1. Consider the deformation $\boldsymbol{\chi}$ given by the following formulae

$$
\begin{aligned}
r &= f(R), \\
\varphi &= \Phi, \\
z &= Z.
\end{aligned}
$$

This means that the deformation $\boldsymbol{\chi}$ is given as a function that takes a point with the coordinates $[R, \Phi, Z]$ in the reference configuration—with respect to the cylindrical coordinate system—and returns the position of that point in terms of polar coordinates in the current configuration, see Figure 1.

Show that the deformation gradient $\mathbb{F}$ is given by the formula

$$
\mathbb{F} = \frac{\mathrm{d}f}{\mathrm{d}R} \boldsymbol{e}_{\hat{r}} \otimes \boldsymbol{E}_{\hat{R}} + \frac{f}{R} \boldsymbol{e}_{\hat{\varphi}} \otimes \boldsymbol{E}_{\hat{\Phi}} + \boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}_{\hat{Z}}.
$$

that is

$$
\mathbb{F} = \begin{bmatrix} \frac{\mathrm{d}f}{\mathrm{d}R} & 0 & 0 \\ 0 & \frac{f}{R} & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

In other words, show that if we have a vector $\boldsymbol{V} = \mathrm{V}^{\hat{R}} \boldsymbol{E}_{\hat{R}} + \mathrm{V}^{\hat{\Phi}} \boldsymbol{E}_{\hat{\Phi}} + \mathrm{V}^{\hat{Z}} \boldsymbol{E}_{\hat{Z}}$ in the reference configuration and a corresponding vector $\boldsymbol{c} = \mathrm{v}^{\hat{r}} \boldsymbol{e}_{\hat{r}} + \mathrm{v}^{\hat{\varphi}} \boldsymbol{e}_{\hat{\varphi}} + \mathrm{v}^{\hat{z}} \boldsymbol{e}_{\hat{z}}$ in the current configuration, then the relation between the components of the vectors reads

$$
\begin{bmatrix} \mathrm{v}^{\hat{r}} \\ \mathrm{v}^{\hat{\varphi}} \\ \mathrm{v}^{\hat{z}} \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}f}{\mathrm{d}R} & 0 & 0 \\ 0 & \frac{f}{R} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{V}^{\hat{R}} \\ \mathrm{V}^{\hat{\Phi}} \\ \mathrm{V}^{\hat{Z}} \end{bmatrix}.
$$

(Recall that by the vector we mean an infinitesimal line segment placed at the given point, or, more precisely it is a tangent vector to the corresponding curve passing through the given point.)

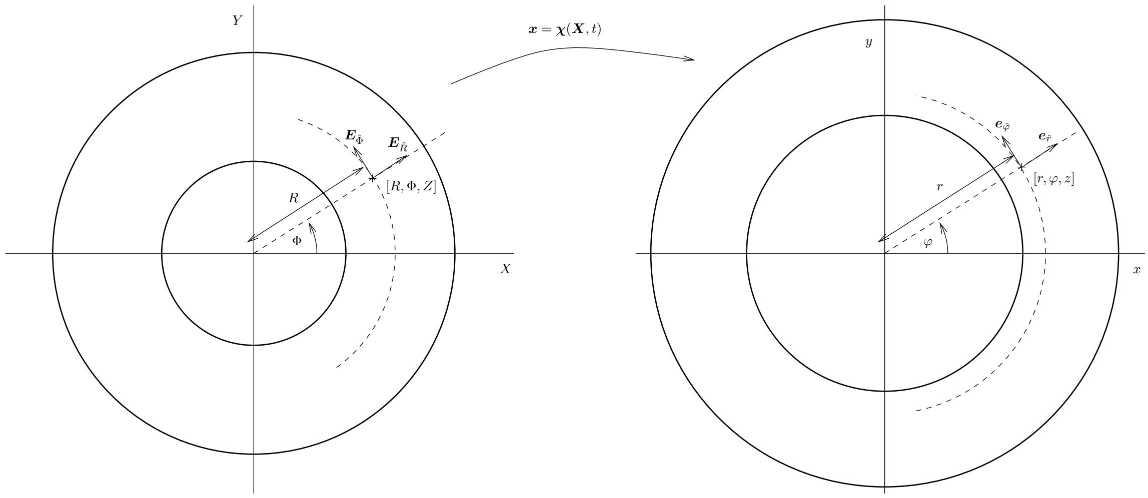

Figure 1: Problem geometry.

2. Prove the following lemma. Let $\boldsymbol{v}$ be a smooth vector field in $\Omega \subset \mathbb{R}^3$, and let $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary. Then there exist a scalar field φ and a vector field $\boldsymbol{A}$ such that

$$
\boldsymbol{v} = -\nabla \varphi + \operatorname{rot} \boldsymbol{A}.
$$

Function φ is called the *scalar potential* and function $\boldsymbol{A}$ is called the *vector potential* of the vector field $\boldsymbol{v}$.

Remark: The decomposition of $\boldsymbol{v}$ is called the *Helmholtz decomposition.* If necessary, you can look up the proof in your favourite book on vector calculus.