. Similarly, there exists a unique $\boldsymbol{\alpha} \in [0,1]^I$ so that $\boldsymbol{x} \in \boldsymbol{\alpha}\text{-}JNE(G)$ and $\forall \hat{\boldsymbol{\alpha}} \in [0,1]^I$ with $\boldsymbol{x} \in \hat{\boldsymbol{\alpha}}\text{-}JNE(G)$ it holds that $\hat{\boldsymbol{\alpha}} \leq \boldsymbol{\alpha}$.

We can define $\boldsymbol{\alpha}(\boldsymbol{x})$ as the maximal levels $\boldsymbol{\alpha}$ so that $\boldsymbol{x} \in \boldsymbol{\alpha}\text{-}NE(G)$.

The Most likely equilibria

Definition

Let $G = (I, \{X_i\}_{i\in I}, \{u_i\}_{i\in I})$ be a game with random payoff on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Denote $\lambda(G) = \{\boldsymbol{\alpha} \in [0,1]^I \mid \boldsymbol{\alpha}\text{-}INE(G) \neq \emptyset\}$. If $\boldsymbol{x}^* \in \boldsymbol{\alpha}\text{-}INE(G)$ such that $\forall \hat{\boldsymbol{\alpha}} \in \lambda(G) : \hat{\boldsymbol{\alpha}} \ngeq \boldsymbol{\alpha}$ we will call $\boldsymbol{x}^*$ the most likely equilibrium. The set of most likely equilibria will be denoted as $MLE(G)$. The MLE is called non-degenerated if $\forall i \in I : \alpha_i > 0$ in this equilibrium. Set of the non-degenerated equilibria will be denoted as $MLE^+(G)$. For a $\boldsymbol{x}^* \in MLE(G)$ we will write $\boldsymbol{\alpha}(\boldsymbol{x}^*)$ to denote the set of maximal confidence levels on which $\boldsymbol{x}^*$ is achieved.

The set of most likely equilibria is a set of efficient solutions of

$$\max_{\boldsymbol{x}\in X} \boldsymbol{\alpha}(\boldsymbol{x}).$$

Learning the most likely equilibria

Restrict ourselves to the study of $\boldsymbol{\alpha}$-$JNE(G)$. In this case, we want to find the efficient values of $\phi : X \to [0,1]^I$ defined as

$$\phi(\boldsymbol{x}) = \left(\mathbb{P}\left[u_i(\boldsymbol{x}) \geq u_i(p, \boldsymbol{x}_{-i}),\, p \in P_i\right]\right)_{i \in I}.$$

This is $\phi(\boldsymbol{x}) = \mathsf{E}_{\mathbb{P}}\, \boldsymbol{f}(\boldsymbol{x}) = \mathsf{E}_{\mathbb{P}}\left(\mathbb{I}(u_i(\boldsymbol{x}) \geq u_i(p, \boldsymbol{x}_{-i}),\, p \in P_i)\right)_{i \in I}$.

Learning the most likely equilibria

By a repeated version of G we mean a collection of games with random payoff $\Gamma = \{G^t\}_{t\in\mathbb{N}}$ such that $\forall t \in \mathbb{N} : I^t = I$, $\forall i \in I : X_i^t = X_i$ and $\forall \{\boldsymbol{x}^t\}_{t\in\mathbb{N}} \subset X$ the associated payoff process $\{u_i^t(\boldsymbol{x}^t)\}_{t\in\mathbb{N}}$ is i.i.d..

During the play of the game, players learn the true value of $\mathbb{P}$ by approximating it based on the empirical measure $\mathbb{P}_T = \frac{1}{T}\sum_{t=1}^{T} \delta_{\omega_t}$, that is known at the round $T+1$.

Learning the most likely equilibria

Our question is whether

$$EF(\hat{\phi}_T, X) \to EF(\phi, X),\, T \to \infty,\, \mathbb{P}-a.s., \tag{1}$$

where $\hat{\phi}_T = \mathsf{E}_{\mathbb{P}_T} \boldsymbol{f}$.

