A thermodynamic framework for heat-conducting flows of mixtures of two intercating fluids

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Joint work with Ondřej Souček

Multicomponent fluids

oil-water emulsions, biological fluids, air bubbles in water, ...

Single fluid vs N interacting fluids

Mass ϱ	$\varrho_1, \varrho_2, \dots, \varrho_N$	$m_1, m_2, \dots, m_N$
Linear momentum $\mathbf{v}$	$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N$	$\mathbf{I}_1, \mathbf{I}_2, \dots, \mathbf{I}_N$
Energy θ	$\theta_1, \theta_2, \dots, \theta_N$	$i_1, i_2, \dots, i_N$

Second law of thermodynamics for the whole material

Goal: **to develop multicomponent models that require to consider different velocities of the individual components**

Specification of a studied class of multicomponent fluids

The most general mixture framework

$\varrho_1, \varrho_2, \dots, \varrho_N \quad \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N \quad \theta_1, \theta_2, \dots, \theta_N$

reaction-advection-diffusion

$\varrho_1, \varrho_2, \dots, \varrho_N \quad \mathbf{v}, \theta$

(Maxwell- Stefan)

Class I mixtures

Fick (1855) molecular diffusion

multiphase fluid flows

$\varrho_1, \varrho_2, \dots, \varrho_N \quad \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N \quad \theta$

binary emulsions (Málek-Souček)

$N = 2$ $\quad \varrho_1, \varrho_2, \quad \mathbf{v}_1, \mathbf{v}_2, \quad \theta$

Class II mixtures

Darcy (1856) porous media flow

Challenges and objectives

Challenges:

- a high number of constitutive quantities including interacting mechanisms
- constitutive equations with a high number of model parameters

Linear models that come out from rational thermodynamics (Rajagopal&Tao (1995)):

$$\mathbb{T}_1 = (-\rho_1 + \lambda_1 \operatorname{div} \mathbf{v}_1 + \lambda_2 \operatorname{div} \mathbf{v}_2)\mathbb{I} + 2\mu_1 \mathbb{D}(\mathbf{v}_1) + 2\mu_2 \mathbb{D}(\mathbf{v}_2) + \lambda_5 \mathbb{V}_{12}$$

$$\mathbb{T}_2 = (-\rho_2 + \lambda_3 \operatorname{div} \mathbf{v}_1 + \lambda_4 \operatorname{div} \mathbf{v}_2)\mathbb{I} + 2\mu_3 \mathbb{D}(\mathbf{v}_1) + 2\mu_4 \mathbb{D}(\mathbf{v}_2) - \lambda_5 \mathbb{V}_{12}$$

$$\boxed{\mu_1, \mu_2, \mu_3, \mu_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5} \qquad \mathbb{V}_{12} := \frac{\nabla \mathbf{v}_1 - (\nabla \mathbf{v}_1)^T}{2} - \frac{\nabla \mathbf{v}_2 - (\nabla \mathbf{v}_2)^T}{2}$$

- identification of boundary conditions for individual components

Objectives:

- To develop a simple/minimalist model knowing the **shear viscosity**, **bulk viscosity** and **heat conductivity** of the mixture as a whole and to consider merely one **drag force** interaction mechanics between two constituents
- To develop a PDE theory for a simple transparent Class II mixture model

Goal, co-authors and main references

- To develop a model for **heat-conducting binary fluid mixtures** described in the terms of the **densities** and the **velocities** for **each fluid** and the **temperature** field for the **mixture as a whole** based on
  - theory of interacting continua (theory of mixtures)
  - the requirement that the response of the whole mixture is determined from a **small (minimal) set of material parameters**
  - a simple yet general thermodynamical approach

J. Málek, **K. R. Rajagopal**: A thermodynamic framework for a mixture of two liquids. *Nonlinear Analysis: Real World Applications*, Vol. 9 (2008) 1649–1649.

J. Málek, O. Souček: A thermodynamic framework for heat-conducting flows of mixtures of two interacting fluids. *ZAMM Z. Angew. Math. Mech.*, 102 (2022), no. 11, Paper No. e202100389, 27 pp.

O. Souček, **V. Průša**, J. Málek, **K. R. Rajagopal**: On the natural structure of thermodynamic potentials and fluxes in the theory of chemically non-reacting binary mixtures. *Acta Mechanica*, Vol. 225 (2014) 3157–3186.

