Analysis of entropic cross-diffusion systems of hyperbolic-parabolic type

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Thermodynamic framework

$$\dot{\boldsymbol{z}} = -\mathcal{K}(\boldsymbol{z})D\mathcal{E}(\boldsymbol{z})$$

- State variable $\boldsymbol{z} = \boldsymbol{z}(t)$
- Driving functional $\mathcal{E} = \mathcal{E}(\boldsymbol{z})$
- Onsager operator $\mathcal{K}$ such that $\mathcal{K}(\boldsymbol{z})$ is symmetric and positive semi-definite for all $\boldsymbol{z}$

Example (multi-component diffusion)

$\boldsymbol{z} = u$, $u = (u_1, \dots, u_n)$; $\mathcal{E} = H$ with $H(u) = \int h(u)\,\mathrm{d}x$ with $h \in C^2$ locally strongly convex

$$\mathcal{K}(u)\square = -\operatorname{div}(\mathrm{M}(u)\nabla\square), \qquad \mathrm{M}(u) \in \mathbb{R}^{n\times n}_{\mathrm{sym}}, \quad \mathrm{M}(u) \geq \mathrm{O}$$

$$\Longrightarrow \quad \text{Evolution equation} \quad \partial_t u = \operatorname{div}(\mathrm{A}(u)\nabla u) \text{ with } \mathrm{A}(u) = \mathrm{M}(u)D^2 h(u)$$

If $\operatorname{rank}(\mathrm{A}(u)) = n$, then $\sigma(\mathrm{A}(u)) \subset \mathbb{R}_+ \rightsquigarrow$ parabolic.

Focus of this talk: rank $\mathrm{M}(u) < n$

Outline

1. **Population dynamics**
   - Hyperbolic–parabolic normal form
   - Young measure framework
2. **Viscoelastic phase separation**
   - Formal sharp-interface asymptotics towards nonlocal/fractional surface diffusion laws

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Directed motion to avoid crowding / Contact inhibition

System of n species with partial densities $u = (u_1, \ldots, u_n)$ and partial velocity fields $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

$$\partial_t u_i + \operatorname{div}(u_i \mathbf{v}_i) = 0, \qquad i=1,\ldots,n,$$

for $t > 0, x \in \Omega \subseteq \mathbb{R}^d$.
Phenomenological closure Gurtin, Pipkin (1984), Busenberg, Travis (1983):

$$\mathbf{v}_i = -k_i \nabla p, \qquad p := \sum_{i=1}^n u_i,$$

where $k_i > 0$, $i = 1, \ldots, n$.

- If $\mathbf{k_i} = \mathbf{k}$ **for all i**: **Globally WP** Bertsch, Hilhorst, Mimura, Izuhara (2012) linear transport eq.
- For **unequal** $\mathbf{k_i}$ only few results on the initial-value problem:
  - Bertsch et al. (1985), segregated, 1D: Global existence and uniqueness
  - Lorenzi, Lorz, Perthame (2017), segregated, source: Instabilities in 2D sim. and 1D travelling wave
  - Kim, Tong (2021), segregated, 2D, incompressible with source: LWP for nearly radial interface.
  - Druet, H, Jüngel (2023), mixed data: Local Cauchy theory H, Jüngel: Global measure-valued solutions

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Cauchy theory for mixed initial data

Write PDE system as

$$\partial_t u - \mathrm{div}((\vec{\kappa}(u) \otimes \mathbf{1})\nabla u) = 0\,,$$

where $\vec{\kappa}(u) = (k_1 u_1, \dots, k_n u_n)^T$, $\mathbf{1} = (1, \dots, 1)^T$. Space domain $\mathbb{T}^d$, for simplicity.

