Efficient linear semi-implicit finite element scheme for fluid-shell interaction

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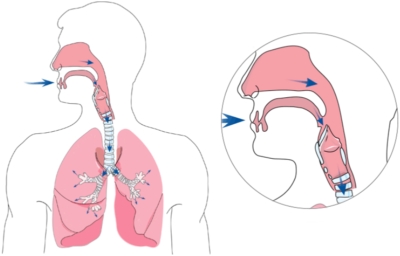
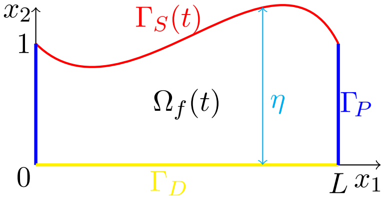



Problem formulation – time dependent domain

- Time dependent domain

$$\Omega_f(t) = \{\mathbf{x} = (x_1, x_2) \in \Sigma \times (0, \eta(t, x_1))\} \subset \mathbb{R}^2,$$

$$\Sigma = (0, L)$$



- Incompressible Newtonian fluid in $\Omega_f(t)$
- Thin elastic structure on $\Gamma_S(t)$

Material description

Incompressible Newtonian fluid

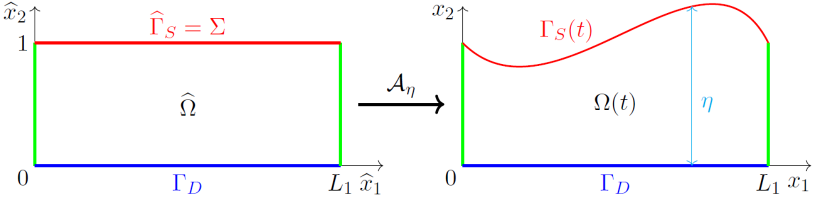
$$\begin{aligned} \operatorname{div} \mathbf{u} &= 0, \\ \rho_f \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla \mathbf{u}) \right) &= \operatorname{div} \mathbb{T}, \\ \mathbb{T} &= -p\mathbb{I} + 2\mu \mathbb{D}(\mathbf{u}). \end{aligned}$$

Thin elastic structure

$$\begin{aligned} \rho_s \frac{\partial \xi}{\partial t} + \mathcal{L}(\eta) = f, \quad \xi = \frac{\partial \eta}{\partial t}, \\ \mathcal{L}(\eta) = -\gamma_1 \Delta_{x_1} \eta - \gamma_2 \Delta_{x_1} \zeta - \gamma_3 \Delta_{x_1} \xi, \quad \zeta = -\Delta_z \eta. \end{aligned}$$

Coupling fluid and structure

- ALE mapping $\mathcal{A}_\eta$.
- Its Jacobian $\mathcal{F}$ and determinant $J = \det \mathbb{F}$.
- Reformulate everything into the fixed configuration $\hat{\Omega}$.



Coupling conditions

$$\text{Kinematic coupling :} \qquad \mathbf{u} = \xi \mathbf{e}_2,$$

$$\text{Dynamic coupling :} \qquad f = -\mathbf{e}_2 \cdot \left(J(\mathbb{T} \circ \mathcal{A})\mathbb{F}^{-\mathrm{T}} \right) \mathbf{e}_2.$$

Weak formulation of FSI on Ω_η

Let us define

$$W_\eta = \{(\boldsymbol{\varphi}, \boldsymbol{\psi}) \in W^{1,2}(\Omega_\eta) \times L^2(\Sigma) : \psi(x)\mathbf{e}_2 = \boldsymbol{\varphi}(x, \eta(x)), \boldsymbol{\varphi} = \mathbf{0} \text{ on } \Gamma_{\mathrm{D}}\}.$$