Learning the most likely equilibria

If we take $F_0(\boldsymbol{x}, \omega; \boldsymbol{c}) = - < \boldsymbol{c}, \boldsymbol{f}(\boldsymbol{x}, \omega) >$ then from (Römisch, 2003) we have that

$$|\min_{\boldsymbol{x} \in X} \int F_0(\boldsymbol{x}, \omega; \boldsymbol{c}) dP(\omega) - \min_{\boldsymbol{x} \in X} \int F_0(\boldsymbol{x}, \omega; \boldsymbol{c}) dQ(\omega)| \leq d_{\mathcal{F}}(P, Q),$$

where $d_{\mathcal{F}}(P, Q) = \sup_{\boldsymbol{x} \in X} | \int F_0(\boldsymbol{x}, \omega; \boldsymbol{c})(P - Q)(d\omega)|$. Now the question is whether $d_{\mathcal{F}}(\mathbb{P}, \mathbb{P}_T) \to 0$, $T \to \infty$, $\mathbb{P}$ $-a.s.$? Or in other words if the collection of functions $\mathcal{F} = \{F_0(\boldsymbol{x}; \boldsymbol{c}) \,|\, \boldsymbol{x} \in X\}$ is strong Glivenko-Cantelli?

Learning the most likely equilibria

We need to consider the Vapnik-Červonekis dimension of collection of sets $\mathcal{C}_i(p) = \{C_i(\boldsymbol{x}, p) \,|\, \boldsymbol{x} \in X\}$, where $C_i(\boldsymbol{x}, p) := \{\omega \in \Omega \,|\, u_i(\boldsymbol{x}, \omega) < u_i(p, \boldsymbol{x}_{-i}, \omega)\}$. From the properties of the VC dimension, it then holds that finite unions, intersections, and complements of collections with finite VC dimensions have finite VC dimensions. In this way, we may construct $\mathcal{F}$ as a collection of linear combinations of indicators over sets with finite VC dimension. Such collections of functions are strong Glivenko-Cantelli.

What is the Vapnik-Červonekis dimension?

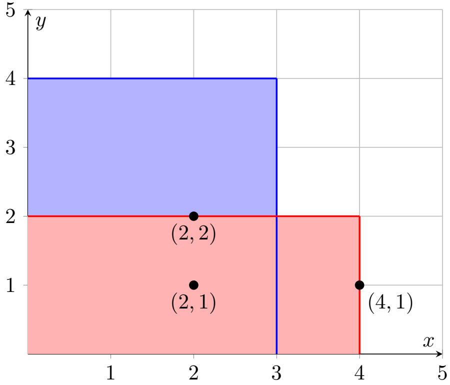

Figure 1: Blue rectangle picks out subset $\{(2,2),(2,1)\}$ from $\{(2,1),(2,2),(4,1)\}$ but it is not possible to find values of x and y so that $[-\infty, x] \times [-\infty, y]$ picks out subset $\{(2,2),(4,1)\}$ therefore system $\mathcal{C} = \{[-\infty, x] \times [-\infty, y] \mid (x, y) \in \mathbb{R}^2\}$ has VC dimension of at most 3.

What is the Vapnik-Červonekis dimension in our case?

We use the Lemma 2.6.15. from (van der Vaart and Wellner, 2023) which states that $VC(\mathcal{V}) \leq \dim \mathcal{V} + 2$. We show that $\dim \mathcal{V}_i(p) < K + 1$ for a collection of functions $\mathcal{V}_i(p) = \{v(\boldsymbol{x}) = u_i(p, \boldsymbol{x}_{-i}) - u_i(\boldsymbol{x}) \,|\, \boldsymbol{x} \in X\}$, where K is the number of pure strategy profiles. Consequently $\mathcal{C}_i(p) = \{\mathcal{V}_i(p) > 0\} = \{\{v > 0\} \,|\, v \in \mathcal{V}_i(p)\}$ has a finite VC dimension.