- Kawashima–Shizuta theory **not** applicable ($\ker \mathrm{M}(u)$ depends on u)
- 'Ad hoc strategy': cancel 2nd order spatial derivatives in $(n-1)$ components by suitable change of variables $w = \Phi(u)$, where $\Phi : (\mathbb{R}_+)^n \to \Phi((\mathbb{R}_+)^n)$. For $n = 2$, obtain with $\mathsf{w} =: (\mathsf{z}, \mathsf{p})$,

$$\partial_t \mathsf{z} - c(\mathsf{w})\nabla \mathsf{p} \cdot \nabla \mathsf{z} = -\frac{(k_2 - k_1)}{\mathfrak{a}(\mathsf{w})}|\nabla \mathsf{p}|^2, \qquad \partial_t \mathsf{p} = \mathrm{div}(\mathfrak{a}(\mathsf{w})\nabla \mathsf{p}),$$

  where $c(\mathsf{w}) = k_1 + (k_2 - k_1)\frac{k_1 u_1}{\mathfrak{a}(\mathsf{w})} > 0, \quad \mathfrak{a}(\mathsf{w}) = \sum_{i=1}^2 k_i u_i$

- If $n \geq 3$, need to ensure symmetrisability of hyperbolic subsystem

Generalisation to rank-r diffusions

Consider, more generally,

$$\partial_t u_i + \operatorname{div}(u_i \mathbf{v}_i) = 0, \qquad i = 1, \dots, n,$$

with

$$\mathbf{v}_i = -\nabla\Big(\sum_{j=1}^{n} b_{ij} u_j\Big),$$

where $B := (b_{ij}) \in \mathbb{R}^{n\times n}$ is such that $B\,\mathrm{diag}(\lambda) \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ is positive semi-definite for some $\lambda \in (0,\infty)^n$.

Let $\mathbf{r} := \mathrm{rank}\, B \in \{1, \dots, n\}$.

Theorem (P.-E. Druet, KH, A. Jüngel, CPDE'23)

There exists a domain $\hat{\mathcal{D}} \subset \mathbb{R}^n$ and a C^∞-diffeom. $\Phi : (\mathbb{R}_+)^n \to \hat{\mathcal{D}}$, $u \mapsto w$, such that in the w-variables, the cross-diffusion system can be recast in **symmetric hyperbolic-parabolic form**:

$$\mathsf{A}_0^{\mathrm{I}}(w)\,\partial_t w_{\mathrm{I}} \;+\; \sum_{\nu=1}^{d} \mathsf{A}_1^{\mathrm{I}}(w, \partial_{x_\nu} w_{\mathrm{II}})\,\partial_{x_\nu} w_{\mathrm{I}} \;=\; f^{\mathrm{I}}(w, \nabla w_{\mathrm{II}}),$$

$$\mathsf{A}_0^{\mathrm{II}}\,\partial_t w_{\mathrm{II}} \;-\; \mathrm{div}\left(\mathsf{A}_*^{\mathrm{II}}(w)\,\nabla w_{\mathrm{II}}\right) \;=\; 0,$$

where $\mathsf{A}_0^{\mathrm{I}} : \mathcal{D} \longrightarrow \mathbb{R}_{\mathrm{spd}}^{(n-\mathsf{r})\times(n-\mathsf{r})}$, and $\mathsf{A}_1^{\mathrm{I}} : \mathcal{D} \times \mathbb{R}^{\mathsf{r}} \longrightarrow \mathbb{R}_{\mathrm{sym}}^{(n-\mathsf{r})\times(n-\mathsf{r})}$ is linear in 2nd argument.
$\mathsf{A}_0^{\mathrm{II}} \in \mathbb{R}_{\mathrm{spd}}^{\mathsf{r}\times\mathsf{r}}$ is constant, and $\mathsf{A}_*^{\mathrm{II}} : \mathcal{D} \longrightarrow \mathbb{R}_{\mathrm{spd}}^{\mathsf{r}\times\mathsf{r}}$, $f^{\mathrm{I}} : \mathcal{D} \times \mathbb{R}^{\mathsf{r}} \longrightarrow \mathbb{R}^{n-\mathsf{r}}$ is quadratic in 2nd argument.