O. Souček, M. Heida, J. Málek: On a thermodynamic framework for developing boundary conditions for Korteweg-type fluids. *International Journal of Engineering Science*, Vol. 154 (2020) 103316.

Methodology

- A general thermodynamical approach based on the idea that **knowing how the material stores the energy and how the entropy is produced suffices** to identify constitutive equations

K. R. Rajagopal, A. R. Srinivasa: On thermomechanical restrictions of continua. *Proc. R. Soc. Lond A: Math Phys Eng Sci*, 460 (2004), 631-651.

V. Pruša, J. Málek: Derivation of equations for continuum mechanics and thermodynamics of fluids. In: Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, pp. 3-72. Springer, Cham (2018).

A thermodynamic framework

Balance equations of continuum thermomechanics

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) = \operatorname{div} \mathbb{T} + \rho \mathbf{f}$$

$$\frac{\partial (\rho E)}{\partial t} + \operatorname{div}(\rho E \mathbf{v}) = \operatorname{div}(\mathbb{T}\mathbf{v} - \mathbf{j}_e) \qquad E = \frac{1}{2}|\mathbf{v}|^2 + e$$

$$\frac{\partial (\rho \eta)}{\partial t} + \operatorname{div}(\rho \eta \mathbf{v}) = \operatorname{div} \mathbf{j}_\eta + \xi \quad \text{and} \quad \xi \geq 0$$

Balance equations of continuum thermomechanics

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Constitutive relations

stress tensor $\mathbb{T}$, energy flux $\mathbf{j}_e$, entropy flux $\mathbf{j}_\eta$

Balance equations of continuum thermomechanics

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$$\frac{\partial (\rho \eta)}{\partial t} + \operatorname{div}(\rho \eta \mathbf{v}) = \operatorname{div} \mathbf{j}_\eta + \xi \quad \text{and} \quad \xi \geq 0$$

Constitutive relations

stress tensor $\mathbb{T}$, energy flux $\mathbf{j}_e$, entropy flux $\mathbf{j}_\eta$

can be determined from the knowledge of constitutive equations for

entropy	η	(e, ψ, H, G)
entropy production	ξ	

Example (compressible Navier-Stokes-Fourier fluid)

$$\psi = \widetilde{\psi}(\rho, \theta)$$

$$\theta\xi = \mathbb{T}_\delta \cdot \mathbb{D}_\delta + (\frac{1}{3}\operatorname{tr}\mathbb{T} + p)\operatorname{div}\mathbf{v} + \mathbf{j}_\eta \cdot (-\nabla\theta) \qquad \text{(NSF-TI)}$$

processes that are isochoric

volume changing processe

heat conduction

$$\mathbb{D} := \mathbb{D}v = \frac{1}{2}\left[\nabla v + (\nabla v)^T\right]$$

$$\mathbb{D}_\delta := \mathbb{D} - \left(\frac{1}{3}\operatorname{tr}\mathbb{D}\right)\mathbb{I}$$

Example (compressible Navier-Stokes-Fourier fluid)

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$$\boxed{\theta\xi = \mathbb{T}_\delta \cdot \mathbb{D}_\delta + (\frac{1}{3} \operatorname{tr} \mathbb{T} + p) \operatorname{div} \mathbf{v} + \mathbf{j}_\eta \cdot (-\nabla\theta)} \qquad \text{(NSF-TI)}$$

$$\mathbb{T}_\delta = 2\nu\mathbb{D}_\delta \qquad \nu > 0$$

$$\frac{1}{3} \operatorname{tr} \mathbb{T} + p = \lambda \operatorname{div} \mathbf{v} \qquad \lambda > 0$$

$$\mathbf{j}_\eta = -\kappa\nabla\theta \qquad \kappa > 0$$

$$\boxed{\theta\xi = 2\nu|\mathbb{D}_\delta|^2 + \lambda(\operatorname{div} \mathbf{v})^2 + \kappa|\nabla\theta|^2 =: \tilde{\zeta}(\mathbb{D}_\delta, \operatorname{div} \mathbf{v}, \nabla\theta)}$$