What is the Vapnik-Červonekis dimension in our case?

We have $\dim X = K$. Suppose that $\dim \mathcal{V}_i(p) \geq K + 1$ this means that there exists $v_1, \ldots, v_{K+1} \in \mathcal{V}_i(p)$ for which

$$\sum_{k=1}^{K+1} a_k v_k = 0 \implies a_1 = \ldots a_{K+1} = 0$$

and so there must be $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{K+1} \in X$ such that $v_1 = v(\boldsymbol{x}_1), \ldots, v_{K+1} = v(\boldsymbol{x}_{K+1})$. From which and linearity of $v(.,.)$ in the first argument, we can find non-trivial $a_1, \ldots, a_{K+1}$ so that

$$0 = v(\mathbf{0}) = v(\sum_{k=1}^{K+1} a_k \boldsymbol{x}_k) = \sum_{k=1}^{K+1} a_k v(\boldsymbol{x}_k) = \sum_{k=1}^{K+1} a_k v_k$$

$\implies a_1 = \cdots = a_{K+1} = 0$. Which is a contradiction, therefore $\dim \mathcal{V}_i(p) < K + 1$.

Back to the original problem

$$\max_{\boldsymbol{x}\in X} \mathsf{E}_{\mathbb{P}_T} < \boldsymbol{c}, \boldsymbol{f}(\boldsymbol{x}) > \to \max_{\boldsymbol{x}\in X} \mathsf{E}_{\mathbb{P}} < \boldsymbol{c}, \boldsymbol{f}(\boldsymbol{x}) >, \, T \to \infty, \, \mathbb{P} - a.s. \quad (2)$$

$$\liminf_{T\to\infty} \arg\max_{\boldsymbol{x}\in X} \mathsf{E}_{\mathbb{P}_T} < \boldsymbol{c}, \boldsymbol{f}(\boldsymbol{x}) > = \arg\max_{\boldsymbol{x}\in X} \mathsf{E}_{\mathbb{P}} < \boldsymbol{c}, \boldsymbol{f}(\boldsymbol{x}) >, \, \mathbb{P} - a.s.. \quad (3)$$

Theorem

Let Γ be a repeated version of the game with random payoff G on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ then (2) and (3) hold.

This is not enough...

Unfortunately, the most likely equilibrium is not well-equipped to be the best notion of stability.

Back to the basics

The classical concept of stability based on the expected payoff

$$\mathsf{E}_{\mathbb{P}}\, u_i(\boldsymbol{x}, \omega) \geq \mathsf{E}_{\mathbb{P}}\, u_i(p, \boldsymbol{x}_{-i}, \omega),\ \forall p \in P_i,\ \forall i \in I. \tag{4}$$

This is a very simple stochastic preference...

Stochastic preference

Definition

We will say that the random preference relation $\succeq^*$ on $L_1(\Sigma)$ is consistent with stochastic dominance relation generated by $\mathcal{G} \subseteq U$ if $X \succeq_{\mathcal{G}} Y$ implies $X \succeq^* Y, \forall X, Y \in L_1(\Sigma)$.

Let $\rho : L_1(\Sigma) \to \mathbb{R}$ be a risk functional and we define the natural ρ-preference as $X \succeq_\rho Y \iff -\rho(-X) \geq -\rho(-Y)$. (Notice that if $\rho = \mathsf{E}$ then we receive exactly the classical approach.)

What are the reasonable choices of ρ?

Definition

We will say that the random preference relation $\succeq^*$ on $L_1(\Sigma)$ is consistent with stochastic dominance relation generated by $\mathcal{G} \subseteq U$ if $X \succeq_{\mathcal{G}} Y$ implies $X \succeq^* Y, \forall X, Y \in L_1(\Sigma)$.

We want to know, for what measures ρ the associated preference is consistent with the FSD?

Distortion measures

It turns out that ρ has to be a distortion measure. For more detail see (Föllmer and Schied, 2016).