Theorem ("Corollary")

Smooth positive initial data in $H^s(\mathbb{T}^d)$, $s > \frac{d}{2} + 1$ $\Longrightarrow$ Exists unique local classical solution.

Remark: Threshold $s > \frac{d}{2} + 1$ is classical for Friedrichs-symmetrisable hyperbolic systems.

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Nonlinear structure

For $t > 0$, $x \in \mathbb{T}^d$:

$$\partial_t u_i + \operatorname{div}(u_i \mathbf{v}_i) = 0, \qquad \mathbf{v}_i = -\nabla(Bu)_i \qquad \text{for } \ i = 1, \dots, n$$

for $B = (b_{ij}) \in \mathbb{R}^{n\times n}$ symmetric positive semi-definite with $b_{ij} \geq 0$, $b_{ii} > 0$.

Formal **gradient-flow** structure $\dot{u} = -\mathcal{K}(u) D\mathcal{E}(u)$, $\mathcal{K}(u)\Box = -\operatorname{div}(\mathrm{M}(u)\nabla\Box)$. Two options:

Driving functional	$\mathcal{E}(u)$		Mobility $\mathrm{M}(u)$	
Boltzmann	$H(u) = \int_{\mathbb{T}^d} h(u)\,\mathrm{d}x$	$h(u) = \sum_i u_i \log u_i$	$\mathrm{M}_{ik}(u) = u_i b_{ik} u_k$	
Rao	$Q(u) = \frac{1}{2}\int_{\mathbb{T}^d} u^T B u\,\mathrm{d}x$		$\mathrm{M}_{ik}(u) = u_i \delta_{ik}$	"Otto"

Thus, along smooth positive solutions

$$\frac{\mathrm{d}}{\mathrm{d}t} H(u) = -\int_{\mathbb{T}^d} |\nabla\sqrt{B}u|^2\,\mathrm{d}x, \qquad \frac{\mathrm{d}}{\mathrm{d}t} Q(u) = -\int_{\mathbb{T}^d} \sum_{i=1}^n u_i |\mathrm{v}_i|^2\,\mathrm{d}x.$$

A priori estimates ... and the problem of compactness

Suppose $(u^{(m)})_m$ is sequence of (approximate) solutions with $u_i^{(m)} \geq 0$ for all i and

$$\operatorname{essup}_t H(u^{(m)}(t)) \leq C, \qquad \operatorname{essup}_t Q(u^{(m)}(t)) \leq C,$$

$$\int_0^\infty \int_{\mathbb{T}^d} |\nabla \sqrt{B} u^{(m)}|^2 \, \mathrm{d}x \mathrm{d}\tau \leq C, \qquad \int_0^\infty \int_{\mathbb{T}^d} \sum_i u_i^{(m)} |\mathrm{v}_i^{(m)}|^2 \, \mathrm{d}x \mathrm{d}\tau \leq C,$$

where $\mathrm{v}_i^{(m)} = -\nabla (Bu^{(m)})_i$.

This implies boundedness of $u_i^{(m)}, \mathrm{v}_i^{(m)}$, and $u_i^{(m)} \mathrm{v}_i^{(m)}$ in suitable $L^p(L^q)$, $p, q > 1$.

After passing to a subsequence

$$u^{(m)} \overset{*}{\rightharpoonup} u \;\text{ in }\; L^\infty(L^2), \quad \mathrm{v}_i^{(m)} \rightharpoonup \mathrm{v}_i = \nabla(Bu)_i \;\text{ in }\; L^2(L^2), \quad \text{and } u_i^{(m)} \mathrm{v}_i^{(m)} \rightharpoonup \overline{u_i^{(m)} \mathrm{v}_i^{(m)}} \;\text{ in }\; L^2(L^{4/3}).$$

If $\ker B$ is non-trivial, it is unclear whether $\overline{u_i^{(m)} \mathrm{v}_i^{(m)}}$ and $u_i \mathrm{v}_i$ coincide.