3 material parameters

Example (compressible Navier-Stokes-Fourier fluid)

$$\boxed{\psi = \widetilde{\psi}(\rho, \theta)}$$

$$\boxed{\theta\xi = \mathbb{T}_\delta \cdot \mathbb{D}_\delta + (\frac{1}{3}\operatorname{tr}\mathbb{T} + p)\operatorname{div}\mathbf{v} + \mathbf{j}_\eta \cdot (-\nabla\theta)} \qquad \text{(NSF-TI)}$$

$$\begin{aligned} \mathbb{T}_\delta &= 2\nu\mathbb{D}_\delta & \nu > 0 \\ \frac{1}{3}\operatorname{tr}\mathbb{T} + p &= \lambda\operatorname{div}\mathbf{v} & \lambda > 0 \\ \mathbf{j}_\eta &= -\kappa\nabla\theta & \kappa > 0 \end{aligned}$$

$$\boxed{\theta\xi = 2\nu|\mathbb{D}_\delta|^2 + \lambda(\operatorname{div}\mathbf{v})^2 + \kappa|\nabla\theta|^2 =: \tilde{\zeta}(\mathbb{D}_\delta, \operatorname{div}\mathbf{v}, \nabla\theta)}$$

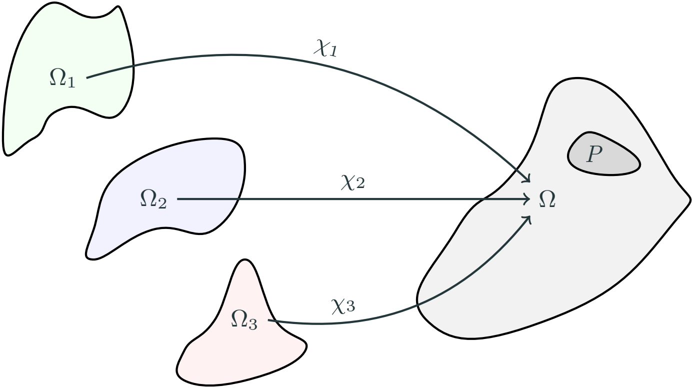
The same constitutive equations achieved by a constrained maximization

$$\max_{(\mathbb{D}_\delta, \operatorname{div}\mathbf{v}, \nabla\theta) \in \mathcal{A}} \tilde{\zeta}(\mathbb{D}_\delta, \operatorname{div}\mathbf{v}, \nabla\theta) \;\; \text{where} \;\; \mathcal{A} := \{\mathbb{D}_\delta, \operatorname{div}\mathbf{v}, \nabla\theta; \tilde{\zeta} = \text{RHS(NSF-TI)}\}$$

Easy to incorporate incompressibility $\operatorname{div}\mathbf{v} = 0$

Theory of interacting continua

Co-existence of individual constituents



Truesdel (1962), Müller (1968), Atkin and Craine (1976), Bowen (1976), Bothe and Dryer (2015) and books by Samohýl (1987), de Groot and Mazur (1984), Rajagopal and Tao (1985), Hutter and Jöhnk (2004) or Pekař and Samohýl (2014).

Theory of Mixtures - mass and volume densities

- $\mathcal{M}(P)$, $\mathcal{V}(P)$ - the total mass and the volume of P
- $\mathcal{M}_\alpha(P)$, $\mathcal{V}_\alpha(P)$ - the mass and the volume of the α-constituent associated with P

Natural requirements

$$\mathcal{M} \ll \mathcal{V}, \quad \mathcal{M}_\alpha \ll \mathcal{V}, \quad \mathcal{M}_\alpha \ll \mathcal{V}_\alpha, \quad \mathcal{M}_\alpha \ll \mathcal{M}, \quad \mathcal{V}_\alpha \ll \mathcal{V}$$

lead to ρ (the density of the mixture as a whole), ρ_α (the density of the α-constituent), $\rho_\alpha^{\mathrm{tr}}$ (the true density of the α-constituent), c_α (the mass fraction/concentration) and ϕ_α (the volume fraction) so that for all P

$$\mathcal{M}(P) = \int_P \rho \, d\mathcal{V}, \quad \mathcal{M}_\alpha(P) = \int_P \rho_\alpha \, d\mathcal{V}, \quad \mathcal{M}_\alpha(P) = \int_P \rho_\alpha^{\mathrm{tr}} \, d\mathcal{V}_\alpha,$$
$$\mathcal{M}_\alpha(P) = \int_P c_\alpha \, d\mathcal{M}, \quad \mathcal{V}_\alpha(P) = \int_P \phi_\alpha \, d\mathcal{V}.$$

$$\boxed{\rho_\alpha = \rho c_\alpha \qquad \rho_\alpha = \phi_\alpha \rho_\alpha^{\mathrm{tr}}}$$

