1. Consider a linearised homogeneous isotropic elastic solid, that is a continuous medium where the (linearised) stress tensor is given by the formula

$$\mathbb{T} = \lambda \left(\operatorname{Tr} \mathbb{E}\right) \mathbb{I} + 2\mu\mathbb{E},$$

where $\mathbb{E} =_{\text{def}} \frac{1}{2}\left(\nabla \boldsymbol{U} + \left(\nabla \boldsymbol{U}\right)^{\top}\right)$ is the linearised strain. Assume the displacement field in the form

$$\boldsymbol{U} = \begin{bmatrix} \mathrm{U}^{\hat{X}}(X,Y) \\ \mathrm{U}^{\hat{Y}}(X,Y) \\ 0 \end{bmatrix}.$$

- Find the corresponding linearised strain $\mathbb{E}$ and show that the strain has nonzero components only in X and Y plane. (The strain is effectively restricted to $\mathbb{R}^2$.)
- Find an explicit formula for the corresponding stress tensor $\mathbb{T}$ in terms of $\mathrm{U}^{\hat{X}}$ and $\mathrm{U}^{\hat{Y}}$. Is it true that the stress tensor has also nonzero components only in X and Y plane?
- Show that the (linearised) governing equations (no specific body force)

$$\operatorname{div} \mathbb{T} = \mathbf{0},$$

for a steady state in $\mathbb{R}^3$ reduce to two nontrivial equations,

$$\frac{\partial \tau^{\hat{X}\hat{X}}}{\partial X} + \frac{\partial \tau^{\hat{X}\hat{Y}}}{\partial Y} = 0, \tag{1a}$$

$$\frac{\partial \tau^{\hat{Y}\hat{X}}}{\partial X} + \frac{\partial \tau^{\hat{Y}\hat{Y}}}{\partial Y} = 0. \tag{1b}$$

- Show that if the stress $\mathbb{T}$ is generated by the means of Airy stress function ψ,

$$\mathbb{T} =_{\text{def}} \begin{bmatrix} \frac{\partial^2 \psi}{\partial Y^2} & -\frac{\partial^2 \psi}{\partial X \partial Y} & \cdot \\ -\frac{\partial^2 \psi}{\partial X \partial Y} & \frac{\partial^2 \psi}{\partial X^2} & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix},$$

then equations (1) are automatically fulfilled.
- Use the compatibility conditions for linearised strain tensor $\mathbb{E}$ in $\mathbb{R}^2$, and show that the compatibility conditions imply

$$\Delta\Delta\psi = 0.$$

The moral of this example is the following. The governing equations for plane strain problems can be, in some cases, converted to a single linear partial differential equation for scalar quantity ψ.