Young measures (YM)

YM as PDE *solution* concept were first considered by DiPerna (1985) for hyperbolic conservation laws.

Associate with each $u^{(m)}$ a parametrised probability measure $\mu^{(m)} = (\mu^{(m)}_{t,x})_{t,x}$ via

$$\mu^{(m)} := \delta_{(u^{(m)}, \nabla \widehat{u}^{(m)})}$$

where $\widehat{u}^{(m)} := P_{(\ker B)^\perp} u^{(m)}$ and $P_{(\ker B)^\perp}$ the projection onto $(\ker B)^\perp \subseteq \mathbb{R}^n$.
Then

$$u_i^{(m)} \mathrm{v}_i^{(m)} = -\int_W s_i (Bp)_i \mathrm{d}\mu^{(m)}(s,p) =: -\langle \mu^{(m)}, s_i (Bp)_i \rangle$$

where $W := [0,\infty)^n \times (\ker B^\perp)^d$. For every $T > 0$, the sequence

$$(\mu^{(m)})_m \subset L^\infty_{w*}((0,T)\times\mathbb{T}^d; \mathcal{M}(W)) \simeq \left(L^1((0,T)\times\mathbb{T}^d; C_0(W))\right)^*$$

is bounded and hence weakly-$*$ convergent along subsequence.

Dissipative measure-valued–strong uniqueness

We call $U \in C^1([0,T] \times \mathbb{T}^d)^n$ a *strong solution* if it satisfies $\partial_t U_i - \operatorname{div}\big(U_i \nabla (BU)_i\big) = 0$ in the weak sense and if it is strictly positive componentwise.

Theorem (KH, A. Jüngel)

Let $U \in C^1([0,T] \times \mathbb{T}^d)^n$ be a strong solution with initial datum $U(0,\cdot) = u^{\mathrm{in}}$, and let μ be a dissipative measure-valued solution. Then

$$\mu_{t,x} = \delta_{U(t,x)} \otimes \delta_{\nabla \widehat{U}(t,x)} \quad \text{for a.e. } (t,x) \in (0,T) \times \mathbb{T}^d.$$

Remarks:

- Weak-strong uniqueness of measure-valued solutions was first obtained by Brenier, De Lellis, Székelyhidi (2011) for the incompressible Euler equations.
- The proof is based on the **relative entropy technique** (Dafermos 1979, DiPerna 1979).

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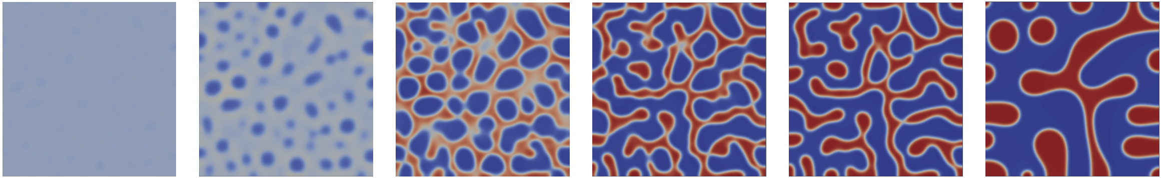
Viscoelastic phase separation (VPS)