Mass and volume measures and related constraints

- $\mathcal{M}(P)$, $\mathcal{V}(P)$ - the total mass and the volume of P
- $\mathcal{M}_\alpha(P)$, $\mathcal{V}_\alpha(P)$ - the mass and the volume of the α-constituent associated with P

Mass additivity constraint $\mathcal{M}(P) = \sum_\alpha \mathcal{M}_\alpha(P)$ for any P

$$\rho = \sum_\alpha \rho_\alpha \quad \Longrightarrow \quad 1 = \sum_\alpha c_\alpha$$

Volume additivity constraint $\mathcal{V}(P) = \sum_\alpha \mathcal{V}_\alpha(P)$ for any P

$$1 = \sum_\alpha \phi_\alpha$$

Molar Masses

- M_α molar mass of the α-constituent

Molar concentrations c_α^{M} of the α-constituent

Molar concentration of the whole mixture c^{M}

$$c_\alpha^{\mathsf{M}} := \frac{\rho_\alpha}{M_\alpha}\,, \qquad \alpha = 1, \ldots, N, \qquad c^{\mathsf{M}} := \sum_\alpha c_\alpha^{\mathsf{M}}\,, \tag{1}$$

Molar fractions x_α by

$$x_\alpha := \frac{c_\alpha^{\mathsf{M}}}{c^{\mathsf{M}}}\,, \qquad \alpha = 1, \ldots, N. \tag{2}$$

Molar additivity constraint

$$\boxed{1 = \sum_\alpha x_\alpha}$$

Velocity associated with the mixture as a whole

$$\mathbf{v} = \frac{1}{\rho}\sum_{\alpha} \rho_\alpha \mathbf{v}_\alpha = \sum_{\alpha} c_\alpha \mathbf{v}_\alpha$$

$$\mathbf{v}^{\mathsf{M}} = \sum_{\alpha} x_\alpha \mathbf{v}_\alpha$$

$$\mathbf{v}^{\phi} = \sum_{\alpha} \phi_\alpha \mathbf{v}_\alpha$$

- What is the right-concept?

Balance equations of the theory of interacting continua

$\alpha = 1, \ldots, N$

$$\frac{\partial \rho_\alpha}{\partial t} + \operatorname{div}(\rho_\alpha \mathbf{v}_\alpha) = 0$$

$$\frac{\partial (\rho_\alpha \mathbf{v}_\alpha)}{\partial t} + \operatorname{div}(\rho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha) = \operatorname{div} \mathbb{T}_\alpha + \mathbf{I}_\alpha$$

$$\frac{\partial}{\partial t}\left(\sum_\alpha \rho_\alpha E_\alpha\right) + \operatorname{div}\left(\sum_\alpha \rho_\alpha E_\alpha \mathbf{v}_\alpha\right) = \operatorname{div}\left(\sum_\alpha \mathbb{T}_\alpha \mathbf{v}_\alpha - \mathbf{j}_e\right)$$

$$\frac{\partial}{\partial t}\left(\sum_\alpha \rho_\alpha \eta_\alpha\right) + \operatorname{div}\left(\sum_\alpha \rho_\alpha \eta_\alpha \mathbf{v}_\alpha\right) = \operatorname{div} \mathbf{j}_\eta + \xi \quad \text{and} \quad \xi \geq 0$$

$E_\alpha = \frac{1}{2}|\mathbf{v}_\alpha|^2 + e_\alpha$

$$\sum_\alpha \mathbf{I}_\alpha = \mathbf{0}$$

Balance equations of the theory of interacting continua

$\alpha = 1, \dots, N$

$$\frac{\partial \rho_\alpha}{\partial t} + \operatorname{div}(\rho_\alpha \mathbf{v}_\alpha) = 0$$

$$\frac{\partial (\rho_\alpha \mathbf{v}_\alpha)}{\partial t} + \operatorname{div}(\rho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha) = \operatorname{div} \mathbb{T}_\alpha + \mathbf{I}_\alpha$$

$$\frac{\partial}{\partial t}\left(\sum_\alpha \rho_\alpha E_\alpha\right) + \operatorname{div}\left(\sum_\alpha \rho_\alpha E_\alpha \mathbf{v}_\alpha\right) = \operatorname{div}\left(\sum_\alpha \mathbb{T}_\alpha \mathbf{v}_\alpha - \mathbf{j}_e\right)$$

