1. Let $\mathbb{A} \in \mathbb{R}^{3\times 3}$ be an invertible matrix. Show that

$$\mathrm{I}_2(\mathbb{A}) = \mathrm{Tr}\,(\mathrm{cof}\,\mathbb{A}) \,,$$

where $\mathrm{cof}\,\mathbb{A} =_{\mathrm{def}} (\det \mathbb{A})\, \mathbb{A}^{-\top}$ denotes the cofactor matrix of matrix $\mathbb{A}$, and $\mathrm{I}_2(\mathbb{A}) =_{\mathrm{def}} \frac{1}{2}\left((\mathrm{Tr}\,\mathbb{A})^2 - \mathrm{Tr}\left(\mathbb{A}^2\right)\right)$ is the second invariant of matrix $\mathbb{A}$.

2. Let $\mathbb{A} \in \mathbb{R}^{3\times 3}$ a $\mathbb{B} \in \mathbb{R}^{3\times 3}$ be invertible matrices. Show that

$$\det(\mathbb{A} + \mathbb{B}) = \det \mathbb{A} + \mathrm{Tr}\left(\mathbb{A}^\top \mathrm{cof}\,\mathbb{B}\right) + \mathrm{Tr}\left(\mathbb{B}^\top \mathrm{cof}\,\mathbb{A}\right) + \det \mathbb{B},$$

where $\mathrm{cof}\,\mathbb{C} =_{\mathrm{def}} (\det \mathbb{C})\, \mathbb{C}^{-\top}$ denotes the cofactor matrix of matrix $\mathbb{C}$.

3. Let us assume that the curve $\boldsymbol{\gamma}$ in $\mathbb{R}^2$ is given in terms of polar coordinates $[r, \varphi]$, see Figure 1, which means that

$$\boldsymbol{\gamma} : s \in (a,b) \mapsto \begin{bmatrix} r(s)\cos\varphi(s) \\ r(s)\sin\varphi(s) \end{bmatrix},$$

where s is the parametrisation of the curve. Further let us assume that $f : \boldsymbol{x} = [x,y] \in \mathbb{R}^2 \mapsto f(\boldsymbol{x}) \in \mathbb{R}$ is a given function. Show that

$$\int_{\boldsymbol{\gamma}} f(\boldsymbol{x})\,\mathrm{d}l = \int_{s=a}^{b} f(r(s)\cos\varphi(s), r(s)\sin\varphi(s)) \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + r^2 \left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2}\,\mathrm{d}s.$$

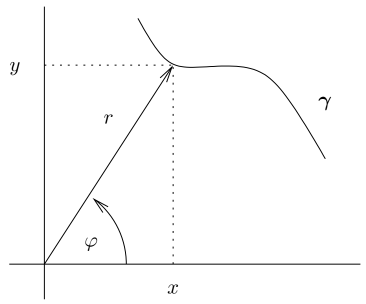

Figure 1: Line integral in $\mathbb{R}^2$ – parametrisation in polar coordinates.