Phase separation in polymer–solvent mixture



- Zhou, Zhang, E (2006): $\mathcal{E}(u,q) = F(u) + \frac{1}{2}\int_\Omega q^2 \,\mathrm{d}x$, $F(u) = \int_\Omega \left(\frac{\varepsilon^2}{2}|\nabla u|^2 + f(u)\right)\mathrm{d}x$, f Flory–Hug.
  $u = u(t,x) \in [-1,1]$ order parameter; $q = q(t,x) \in \mathbb{R}$ bulk stress variable

$$
\begin{aligned}
\partial_t u &= -\operatorname{div}\left((1-u^2)\boldsymbol{j}\right), \qquad \boldsymbol{j} = -[(1-u^2)\nabla \tfrac{\delta F}{\delta u} - \nabla(A(u)q)], && t>0, x\in\Omega,\\
\partial_t q &= -\frac{1}{\tau(u)} q + A(u)\operatorname{div}\boldsymbol{j}, && t>0, x\in\Omega,\\
(1-u^2)\boldsymbol{j}\cdot\nu &= 0, \qquad \nabla u\cdot\nu = 0, && t>0, x\in\partial\Omega,
\end{aligned}
$$

  $A(u)$: bulk modulus; $\tau(u)$: relaxation time; $\Omega \subset \mathbb{R}^{\mathrm{d}}$ smooth bounded domain, $\mathrm{d} \geq 2$.
- Brunk, Lukáčová-Medvid'ová (2022) with hydrodynamics $\longrightarrow$ later today

Entropic structure

$$\dot{\boldsymbol{z}} = -\mathcal{K}(\boldsymbol{z}) D\mathcal{E}(\boldsymbol{z})$$

- State $\boldsymbol{z} = (u, q)$
- Free energy $\mathcal{E}(u, q) = F(u) + \frac{1}{2}\int_\Omega q^2 \,\mathrm{d}x$ with $\nabla u \cdot \nu = 0$ on $\partial\Omega$
- Onsager operator $\mathcal{K}$: $\mathsf{M}(u)$, $\mathsf{L}(u) \in \mathbb{R}^{2\text{x}2}_{\text{sym}}$ positive semi-definite, $N_1(u) \in \mathbb{R}^{2\text{x}2}$

$$\mathcal{K}(u, q)\square = -N_1(u)^T \operatorname{div}\left(\mathsf{M}(u)\nabla(N_1(u)\square)\right) + \mathsf{L}(u)\square \quad \text{with no-flux b.c.}$$

ZZE model:

$$\mathsf{M}(u) = N_2(u) m(u)(\mathbf{1}\otimes\mathbf{1})N_2(u), \quad \text{where } \mathbf{1} = (1,1)^T, \qquad \mathsf{L}(u) = \operatorname{diag}\left(0, \frac{1}{\tau(u)}\right)$$

$$N_1(u) = \operatorname{diag}(1, -A(u)), \quad N_2(u) = \operatorname{diag}\left(1, \frac{1}{n(u)}\right)$$

- with $m(u) = (1-u^2)^2, \qquad n(u) = 1 - u^2$

Entropic structure

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Entropic structure

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- Onsager operator $\mathcal{K}$: $\mathrm{M}(u)$, $\mathrm{L}(u) \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ positive semi-definite, $N_1(u) \in \mathbb{R}^{2\times 2}$

$$\mathcal{K}(u,q)\Box = -N_1(u)^T \operatorname{div}\left(\mathrm{M}(u)\nabla(N_1(u)\Box)\right) + \mathrm{L}(u)\Box \quad \text{with no-flux b.c.}$$

Modified **ZZE model:**

$$\mathrm{M}(u) = N_2(u)m(u)(\mathbf{1}\otimes\mathbf{1})N_2(u), \quad \text{where } \mathbf{1} = (1,1)^T, \qquad \mathrm{L}(u) = \operatorname{diag}\left(0, \frac{1}{\tau(u)}\right)$$

$$N_1(u) = \operatorname{diag}(1, -A(u)), \quad N_2(u) = \operatorname{diag}\left(1, \frac{1}{n(u)}\right)$$

- with $m(u) = 1 - u^2$, $u \mapsto n(u)$ monotonic; $\min_{[-1,1]}|n| > 0$; either $n' \equiv 0$ or $|n'| > 0$ on $[-1,1]$