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Goal: to determine the form of constitutive relations for

stress tensors $\mathbb{T}_\alpha$, interaction terms $\mathbf{I}_\alpha$, energy flux $\mathbf{j}_e$, entropy flux $\mathbf{j}_\eta$

Mixtures of two fluids

$$N = 2$$

$$\rho = \rho_1 + \rho_2$$

$$c := c_1$$

$$\mathbf{I} := \mathbf{I}_1$$

$$\rho e = \rho_1 e_1 + \rho_2 e_2 \qquad \rho\eta = \rho_1\eta_1 + \rho_2\eta_2$$

Helmholtz free energy

$$\rho\psi = \widehat{\rho\psi}(\theta, \rho_1, \rho_2) = \widehat{\rho_1\psi_1}(\theta, \rho_1, \rho_2) + \widehat{\rho_2\psi_2}(\theta, \rho_1, \rho_2)$$

$$\rho_\alpha\psi_\alpha := \rho_\alpha e_\alpha - \theta\rho_\alpha\eta_\alpha$$

$$\rho\eta := -\frac{\partial\widehat{\rho\psi}}{\partial\theta}$$

$$\mu_\alpha := \frac{\partial\widehat{\rho\psi}}{\partial\rho_\alpha} \qquad \alpha = 1, 2$$

$$p := -\rho e + \theta\rho\eta + \sum_\alpha \rho_\alpha\mu_\alpha$$

Entropy production

$$\theta\xi = \widetilde{\zeta}(\mathbb{D}_\delta^{\text{mixt}}, \operatorname{div}\mathbf{v}^{\text{mixt}}, \nabla\theta, \mu, \mathbf{v}_1 - \mathbf{v}_2)$$
$$= 2\nu|\mathbb{D}_\delta^{\text{mixt}}|^2 + \lambda(\operatorname{div}\mathbf{v}^{\text{mixt}})^2 + \kappa|\nabla\theta|^2 + \alpha|\mathbf{v}_1 - \mathbf{v}_2|^2$$

4 material parameters

$$\mathbb{D}^{\text{mixt}} = \frac{1}{2}\Big((\nabla\mathbf{v}^{\text{mixt}}) + (\nabla\mathbf{v}^{\text{mixt}})^T\Big) \qquad \text{where } \mathbf{v}^{\text{mixt}} := \omega\mathbf{v}_1 + (1-\omega)\mathbf{v}_2$$

Three forms for the velocity $\mathbf{v}^{\mathsf{mixt}}$ associated with the mixture as a whole:

$$\mathbf{v} = c\mathbf{v}_1 + (1-c)\mathbf{v}_2 \qquad \omega = c := c_1$$

$$\mathbf{v}^{\mathsf{M}} = x\mathbf{v}_1 + (1-x)\mathbf{v}_2 \qquad \omega = x := x_1$$

$$\mathbf{v}^{\phi} = \phi\mathbf{v}_1 + (1-\phi)\mathbf{v}_2 \qquad \omega = \phi := \phi_1$$

The constraint that all admissible processes are volume conserving

$$\operatorname{div} \mathbf{v}^{\mathsf{mixt}} = 0$$

Material derivative associated with the mixture as the whole

$$\dot{z} := \frac{\partial z}{\partial t} + \mathbf{v}^{\mathsf{mixt}} \cdot \nabla z$$

Application of the constrained maximization:

$$\max_{\mathbb{D}_\delta(\mathbf{v}_1),\operatorname{div}\mathbf{v}_1,\mathbb{D}_\delta(\mathbf{v}_2),\operatorname{div}\mathbf{v}_2,\nabla\theta,\mathbf{v}_1-\mathbf{v}_2\in\mathcal{A}} \tilde{\zeta}(\mathbb{D}_\delta(\mathbf{v}^{\text{mixt}}), \operatorname{div}\mathbf{v}^{\text{mixt}}, \nabla\theta, \mathbf{v}_1-\mathbf{v}_2)$$

where

$$\mathcal{A} := \{\mathbb{D}_\delta(\mathbf{v}_1), \operatorname{div}\mathbf{v}_1, \mathbb{D}_\delta(\mathbf{v}_2), \operatorname{div}\mathbf{v}_2, \nabla\theta, \mathbf{v}_1-\mathbf{v}_2; \quad \tilde{\zeta} = \text{RHS(TI-BMixt)}\}$$