Definition

Let $(p, \mathbf{u}, \xi, \eta)$ be a solution to the coupled FSI problem. The weak form then reads

$$\int_{\Omega_\eta} \operatorname{div} \mathbf{u}\, q \,\mathrm{d}x = 0 \quad \text{for all } q \in L^2(\Omega_\eta)$$

$$\rho_f \int_{\Omega_\eta} \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{w} \cdot \nabla)\mathbf{u} + \operatorname{div} \mathbf{w}\, \frac{\mathbf{u}}{2} \right) \cdot \boldsymbol{\varphi} \,\mathrm{d}x + \frac{\rho_f}{2} \int_{\Omega_\eta} (\boldsymbol{\varphi} \cdot (\nabla \mathbf{u}) - \mathbf{u} \cdot (\nabla \boldsymbol{\varphi})) \cdot \mathbf{v} \,\mathrm{d}x + \int_{\Omega_\eta} \mathbb{T} \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x + \rho_s \int_\Sigma \frac{\partial \xi}{\partial t} \psi \,\mathrm{d}x_1 + a_s(\eta, \zeta, \xi, \psi) = 0$$

for all $(\boldsymbol{\varphi}, \psi) \in W_\eta$, where $\mathbf{w}$ is the speed of deformation, $\mathbf{v} = \mathbf{u} - \mathbf{w}$ and

$$a_s(\eta, \zeta, \xi, \psi) = \int_\Sigma \left(\gamma_1 \frac{\partial \eta}{\partial x_1} \frac{\partial \psi}{\partial x_1} + \gamma_2 \frac{\partial \zeta}{\partial x_1} \frac{\partial \psi}{\partial x_1} + \gamma_3 \frac{\partial \xi}{\partial x_1} \frac{\partial \psi}{\partial x_1} \right) \mathrm{d}x_1.$$

Weak formulation of FSI on $\hat{\Omega}$

Definition

Let $(p, \mathbf{u}, \xi, \eta)$ satisfy the weak formulation on Ω_η with the test functions $(q, \boldsymbol{\varphi}, \psi) \in L^2 \times W_\eta$. Let $(\hat{p}, \hat{q}, \hat{\mathbf{u}}, \hat{\boldsymbol{\varphi}}) = (p, q, \mathbf{u}, \boldsymbol{\varphi}) \circ \mathcal{A}_\eta$. Then it holds

$$\int_{\hat{\Omega}} J \nabla \hat{\mathbf{u}} \cdot \mathbb{F}^{-\mathrm{T}} \, \hat{q} \, \mathrm{d}\hat{x} = 0,$$

$$\rho_f \int_{\hat{\Omega}} \left(J \frac{\partial \hat{\mathbf{u}}}{\partial t} + \frac{\partial J}{\partial t} \frac{\hat{\mathbf{u}}}{2} \right) \cdot \boldsymbol{\varphi} \, \mathrm{d}\hat{x} + \frac{\rho_f}{2} \int_{\hat{\Omega}} J \left(\hat{\boldsymbol{\varphi}} \cdot (\nabla \hat{\mathbf{u}}) - \hat{\mathbf{u}} \cdot (\nabla \hat{\boldsymbol{\varphi}}) \right) \cdot \mathbb{F}^{-1} \hat{\mathbf{v}} \, \mathrm{d}\hat{x} + \int_{\hat{\Omega}} J \hat{\mathbb{T}} \mathbb{F}^{-T} \cdot \nabla \hat{\boldsymbol{\varphi}} \, \mathrm{d}\hat{x} + \rho_s \int_{\Sigma} \frac{\partial \xi}{\partial t} \psi \, \mathrm{d}x_1 + a_s(\eta, \zeta, \xi, \psi) = 0.$$

Finite element method on $\hat{\Omega}$

- Numerical approximation denoted by $(\hat{p}_h^k, \hat{\mathbf{u}}_h^k, \hat{\eta}_h^k, \hat{\xi}_h^k)$ at time t^k.
- Time step τ, $t^k = k\tau$.
- Backward Euler

$$D_t v_h^k = \frac{v_h^k - v_h^{k-1}}{\tau}.$$

- Ω_h triangulated uniformly.
- Pair $(\hat{\mathbf{u}}_h^k, \hat{p}_h^k) \in \hat{V}_h^f \times \hat{Q}_h^f$ inf-sup stable mini elements (P1-bubble + P1) .
- Unknowns $\hat{\eta}_h^k, \hat{\xi}_h^k$ P1 elements ($\hat{V}_h^s$).

$$\hat{V}_h^{\text{fsi}} = \{(\hat{\boldsymbol{\varphi}}, \hat{q}, \psi) \in \hat{V}_h^f \times \hat{Q}_h^f \times \hat{V}_h^s : \hat{\boldsymbol{\varphi}}(x_1, 1) = \psi(x_1)\}$$