Definition

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Non-decreasing function $\psi : [0,1] \to [0,1]$ that satisfies $\psi(0) = 0$, $\psi(1) = 1$ is called distortion function. By the dual distortion function corresponding to ψ we mean $\psi^*(x) := 1 - \psi(1 - x)$, $x \in [0,1]$. The monotone, normalized set function defined on $(\Omega, \mathcal{A})$ as $\Psi(.) = \psi(\mathbb{P}(.))$ is called the distorted version of $\mathbb{P}$ or ψ-distortion of $\mathbb{P}$.

ρ is a distortion measure if it is the 'expectance' with respect to some distorted version of $\mathbb{P}$.

Example of a distortion function

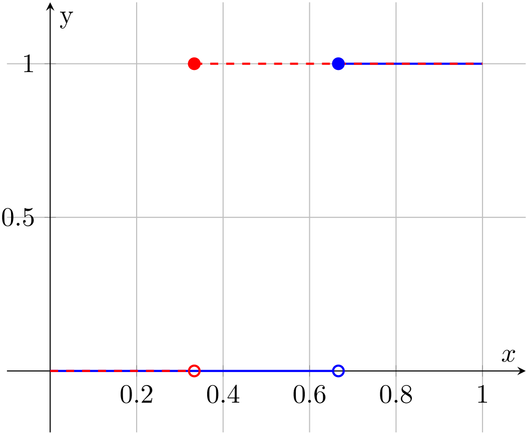

Figure 2: Graph of the distortion function $\psi(x)$ (blue) and its dual $\psi^*(x)$ (red) defined on $[0, 1]$.

Choquet integral

Since Ψ from the previous definition in general isn't a proper measure, we need to define a new type of integral.

Definition (Choquet integral)

Let Ω be an arbitrary set and $\mathcal{A}$ be a collection of subsets of Ω. Finite set function $c : \mathcal{A} \to [0, \infty)$ is called monotone if $\forall A, B \in \mathcal{A} : A \subseteq B$ it holds that $c(A) \leq c(B)$. Let $f : \Omega \to \mathbb{R}$ be any function, the Choquet integral with respect to finite monotone c is defined as

$$\oint f dc := \int_{-\infty}^{0} (c(\{\omega \in \Omega \mid f(\omega) > t\}) - c(\Omega)) dt + \int_{0}^{\infty} c(\{\omega \in \Omega \mid f(\omega) > t\}) dt,$$

where the integrals on the right-hand side are standard Lebesgue integrals.

Choquet integral

If $f \geq 0$ this is simplified to just

$$\oint f dc = \int_0^\infty c(\{\omega \in \Omega \,|\, f(\omega) > t\}) dt.$$

Notice the similarity with

$$\mathsf{E}_{\mathbb{P}}\, X = \int_0^\infty \mathbb{P}\left(\{\omega \in \Omega \,|\, X(\omega) > t\}\right) dt$$

Capacity and measure

In some settings the distorted measure Ψ is a proper probability measure. Namely, we need that ψ is a subadditive function. In general Ψ is just a monotone, normed set function (Capacity).

Some properties of the distorted integration

Theorem

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and Ψ be a distorted version of $\mathbb{P}$ then

1. *If $X \succeq_{FSD} Y$ then $\oint X d\Psi \geq \oint Y d\Psi$*
2. $-\oint X d\Psi = \oint -X d\Psi^*$
3. $\oint X d\Psi = \max_{Q \in \mathcal{Q}} E_Q X \iff \psi$ *is a concave function.*

Consequently we may consider the risk measure $\rho(-X) := \oint -X d\Psi^*$ then clearly $X \succeq_{FSD} Y \implies X \succeq_\rho Y$ because we have

$$-\rho(-X) = -\oint -X d\Psi^* = \oint X d\Psi \geq \oint Y d\Psi = -\oint -Y d\Psi^* = -\rho(-Y)$$

Modules of stability

Definition

Let $G = (I, \{X\}_{i\in I}, \{u_i\}_{i\in I})$ be a game with random payoff on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Mapping $M : [0,1]^I \times X \to [0,1]$ that satisfies following:

