1. Consider a hollow cylinder of initial inner radius R_{in} and outer radius R_{out}, see Figure 1, and assume that the cylinder is in this configuration in a stress free state. Further, assume that the material of which is the cylinder made is a homogeneous isotropic *incompressible* elastic material specified by constitutive relation

$$\mathbb{T} = -p\mathbb{I} + \mu\left(\mathbb{B} - \mathbb{I}\right),$$

where μ is a positive constant and $\mathbb{B}$ denotes the left Cauchy–Green tensor with respect to the initial configuration with the inner radius R_{in} and the outer radius R_{out}.

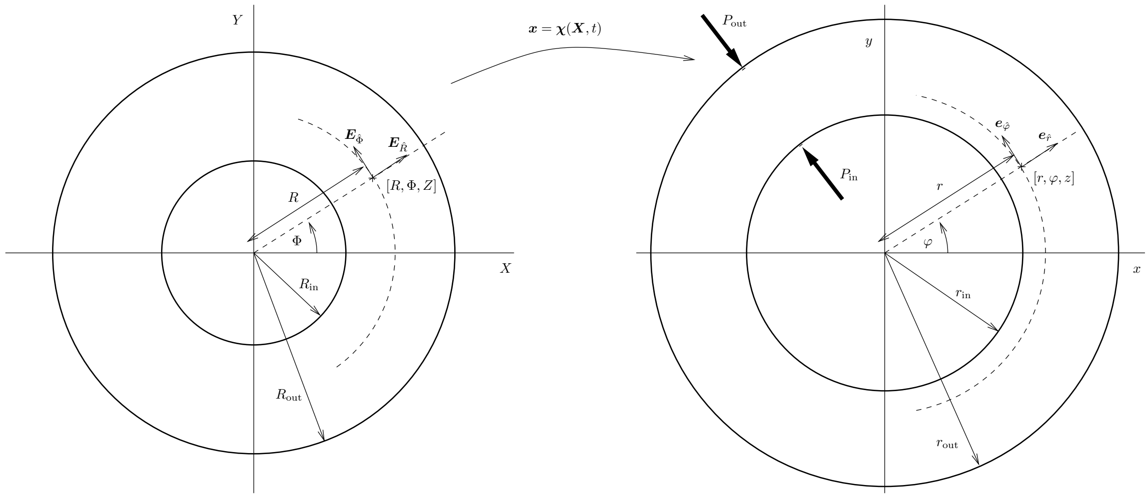

Figure 1: Inflation of a hollow cylinder made of an incompressible elastic material.

Let us now apply a pressure P_{in} inside the cylinder and a pressure P_{out} outside the cylinder. If the inner pressure is higher than the outer pressure, then the cylinder inflates. The task is to find a relation between the relative change in the void area

$$c =_{\text{def}} \frac{r_{\text{in}}^2 - R_{\text{in}}^2}{R_{\text{in}}^2}$$

and the pressure difference $P_{\text{in}} - P_{\text{out}}$.

Find the answer using *linearised elasticity* theory, that is use the governing equations in the form

$$\operatorname{div} \mathbb{\tau} = \mathbf{0},$$
$$\operatorname{Tr}\left(\nabla \boldsymbol{U}\right) = 0,$$

where $\mathbb{\tau} =_{\text{def}} -p\mathbb{I} + 2\mu\mathbb{\epsilon}$ and $\mathbb{\epsilon} =_{\text{def}} \frac{1}{2}\left(\nabla \boldsymbol{U} + \left(\nabla \boldsymbol{U}\right)^{\top}\right)$. The boundary conditions read

$$\mathbb{\tau}\boldsymbol{E}_{\hat{Z}}\big|_{R=R_{\text{in}}} = P_{\text{in}}\boldsymbol{E}_{\hat{Z}}\big|_{R=R_{\text{in}}},$$
$$\mathbb{\tau}\boldsymbol{E}_{\hat{Z}}\big|_{R=R_{\text{out}}} = P_{\text{out}}\boldsymbol{E}_{\hat{Z}}\big|_{R=R_{\text{out}}},$$

while the deformation is assumed to take the form

$$r = f(R),$$
$$\varphi = \Phi,$$
$$z = Z.$$

The result should be identical to the result

$$P_{\text{out}} - P_{\text{in}} \approx \mu \int_{r=R_{\text{in}}}^{R_{\text{out}}} \frac{2cR_{\text{in}}^2}{r^3}\,\mathrm{d}r$$

that we have already obtained via linearisation of the solution to the complete system of nonlinear governing equations.

The formula for the divergence of a tensorial quantity $\mathbb{A}$ in the cylindrical coordinate system reads

$$
\operatorname{div} \mathbb{A} = \begin{bmatrix}
\frac{\partial \mathrm{A}^{\hat{r}}{}_{\hat{r}}}{\partial r} + \frac{1}{r}\left(\frac{\partial \mathrm{A}^{\hat{r}}{}_{\hat{\varphi}}}{\partial \varphi} - \mathrm{A}^{\hat{\varphi}}{}_{\hat{\varphi}} + \mathrm{A}^{\hat{r}}{}_{\hat{r}}\right) + \frac{\partial \mathrm{A}^{\hat{r}}{}_{\hat{z}}}{\partial z} \\
\frac{\partial \mathrm{A}^{\hat{\varphi}}{}_{\hat{r}}}{\partial r} + \frac{1}{r}\left(\frac{\partial \mathrm{A}^{\hat{\varphi}}{}_{\hat{\varphi}}}{\partial \varphi} + \mathrm{A}^{\hat{r}}{}_{\hat{\varphi}} + \mathrm{A}^{\hat{\varphi}}{}_{\hat{r}}\right) + \frac{\partial \mathrm{A}^{\hat{\varphi}}{}_{\hat{z}}}{\partial z} \\
\frac{\partial \mathrm{A}^{\hat{z}}{}_{\hat{r}}}{\partial r} + \frac{1}{r}\left(\frac{\partial \mathrm{A}^{\hat{z}}{}_{\hat{\varphi}}}{\partial \varphi} + \mathrm{A}^{\hat{z}}{}_{\hat{r}}\right) + \frac{\partial \mathrm{A}^{\hat{z}}{}_{\hat{z}}}{\partial z}
\end{bmatrix}.
$$