1. Let $\mathbb{A} \in \mathbb{R}^{3\times 3}$ be a skew-symmetric matrix

$$\mathbb{A} =_{\text{def}} \begin{bmatrix} 0 & \mathrm{A}_{12} & \mathrm{A}_{13} \\ -\mathrm{A}_{12} & 0 & \mathrm{A}_{23} \\ -\mathrm{A}_{13} & -\mathrm{A}_{23} & 0 \end{bmatrix},$$

such that $\mathrm{A}_{12}^2 + \mathrm{A}_{13}^2 + \mathrm{A}_{23}^2 = 1$, and let $\varphi \in [0, 2\pi)$ be an arbitrary number. Show that

$$\mathrm{e}^{\varphi \mathbb{A}} = \mathbb{I} + (\sin\varphi)\,\mathbb{A} + (1 - \cos\varphi)\,\mathbb{A}^2.$$

(Cayley–Hamilton theorem might be useful.) Note the striking similarity of the result with our formula for the determination of the rotation matrix from the axis/angle data. This is not a coincidence. (Skew-symmetric matrices are infinitesimal generators of the group of rotations in $\mathbb{R}^3$.)

2. Show that

$$\begin{aligned}
\frac{\partial^2 \mathrm{I}_1(\mathbb{A})}{\partial \mathbb{A}^2}[\mathbb{B}, \mathbb{C}] &= 0, \\
\frac{\partial^2 \mathrm{I}_2(\mathbb{A})}{\partial \mathbb{A}^2}[\mathbb{B}, \mathbb{C}] &= (\mathrm{Tr}\,\mathbb{C})\,(\mathrm{Tr}\,\mathbb{B}) - \mathrm{Tr}\,(\mathbb{C}\mathbb{B})\,, \\
\frac{\partial^2 \mathrm{I}_3(\mathbb{A})}{\partial \mathbb{A}^2}[\mathbb{B}, \mathbb{C}] &= (\det \mathbb{A})\left(\mathrm{Tr}\left(\mathbb{A}^{-1}\mathbb{B}\right)\mathrm{Tr}\left(\mathbb{A}^{-1}\mathbb{C}\right) - \mathrm{Tr}\left(\mathbb{A}^{-1}\mathbb{B}\mathbb{A}^{-1}\mathbb{C}\right)\right),
\end{aligned}$$

where $\mathrm{I}_1(\mathbb{A})$, $\mathrm{I}_2(\mathbb{A})$ and $\mathrm{I}_3(\mathbb{A})$ denote the principal invariants of matrix $\mathbb{A}$, that is

$$\begin{aligned}
\mathrm{I}_1(\mathbb{A}) &=_{\text{def}} \mathrm{Tr}\,\mathbb{A}, \\
\mathrm{I}_2(\mathbb{A}) &=_{\text{def}} \frac{1}{2}\left((\mathrm{Tr}\,\mathbb{A})^2 - \mathrm{Tr}\left(\mathbb{A}^2\right)\right), \\
\mathrm{I}_3(\mathbb{A}) &=_{\text{def}} \det \mathbb{A}.
\end{aligned}$$

Please note that $\frac{\partial \mathfrak{f}(\mathbb{A})}{\partial \mathbb{A}}[\mathbb{B}]$ is just another notation for Gâteaux derivative, that is

$$\frac{\partial \mathfrak{f}(\mathbb{A})}{\partial \mathbb{A}}[\mathbb{B}] =_{\text{def}} \mathrm{D}_{\mathfrak{f}}(\mathbb{A})\,[\mathbb{B}]\,.$$