Linear monolithic scheme on $\hat{\Omega}$

Definition

For $k = 1, \dots, N$ we seek $(\hat{\mathbf{u}}_h^k, \hat{p}_h^k, \hat{\xi}_h^k, \hat{\eta}_h^k) \in \hat{V}_h^{\mathrm{fsi}} \times \hat{V}_h^s$ with $\hat{\xi}_h^k = D_t \eta_h^k$ such that for all $(\hat{\boldsymbol{\varphi}}, \hat{q}, \psi) \in \hat{V}_h^{\mathrm{fsi}}$ it holds

$$\int_{\hat{\Omega}} J_h^{k-1} \nabla \hat{\mathbf{u}}_h^k \cdot (\mathbb{F}_h^{k-1})^{-\mathrm{T}} \, \hat{q} \, \mathrm{d}\hat{x} = 0,$$

$$\begin{aligned}
\rho_f \int_{\hat{\Omega}} \left(J_h^{k-1} D_t \hat{\mathbf{u}}_h^k + D_t J_h^{k-1} \frac{2\hat{\mathbf{u}}_h^{k-1} - \hat{\mathbf{u}}_h^k}{2} \right) \cdot \boldsymbol{\varphi} \, \mathrm{d}\hat{x} +& \\
\frac{\rho_f}{2} \int_{\hat{\Omega}} J_h^{k-1} \left(\hat{\boldsymbol{\varphi}} \cdot (\nabla \hat{\mathbf{u}}_h^k) - \hat{\mathbf{u}}_h^k \cdot (\nabla \hat{\boldsymbol{\varphi}}) \right) \cdot (\mathbb{F}_h^{k-1})^{-1} \hat{\mathbf{v}}_h^{k-1} \, \mathrm{d}\hat{x} +& \\
\int_{\hat{\Omega}} J_h^{k-1} \hat{\mathbb{T}}(\mathbf{u}_h^k, p_h^k) (\mathbb{F}_h^{k-1})^{-T} \cdot \nabla \hat{\boldsymbol{\varphi}} \, \mathrm{d}\hat{x} +& \\
\rho_s \int_{\Sigma} D_t \xi_h^k \psi \, \mathrm{d}x_1 + a_s(\eta_h^k, \zeta_h^k, \xi_h^k, \psi) = 0.&
\end{aligned}$$

Stability

Theorem

Let $\{(\hat{\mathbf{u}}_h^k, \hat{p}_h^k, \hat{\xi}_h^k, \hat{\eta}_h^k)\}_{k=1}^N$ *be the solution of our numerical scheme. Then the following stability result holds for all* $m = 1, \dots, N$

$$E_h^m + \tau \sum_{k=1}^m 2\mu \int_{\hat{\Omega}} \eta_h^k |(\nabla \mathbf{u}_h^k (\mathbb{F}_h^{k-1})^{-1})^s|^2 \, \mathrm{d}\hat{x} + \gamma_3 \left\| \frac{\partial \xi_h^k}{\partial x_1} \right\|_{L^2(\Sigma)}^2 + \tau D_{\mathrm{num}}^k = E_h^0$$

where for any $k = 0, \dots, N$ *the total energy* E_h^k *and the numerical dissipation* D_{num}^k *read*

$$E_h^k = \frac{\rho_f}{2} \int_{\hat{\Omega}} \eta_h^k |\mathbf{u}_h^k|^2 \, \mathrm{d}\hat{x} + \frac{\rho_s}{2} \|\xi_h^k\|_{L^2(\Sigma)}^2 + \frac{\gamma_1}{2} \left\| \frac{\partial \eta_h^k}{\partial x_1} \right\|_{L^2(\Sigma)} + \frac{\gamma_2}{2} \left\| \frac{\partial^2 \eta_h^k}{\partial x_1^2} \right\|_{L^2(\Sigma)},$$

$$D_{\mathrm{num}}^k = \frac{\rho_f}{2} \int_{\hat{\Omega}} \eta_h^k |D_t \hat{\mathbf{u}}_h^k|^2 \mathrm{d}\hat{x} + \frac{\rho_s}{2} \|D_t \xi_h^k\|_{L^2(\Sigma)}^2 + \frac{\gamma_1}{2} \left\| \frac{\partial \xi_h^k}{\partial x_1} \right\|_{L^2(\Sigma)}^2 + \frac{\gamma_2}{2} \left\| \frac{\partial^2 \xi_h^k}{\partial x_1^2} \right\|_{L^2(\Sigma)}^2.$$

- We need to preserve $\eta_h^k > 0$. This holds due to no contact between the upper and the bottom surface. (Talk by J. Fara, Thursday 16:00.)