Effect of coupling function $n(u)$

$$\partial_t u = -\operatorname{div}\left(m(u)\boldsymbol{j}\right), \qquad \boldsymbol{j} = -\left[\nabla w - \frac{1}{n(u)}\nabla(A(u)q)\right], \ w \in \partial_u F_\varepsilon, \qquad t > 0, x \in \Omega,$$

$$\partial_t q = -\frac{1}{\tau(u)} q + A(u)\operatorname{div}\left(\tfrac{m(u)}{n(u)}\boldsymbol{j}\right), \qquad t > 0, x \in \Omega,$$

$$m(u)\boldsymbol{j}\cdot\nu = 0, \qquad \nabla u \cdot \nu = 0, \qquad t > 0, x \in \partial\Omega,$$

with $F_\varepsilon(u) = \int_\Omega \left(\frac{\varepsilon^2}{2}|\nabla u|^2 + f(u)\right)\mathrm{d}x$.

Effect of coupling function $n(u)$

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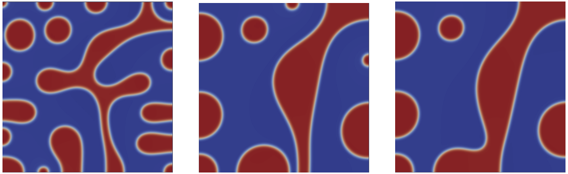
&\partial_t z = -\frac{1}{\tau(u)}q + A(u)\nabla(\frac{1}{n(u)})\cdot m(u)\boldsymbol{j}, \quad z = q + R(u), \quad R' = \frac{A}{n}, && t>0, x\in\Omega,\\
&m(u)\boldsymbol{j}\cdot\nu = 0, \qquad \nabla u\cdot\nu = 0, && t>0, x\in\partial\Omega,
\end{aligned}
$$

with $F_\varepsilon(u) = \int_\Omega \big(\frac{\varepsilon^2}{2}|\nabla u|^2 + f(u)\big)\,\mathrm{d}x$.

Later-stage evolution of Cahn–Hilliard type models

Expect curvature flow:

$$V_\Gamma = \mathcal{G}_\Gamma \kappa_\Gamma, \qquad V_\Gamma: \text{ normal velocity}, \quad \kappa_\Gamma: \text{ mean curvature of interface } \Gamma.$$



Degenerate Cahn–Hilliard equation along $t \mapsto \varepsilon^2 t$ and vanishing temperature θ:

- Formal asymptotics by Cahn, Elliott, Novick–Cohen (1996) yield surface diffusion flow:

$$\mathcal{G}_\Gamma = -\frac{\sigma}{\delta}\Delta_\Gamma, \qquad \frac{\sigma}{\delta} = \frac{16}{\pi^2} > 0,$$

  [for logarithmic potential with $\theta = O(\varepsilon^\alpha)$, $\alpha > 0$; resp. double-obstacle potential $f^{(\mathrm{DO})}$]
- Rigorous limit is open

Interface dynamics in VPS (KH, J. R. King, A. Münch, B. Wagner)

(i) If $n \equiv 1$: intermediate surface diffusion flow [Cahn, Taylor (1994)]

$$\mathcal{G}_\Gamma = -\sigma(\delta \mathrm{Id} - \omega\Delta_\Gamma)^{-1}\Delta_\Gamma, \qquad \sigma,\delta,\omega > 0.$$

