1. Let $\varphi$, $\psi$, $\boldsymbol{u}$, $\boldsymbol{v}$ and $\mathbb{A}$ be smooth scalar, vector and tensor fields in $\mathbb{R}^3$. Show that

$$
\begin{aligned}
\operatorname{div}(\varphi \boldsymbol{v}) &= \boldsymbol{v} \bullet (\nabla \varphi) + \varphi \operatorname{div} \boldsymbol{v}, \\
\operatorname{div}(\boldsymbol{u} \times \boldsymbol{v}) &= \boldsymbol{v} \bullet \operatorname{rot} \boldsymbol{u} - \boldsymbol{u} \bullet \operatorname{rot} \boldsymbol{v}, \\
\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{v}) &= [\nabla \boldsymbol{u}]\, \boldsymbol{v} + \boldsymbol{u} \operatorname{div} \boldsymbol{v}, \\
\operatorname{div}(\varphi \mathbb{A}) &= \mathbb{A} (\nabla \varphi) + \varphi \operatorname{div} \mathbb{A}.
\end{aligned}
$$

Further, show that

$$
\begin{aligned}
\nabla (\varphi \psi) &= \psi \nabla \varphi + \varphi \nabla \psi, \\
\nabla (\varphi \boldsymbol{v}) &= \boldsymbol{v} \otimes \nabla \varphi + \varphi \nabla \boldsymbol{v}, \\
\nabla (\boldsymbol{u} \bullet \boldsymbol{v}) &= (\nabla \boldsymbol{u})^\top \boldsymbol{v} + (\nabla \boldsymbol{v})^\top \boldsymbol{u}, \\
\operatorname{rot}(\varphi \boldsymbol{v}) &= \varphi \operatorname{rot} \boldsymbol{v} - \boldsymbol{v} \times \nabla \varphi.
\end{aligned}
$$

2. Let $\mathbb{U} \in \mathbb{R}^{3\times3}$ be a symmetric positive definite matrix, and let $\mathbb{A} \in \mathbb{R}^{3\times3}$ be a matrix. Show that the solution $\mathbb{X}$ of the matrix equation

$$\mathbb{X}\mathbb{U} + \mathbb{U}\mathbb{X} = \mathbb{A},$$

is given by the formula

$$\mathbb{X} = \int_{u=0}^{+\infty} \mathrm{e}^{-u\mathbb{U}} \mathbb{A} \mathrm{e}^{-u\mathbb{U}} \, \mathrm{d}u.$$

3. Consider a vector field $\boldsymbol{Y}$ in an open simply connected domain in $\mathbb{R}^2$, where the coordinates are labelled by $[T, V]$, that is $[T, V] \in \mathbb{R}^2$. The vector field $\boldsymbol{Y}$ has the form

$$\boldsymbol{Y} = \begin{bmatrix} -c_{\mathrm{V}} \\ -n\frac{R_{\mathrm{m}} T}{V} \end{bmatrix},$$

where $n$, $c_{\mathrm{V}}$ and $R_{\mathrm{m}}$ are positive constants. Is it true that there exists a potential $\phi(T, V)$ such that $\boldsymbol{Y} = \nabla \phi$, that is $\boldsymbol{Y} = \begin{bmatrix} \frac{\partial \phi}{\partial T} & \frac{\partial \phi}{\partial V} \end{bmatrix}^\top$? (Check the condition $\operatorname{rot} \boldsymbol{Y} = \boldsymbol{0}$. How would you use this condition for vector fields in $\mathbb{R}^2$?)

Calculate explicitly the line integral

$$\int_\gamma \boldsymbol{Y} \bullet \mathrm{d}\boldsymbol{l}$$

for two different curves $\gamma_A$ and $\gamma_B$ connecting the points $[T_1, V_1]$ and $[T_2, V_2]$ in the $TV$ space. The first curve $\gamma_A$ is composed of two line segments $\gamma_1$ and $\gamma_2$, while the other curve is composed of two line segments $\gamma_3$ and $\gamma_4$, see Figure 1. Show that the values of line integrals taken along these curves are indeed different.

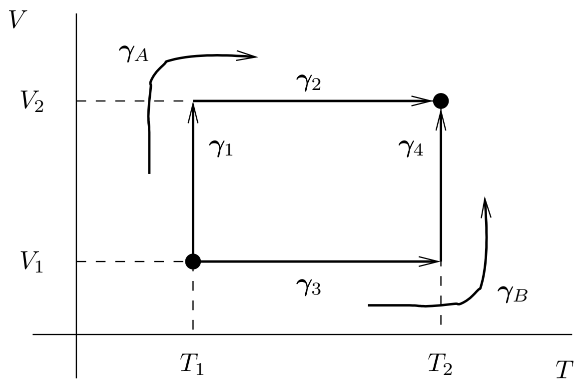


Figure 1: Different paths (curves) in $TV$ space.

Further, is it true that there exists a potential $\psi(T, V)$ such that $\frac{1}{T}\boldsymbol{Y} = \nabla \psi$?

What is the meaning of all of this? We are dealing with an ideal gas with the equation of state $\frac{PV}{T} = nR_{\mathrm{m}}$ (Boyle–Mariotte–Charles) and the internal energy $E = c_{\mathrm{V}}T$ (Joule–Thomson). The gas occupies a cylinder of volume $V$ that is kept at temperature $T$. As the gas undergoes a process that takes it form the state $[T_1, V_1]$ to the state $[T_2, V_2]$, it releases/absorbs heat. For whatever reason the process is assumed to not to induce a substantial inhomogeneity in the gas (no macroscopic motion, no temperature gradients, no spatial variations of density).

The amount of the released/absorbed heat is then given as a sum of the change of the internal energy and the work done/consumed by the gas. (This is the first law of thermodynamics.) As one might easily check, the sum of the change in the internal energy and the work done/consumed by the gas is given by the line integral $\int_{\gamma} \boldsymbol{Y} \bullet \mathrm{d}\boldsymbol{l}$. (The first term in the scalar product in the change in the internal energy, while the other term is the work done/consumed. The curves are parametrised by the variable that is called the time.) Consequently, the line integral represents the heat released/absorbed by the gas.

Now the gas goes from the state $[T_1, V_1]$ to the state $[T_2, V_2]$ by the means of two different processes. Curve $\boldsymbol{\gamma}_A$ represents the isothermal process followed by the isochoric process, while curve $\boldsymbol{\gamma}_A$ represents the isochoric process followed by the isothermal process. You have found that the amount of heat released/absorbed *depends* on the process. In particular, it can not be determined only from knowledge of the initial and the final state.

However, if one deals with a new quantity called "reduced heat" something happens. (The "reduced heat" is by definition the consumed/released heat divided by the temperature at which it has been consumed/released.) One miraculously finds that the "reduced heat" is independent of the process. What is the different name for the "reduced heat"?