Convergence rate

Errors:

$$
\begin{aligned}
e_p^k &= \hat{p}_h^k - \hat{p}^k,\\
e_{\mathbf{u}}^k &= \hat{\mathbf{u}}_h^k - \hat{\mathbf{u}}^k,\\
e_\xi^k &= \xi_h^k - \xi^k,\\
e_\eta^k &= \eta_h^k - \eta^k,\\
e_\zeta^k &= \zeta_h^k - \zeta^k.
\end{aligned}
$$

We study the error between our numerical solution $(\hat{\mathbf{u}}_h, \hat{p}_h, \hat{\xi}_h, \hat{\eta}_h)$ and target smooth solution $(\hat{\mathbf{u}}, \hat{p}, \xi, \eta)$ of FSI problem existing in the following class of strong solutions (Grandmont and Hillairet, ARMA 2016)

$$
\begin{cases}
\eta > \underline{\eta}, \eta \in L^2(0,T; W^{3,2}(\Sigma)) \cap W^{2,2}(0,T; W^{2,2}(\Sigma)),\\
\hat{\mathbf{u}} \in L^\infty(0,T; W^{1,2}(\hat{\Omega}; \mathbb{R}^2)) \cap L^2(0,T; W^{2,2}(\hat{\Omega}; \mathbb{R}^2)),\\
\dfrac{\partial \hat{\mathbf{u}}}{\partial t} \in L^2(0,T; W^{1,2}(\hat{\Omega}; \mathbb{R}^2)),\\
\hat{p} \in L^\infty(0,T; L^2(\hat{\Omega})),\ \nabla p \in L^2((0,T) \times \hat{\Omega}).
\end{cases}
$$

Convergence rate

Theorem

Let $\{(\hat{\mathbf{u}}_h^k, \hat{p}_h^k, \hat{\xi}_h^k, \hat{\eta}_h^k)\}_{k=1}^N$ *be the solution of our numerical scheme, and let* $(\hat{\mathbf{u}}, \hat{p}, \xi, \eta)(t), t \in (0,T)$ *be the strong solution of given FSI problem belonging to the class on the previous slide. Then for any* $k = 1, \ldots, N$ *it holds*

$$\|e_{\mathbf{u}}^k\|_{L^\infty(0,T;L^2(\hat{\Omega}))} + \|e_\xi^k\|_{L^\infty(0,T;L^2(\Sigma))} + \left\|\frac{\partial e_\eta^k}{\partial x_1}\right\|_{L^\infty(0,T;L^2(\Sigma))} +$$

$$\|e_\zeta^k\|_{L^\infty(0,T;L^2(\Sigma))} + \|\nabla e_{\mathbf{u}}^k\|_{L^2((0,T)\times\hat{\Omega})} + \gamma_3\left\|\frac{\partial e_\xi^k}{\partial x_1}\right\|_{L^2((0,T)\times\Sigma)} \lesssim \tau + h.$$

Numerical implementation

- FEniCS finite element code.
- Instead of height of the structure η, we take a shift $\eta = \eta - 1$ and then linearly extend it to the whole domain via $\eta = \eta \hat{x}_2$.
- Displacement η^k is computed on Γ using

$$\eta^k = \eta^{k-1} + \tau u_2^k \quad \text{on } \Gamma.$$

- Direct solver MUMPS.
- Whole simulation consists of two steps:

  Step 1 For known η^{k-1} we solve for velocity $\mathbf{u}^k$, its Laplace ζ^k and pressure p^k using the weak form.