1. For any player $i \in I$ it holds that $\forall \boldsymbol{a}_{-i} \in [0,1]^{-i} : \lim_{a_i \to 0} M(a_i, \boldsymbol{a}_{-i}, .) = M(0, \boldsymbol{a}_{-i}, .) = 0$,
2. M is non-decreasing and right-continuous as a function of the confidence levels,
3. if for some $\boldsymbol{x} \in X$, $i \in I$ and $y_i \in X_i$ it holds $\boldsymbol{\alpha}(\boldsymbol{x}) = \boldsymbol{\alpha}((y_i, \boldsymbol{x}_{-i})) = \boldsymbol{a} \in [0,1]^I$ and we define $\mathcal{G}(\boldsymbol{x}_{-i}, \mathbb{P}) = \{u_i(., \boldsymbol{x}_{-i}, \omega) \,|\, \omega \in Supp(\mathbb{P})\}$ then if $\forall u_i \in \mathcal{G}(\boldsymbol{x}_{-i}, \mathbb{P}) : u_i(\boldsymbol{x}) \geq u_i(y, \boldsymbol{x}_{-i})$ then $M(\boldsymbol{a}, \boldsymbol{x}) \geq M(\boldsymbol{a}, (y_i, \boldsymbol{x}_{-i}))$.

is called the Module of stability of the game G.

Most stable equilibria

We may define the most stable equilibrium as $\boldsymbol{x} \in \boldsymbol{\alpha}\text{-}NE(G)$ that maximizes $M(\boldsymbol{\alpha}(\boldsymbol{x}), \boldsymbol{x})$.
The idea is to create a stochastically robust analog of the variational inequalities to ensure that our model doesn't predict unreasonable equilibria which are never the best responses.

Some more structure for the module?

We want to especially consider modules of the type $M(\boldsymbol{\alpha}(\boldsymbol{x}), \boldsymbol{x}) = \prod_{i \in I} M_i(\alpha_i(\boldsymbol{x}), \boldsymbol{x})$ and more specifically $M_i(\alpha_i, \boldsymbol{x}) := f_i(\alpha_i, \oint u_i(\boldsymbol{x}) d\Psi_i)$ for some distortion measure $\Psi_i(.) = \psi_i(\mathbb{P}(.))$ and $f_i : [0,1] \times [0,1] \to [0,1]$ is non-decreasing in both arguments and continuous and vanishing in 0. f_i may be interpreted as an endogenous probability that the player i will choose to play x_i, given they know that the probability of x_i beating $\boldsymbol{x}_{-i}$ is α_i and their prospect resulting from $\boldsymbol{x}$ is $\oint u_i(\boldsymbol{x}) d\Psi_i$.

Open questions

- What are the properties of the mapping $\oint(.)\psi(.) : X \times \mathcal{D}(\Omega) \to \mathbb{R}$ which maps $(\boldsymbol{x}, Q) \mapsto \oint u_i(\boldsymbol{x}, \omega) d\psi(Q(\omega))$?
- How to design the module of stability for a given use?
- Can we construct a sufficient and necessary condition on the existence of the most stable equilibrium?
- Can we connect the deterministic equivalents based on distortion measures with the $\boldsymbol{\alpha}$-NE?

References

Hans Föllmer and Alexander Schied. *Stochastic Finance : An Introduction in Discrete Time*. De Gruyter Textbook. De Gruyter,, Berlin ;, 4th rev. ed. edition, 2016. ISBN 3-11-046346-6.

Werner Römisch. Stability of stochastic programming problems. In *Handbooks in Operations Research and Management Science*, volume 10, pages 483–554. Elsevier B.V, 2003. ISBN 9780444508546.

A. W van der Vaart and Jon A Wellner. *Weak Convergence and Empirical Processes: With Applications to Statistics*. Springer Series in Statistics. Springer International Publishing AG, Cham, second edition. edition, 2023. ISBN 9783031290381.