and (TI-BMixt) has the form:

$$\begin{aligned}\zeta &= \left(\tfrac{1}{3}\operatorname{tr}\mathbb{T}_1 - \gamma E_{12} + \omega p\right)\operatorname{div}\mathbf{v}_1 + \left(\tfrac{1}{3}\operatorname{tr}\mathbb{T}_2 + \gamma E_{12} + (1-\omega)p\right)\operatorname{div}\mathbf{v}_2 \\ &\quad + (\mathbb{T}_1)_\delta : \mathbb{D}_\delta(\mathbf{v}_1) + (\mathbb{T}_2)_\delta : \mathbb{D}_\delta(\mathbf{v}_2) \\ &\quad - \frac{\mathbf{j}_e + ((1-\gamma)E_{12} - (1-\delta)\mu_{12})(\mathbf{v}_1-\mathbf{v}_2)}{\theta}\cdot\nabla\theta \\ &\quad - \left(\mathbf{I} + \nabla(\gamma E_{12}) - p\nabla\omega + \mathbf{m}_{12} - \frac{\delta\mu_{12}}{\theta}\cdot\nabla\theta\right)\cdot(\mathbf{v}_1-\mathbf{v}_2)\end{aligned} \qquad \text{(TI-BMixt)}$$

with

$$E_{12} := (1-\omega)\rho_1 e_1 - \omega\rho_2 e_2$$

$$\mu_{12} := (1-\omega)\rho_1\mu_1 - \omega\rho_2\mu_2$$

$$\mathbf{m}_{12} := (1-\omega)\rho_1\nabla\mu_1 - \omega\rho_2\nabla\mu_2$$

This results at

$$\mathbb{T}_1 = (\gamma E_{12} - \omega p)\mathbb{I} + \lambda\omega(\operatorname{div} \mathbf{v}^{\text{mixt}})\mathbb{I} + 2\nu\omega\mathbb{D}(\mathbf{v}^{\text{mixt}})$$

$$\mathbb{T}_2 = (-\gamma E_{12} - (1-\omega)p)\mathbb{I} + \lambda(1-\omega)(\operatorname{div} \mathbf{v}^{\text{mixt}})\mathbb{I} + 2\nu(1-\omega)\mathbb{D}(\mathbf{v}^{\text{mixt}})$$

$$\mathbf{I} = -\nabla(\gamma E_{12}) - \mathbf{m}_{12} + \delta\mu_{12}\frac{\nabla\theta}{\theta} - \alpha(\mathbf{v}_1 - \mathbf{v}_2) - (\mathbb{T}_1 + \mathbb{T}_2)\nabla\omega$$

Friction parts

Darcian part

thermal part

Special cases No. 1

$$\omega = x$$

We also set $\nu = \lambda = 0$ and consider ideal gas ansatz for both components:

$$\mathbb{T}_1 = -xp\,\mathbb{I} \qquad \mathbb{T}_2 = -(1-x)p\,\mathbb{I} \qquad \mathbf{I} = -\alpha(\mathbf{v}_1 - \mathbf{v}_2)$$

$$\begin{aligned}
\frac{\partial \rho_1}{\partial t} + \operatorname{div}(\rho_1 \mathbf{v}_1) &= 0 \\
\frac{\partial \rho_2}{\partial t} + \operatorname{div}(\rho_2 \mathbf{v}_2) &= 0 \\
\frac{\partial (\rho_1 \mathbf{v}_1)}{\partial t} + \operatorname{div}(\rho_1 \mathbf{v}_1 \otimes \mathbf{v}_1) &= -\nabla(xp) - \alpha(\mathbf{v}_1 - \mathbf{v}_2) \\
\frac{\partial (\rho_2 \mathbf{v}_2)}{\partial t} + \operatorname{div}(\rho_2 \mathbf{v}_2 \otimes \mathbf{v}_2) &= -\nabla((1-x)p) + \alpha(\mathbf{v}_1 - \mathbf{v}_2)
\end{aligned}$$