(ii) If $n \in C^\infty(\mathbb{R})$ with $\min_{[-1,1]}|n| > 0$, $\min_{[-1,1]}|n'| > 0$, then $\mathcal{G}_\Gamma : \kappa \to \mathrm{V}$ is determined by a **constrained elliptic equation**. Given $\kappa = \kappa(s), s \in \Gamma$, find solution (f,V), $\mathrm{f} = \mathrm{f}(s,u), \mathrm{V} = \mathrm{V}(s)$:

$$-\partial_u(\mathrm{a}\partial_u \mathrm{f}) - \tilde{\mathrm{m}}\Delta_\Gamma \mathrm{f} = \big(1 - \partial_u(\frac{n}{n'})\big)\mathrm{V} \qquad \text{in } \Gamma \times [-1,1],$$

$$-\mathrm{a}\partial_u \mathrm{f} = -\frac{n}{n'}\mathrm{V} \qquad \text{on } \Gamma \times \{\pm 1\},$$

$$\int_{-1}^{+1} (\mathrm{f} + \frac{n}{n'}\partial_u \mathrm{f})\,\mathrm{d}u = \sigma\kappa \qquad \text{on } \Gamma \qquad \text{("solvability condition"; constraint)}$$

For special choice of $m(u), A(u), \tau(u)$:

$$\mathcal{G}_\Gamma = \sigma\eta\sqrt{-\Delta_\Gamma} + \sigma\mathcal{R}(\sqrt{-\Delta_\Gamma}), \quad \mathcal{R} \text{ of lower order}, \qquad \eta = \Big(\big(\tfrac{n(1)}{n'(1)}\big)^2 + \big(\tfrac{n(-1)}{n'(-1)}\big)^2\Big)^{-1}$$

Order of $\mathcal{R}(\sqrt{-\Delta_\Gamma})$ is $-\infty$ if $n = n(u)$ is affine.

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Thank you!

Definition (Dissipative measure-valued dmv solution)

Let $\mu \in L^\infty_{w^*}([0,\infty)\times\mathbb{T}^d;\mathcal{P}(W))$ be parametrised measure and define $u := \langle \mu, s\rangle$ and $\mathbf{y} := \langle \mu, \mathbf{p}\rangle$.
Then, μ is called a dmv solution if for all $T>0$:

- **Basic regularity:** For $i = 1,\dots,n$

$$u_i \in L^\infty(0,\infty;L^2), \quad \partial_t u \in L^2(0,\infty;(W^{1,4})^*), \quad \mathbf{y} \in L^2((0,T)\times\mathbb{T}^d;(\ker B^\perp)^d), \quad \mathbf{y} = \nabla\widehat{u}.$$

  Moreover, μ acts trivially on the $\widehat{s}$-component, i.e. for all $f \in C_0(W)$: $\langle \mu, f(s,\mathbf{p})\rangle = \langle \mu, f(\widehat{u} + P_{\ker B} s, \mathbf{p})\rangle$

- **Dissipation inequalities:** It holds for a.e. $t>0$ that

$$H^{\mathrm{mv}}(u(t)) + \int_0^t \int_{\mathbb{T}^d} \langle \mu_{\tau.x}, |B^{1/2}\mathbf{p}|^2\rangle \mathrm{d}x\mathrm{d}\tau \le H(u^{\mathrm{in}}), \quad \text{where } H^{\mathrm{mv}}(u(t)) := \int_{\mathbb{T}^d} \langle \mu_{t,x}, h(s)\rangle \mathrm{d}x,$$

$$Q(u(t)) + \sum_{i=1}^n \int_0^t \int_{\mathbb{T}^d} \left\langle \mu_{\tau,x}, s_i |(B\mathbf{p})_i|^2 \right\rangle \mathrm{d}x\mathrm{d}\tau \le Q(u^{\mathrm{in}}).$$

- **Evolution equation:** It holds for all $i = 1,\dots,n$ and $\phi \in C^1_c([0,T)\times\mathbb{T}^d)$ that

$$\int_0^T \int_{\mathbb{T}^d} u_i \partial_t \phi \mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}^d} u_i^{\mathrm{in}} \phi(0) \mathrm{d}x = \int_0^T \int_{\mathbb{T}^d} \langle \mu_{t,x}, s_i (B\mathbf{p})_i\rangle \cdot \nabla\phi \mathrm{d}x\mathrm{d}t.$$