  Step 2 We linearly prolongate the displacement η to whole $\hat{\Omega}$ by solving

$$\int_{\hat{\Omega}} \frac{\partial \eta}{\partial x_2} \frac{\partial t}{\partial x_2} \,\mathrm{d}\hat{x} = 0$$

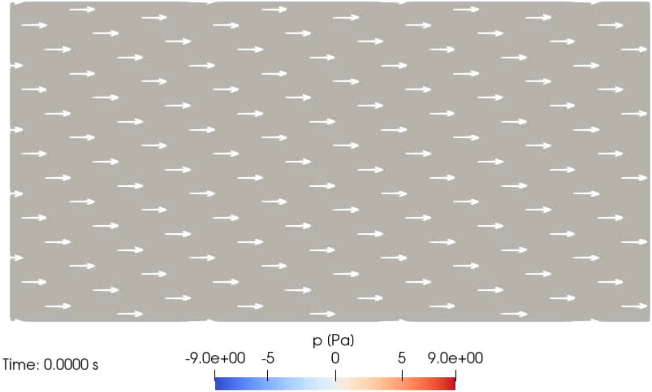
  with zero BC at the bottom, and $\eta = \eta^{k-1} + \tau u_2^k$ at the top, where u_2^k is obtained in Step 1.

Problem description

- Domain $\hat{\Omega}$ is a rectangle 2×1.
- Periodic BC on lateral sides, no-slip at the bottom.
- Parameters $\rho_f = \rho_s = 1$, $\mu = 0.01$, $\gamma_1 = \gamma_2 = 0.1$, $\gamma_3 = 0$.
- Flow driven by force acting on the shell

$$g = \begin{cases} 200t \sin(2\pi x) & t \leq 0.2, \\ 0 & t > 0.2. \end{cases}$$

Simulation



Experimental order of convergence

- $t \in [0, T], T = 1.0$
- 6 different time steps, six different meshes.
- $\tau_{\min} = 1 \times 10^{-4}$, $h_{\min} = 8.84 \times 10^{-3}$ used as reference solution

h	$\|e_{\mathbf{u}}\|_{L^\infty(L^2)}$	$\|e_\xi\|_{L^\infty(L^2)}$	$\|e_\eta\|_{L^\infty(L^2)}$	$\|\nabla e_\eta\|_{L^\infty(L^2)}$	$\|e_\zeta\|_{L^\infty(L^2)}$	$\|\nabla e_{\mathbf{u}}\|_{L^2(L^2)}$
2.83×10^{-1}	1.20×10^{0}	2.84×10^{0}	2.22×10^{-1}	1.41×10^{0}	9.22×10^{0}	1.23×10^{1}
1.41×10^{-1}	3.19×10^{-1}	5.80×10^{-1}	5.99×10^{-2}	3.79×10^{-1}	2.42×10^{0}	7.51×10^{0}
7.07×10^{-2}	1.05×10^{-1}	1.39×10^{-1}	1.52×10^{-2}	1.34×10^{-1}	6.02×10^{-1}	4.11×10^{0}
3.54×10^{-2}	2.78×10^{-2}	3.31×10^{-2}	3.65×10^{-3}	6.57×10^{-2}	1.44×10^{-1}	2.12×10^{0}
1.77×10^{-2}	5.91×10^{-3}	6.64×10^{-3}	7.32×10^{-4}	2.94×10^{-2}	2.89×10^{-2}	1.04×10^{0}

τ	$\|e_{\mathbf{u}}\|_{L^\infty(L^2)}$	$\|e_\xi\|_{L^\infty(L^2)}$	$\|e_\eta\|_{L^\infty(L^2)}$	$\|\nabla e_\eta\|_{L^\infty(L^2)}$	$\|e_\zeta\|_{L^\infty(L^2)}$	$\|\nabla e_{\mathbf{u}}\|_{L^2(L^2)}$
5.00×10^{-3}	2.55×10^{-1}	5.50×10^{-1}	4.23×10^{-2}	2.66×10^{-1}	1.67×10^{0}	1.61×10^{0}
2.50×10^{-3}	1.36×10^{-1}	2.87×10^{-1}	2.21×10^{-2}	1.39×10^{-1}	8.74×10^{-1}	8.52×10^{-1}
1.25×10^{-3}	6.87×10^{-2}	1.43×10^{-1}	1.10×10^{-2}	6.91×10^{-2}	4.35×10^{-1}	4.28×10^{-1}
6.25×10^{-4}	3.25×10^{-2}	6.73×10^{-2}	5.17×10^{-3}	3.25×10^{-2}	2.05×10^{-1}	2.02×10^{-1}
3.12×10^{-4}	1.37×10^{-2}	2.83×10^{-2}	2.17×10^{-3}	1.36×10^{-2}	8.60×10^{-2}	8.45×10^{-2}