Special case No. 2

$$\omega = \phi$$

Binary emulsions with the true densities assumed to be constant and

$$\rho\psi = \widehat{\rho\psi}(\theta, \rho_1, \rho_2) = \phi\Psi_1(\theta, \rho_1^{\mathrm{tr}}) + (1-\phi)\Psi_2(\theta, \rho_2^{\mathrm{tr}})$$

we get

$$\frac{\partial \phi}{\partial t} + \mathrm{div}(\phi \mathbf{v}_1) = 0$$

$$\frac{\partial (1-\phi)}{\partial t} + \mathrm{div}((1-\phi)\mathbf{v}_2) = 0$$

leading to

$$\frac{\partial \phi}{\partial t} + \mathrm{div}(\phi \mathbf{v}_1) = 0$$

$$\mathrm{div}(\phi \mathbf{v}_1 + (1-\phi)\mathbf{v}_2) = 0$$

$$\rho_1^{\mathrm{tr}}\left(\frac{\partial(\phi\mathbf{v}_1)}{\partial t} + \mathrm{div}(\phi\mathbf{v}_1 \otimes \mathbf{v}_1)\right) = \phi\nabla p - \alpha(\mathbf{v}_1 - \mathbf{v}_2) + \phi\,\mathrm{div}(2\nu\mathbb{D}(\mathbf{v}^{\mathrm{mixt}}))$$

$$\rho_2^{\mathrm{tr}}\left(\frac{\partial((1-\phi)\mathbf{v}_2)}{\partial t} + \mathrm{div}((1-\phi)\mathbf{v}_2 \otimes \mathbf{v}_2)\right) = (1-\phi)\nabla p + \alpha(\mathbf{v}_1 - \mathbf{v}_2) + (1-\phi)\,\mathrm{div}(2\nu\mathbb{D}(\mathbf{v}^{\mathrm{mixt}}))$$

Conclusions

Conclusions

- An approach to develop models for mixtures consisting of two constituents (fluids) that are simple in the sense of the number parameters needed to their identification. For comparison, see the formulas for models that come out from the rational thermodynamics:

$$\mathbb{T}_1 = (-\rho_1 + \lambda_1 \operatorname{div} \mathbf{v}_1 + \lambda_2 \operatorname{div} \mathbf{v}_2)\mathbb{I} + 2\mu_1\mathbb{D}(\mathbf{v}_1) + 2\mu_2\mathbb{D}(\mathbf{v}_2) + \lambda_5\mathbb{V}_{12}$$

$$\mathbb{T}_2 = (-\rho_2 + \lambda_3 \operatorname{div} \mathbf{v}_1 + \lambda_4 \operatorname{div} \mathbf{v}_2)\mathbb{I} + 2\mu_3\mathbb{D}(\mathbf{v}_1) + 2\mu_4\mathbb{D}(\mathbf{v}_2) - \lambda_5\mathbb{V}_{12}$$

$$\boxed{\mu_1, \mu_2, \mu_3, \mu_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5}$$

$$\mathbb{V}_{12} := \frac{\nabla \mathbf{v}_1 - (\nabla \mathbf{v}_1)^T}{2} - \frac{\nabla \mathbf{v}_2 - (\nabla \mathbf{v}_2)^T}{2}$$

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$$\boxed{\mu_1, \mu_2, \mu_3, \mu_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5} \qquad \mathbb{V}_{12} := \frac{\nabla \mathbf{v}_1 - (\nabla \mathbf{v}_1)^T}{2} - \frac{\nabla \mathbf{v}_2 - (\nabla \mathbf{v}_2)^T}{2}$$

- Removed drawbacks in Málek and Rajagopal (2008).
  - different functions $\rho_\alpha \psi_\alpha$ for constituents
  - thermal effects included
  - different forms of averaged velocities (choice of molar fractions lead to the generalization of ideal gas mixture)

Conclusions

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$$\mathbb{T}_1 = (-\rho_1 + \lambda_1 \operatorname{div} \mathbf{v}_1 + \lambda_2 \operatorname{div} \mathbf{v}_2)\mathbb{I} + 2\mu_1 \mathbb{D}(\mathbf{v}_1) + 2\mu_2 \mathbb{D}(\mathbf{v}_2) + \lambda_5 \mathbb{V}_{12}$$
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  - different functions $\rho_\alpha \psi_\alpha$ for constituents
  - thermal effects included
  - different forms of averaged velocities (choice of molar fractions lead to the generalization of ideal gas mixture)
- An easy incorporation of the condition $\operatorname{div} \mathbf{v}^{\mathrm{Mixt}} = 0$

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- The framework of mixtures is very reach, covering three possible ways how the velocity of the whole mixture is specified