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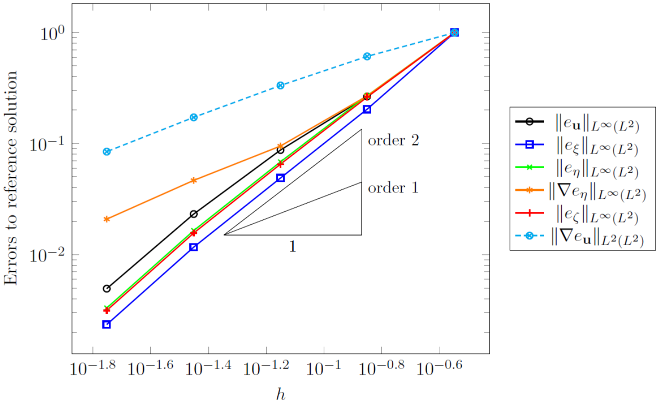

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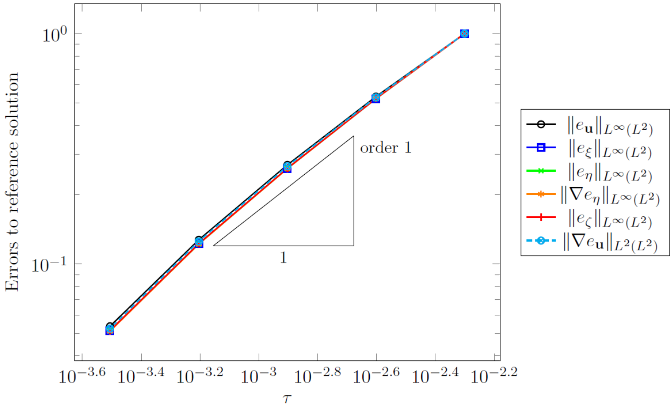
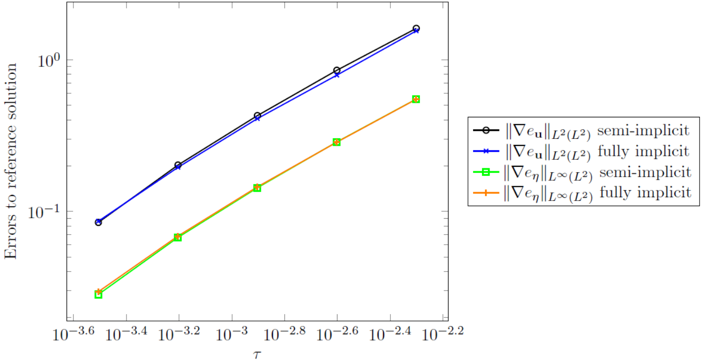


Linear semi-implicit scheme vs fully implicit scheme

- Fully implicit = non-linear scheme with all unknowns.
- Main difference in time splitting, compare errors for different time steps τ on finest mesh.
- Linear semi-implicit scheme: 410 880 DOFs in Step 1 + 153 920 DOFs in Step 2 in every time step.
- Fully implicit scheme: 564 000 DOFs in every Newton step.



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- Fully implicit scheme: 564 000 DOFs in every Newton step.
- CPU time matters! Intel Xeon Gold 6240 CPU.

Scheme	τ	Avg Newton its	CPU time [min]
Fully implicit	5.00×10^{-3}	3	135.5
Semi-implicit	5.00×10^{-3}	–	24.5
Fully implicit	3.12×10^{-4}	2	1 310.7
Semi-implicit	3.12×10^{-4}	–	338.0

Conclusion

- FSI linear semi-implicit scheme.
- Energy stable, linear convergence in space and time.
- Implemented in FEniCS, convergence rates confirmed.
- Our linear scheme outperforms fully implicit scheme.