Compressible heat conducting mixtures - existence analysis: steady & unsteady case

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References

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Goal

We consider a flow of a L-constituent mixture in a space time cylinder $Q := (0, T) \times \Omega$ with $\Omega \subset \mathbb{R}^d$ and $d = 2, 3$. Our goal is to find a model that is

- mathematically treatable - existence of a (weak, very weak, entropy, etc.) solution
- as easy as possible - could be also very naive in some aspects (not reflecting all phenomena)
- able to handle the cases with $L > 2$
- thermodynamically and mechanically consistent (first and second law of thermodynamics, etc.)
- capable to describe the observable laws/effect: Fick law, Ohm law, Peltier effect, Joul heat, Soret effect, Dufour effect, Tomphson effect and the Seebeck effect. But we require the system to be thermodynamically and mechanically compatible.

Balance of mass

- $\rho_i : Q \to \mathbb{R}_+$ - the density of the i-th constituent
- $\boldsymbol{v}_i : Q \to \mathbb{R}^d$ - the velocity of the i-th consitutent
- $\varrho : Q \to \mathbb{R}_+$ - the density and $\mathfrak{c} = (c_1, \dots, c_L) : Q \to [0,1]^L$ the concentration vector, i.e.,

$$\varrho := \sum_{i=1}^{L} \rho_i, \qquad c_i := \frac{\rho_i}{\varrho}$$

- $\boldsymbol{v} : Q \to \mathbb{R}^d$ - the barycentric velocity, i.e.,

$$\boldsymbol{v} := \frac{\sum_{i=1}^{L} \rho_i \boldsymbol{v}_i}{\varrho}$$

- $\mathfrak{r} := (r_1, \dots, r_L) : Q \to \mathbb{R}^L$ with r_i being the production rate of the i-th constituent

$$\boxed{\begin{aligned} &\partial_t \rho_i + \operatorname{div}(\rho_i \boldsymbol{v}_i) = r_i \qquad \text{for } i = 1, \dots, L, \\ &\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{v}) = \sum_{i=1}^{L} r_i. \end{aligned}}$$

Balance of mass - modelling - incompressible

- We consider the mixture that is homogeneous and incompressible, i.e.

$$\boxed{\varrho = const = 1, \qquad \operatorname{div} \boldsymbol{v} = 0, \qquad \rho_i = c_i}$$

- We consider that the only macroscopic velocity is the barycentric one and we intend to describe all quantities and laws in terms of $\boldsymbol{v}$ instead of $\boldsymbol{v}_i$. We introduce the diffusion flux $\mathbf{q}_{\mathfrak{c}} := (\mathbf{q}_{\mathfrak{c}}^1, \dots, \mathbf{q}_{\mathfrak{c}}^L) : Q \to \mathbb{R}^{d\times L}$ which models

$$\boxed{\mathbf{q}_{\mathfrak{c}}^i :\overset{\sim}{=} c_i \boldsymbol{v}_i - c_i \boldsymbol{v}, \qquad i = 1, \dots, L}$$

and the balance of mass for the i-th constituent takes the form

$$\boxed{\partial_t \mathfrak{c} + \operatorname{div}(\mathfrak{c}\boldsymbol{v}) + \operatorname{div} \mathbf{q}_{\mathfrak{c}} = \mathfrak{r}}$$

- total charge balance ($\mathfrak{z} := (z_1, \dots, z_L)$ where z_i is the specific charge of c_i)

$$\boxed{\partial_t Q + \operatorname{div}(Q\boldsymbol{v}) + \operatorname{div}(\mathbf{q}_{\mathfrak{c}}\mathfrak{z}) = \mathfrak{r} \cdot \mathfrak{z}}$$

- The necessary compatibility conditions

$$\boxed{\sum^L r_i = 0, \qquad \sum^L r_i z_i = 0, \qquad \sum^L \mathbf{q}_{\mathfrak{c}}^i = \mathbf{0}}$$

Linear momentum and electric potential

We consider the possibly non-Newtonian electrically charged fluid described by

$$\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \mathbf{S} = -\nabla p + \boldsymbol{f} - Q\nabla\varphi$$

$$-\Delta\varphi = Q := \sum_{i=1}^{L} z_i c_i = \mathfrak{z} \cdot \mathfrak{c}, \qquad \operatorname{div} \boldsymbol{v} = 0,$$

where

- p - is the pressure
- $\boldsymbol{f}$ - the external body forces; $-Q\nabla\varphi$ - Lorentz force
- $\mathbf{S}$ - the constitutively determined part of the Cauchy stress, eg.,

$$\mathbf{S} = \mathbf{S}^*(\text{invariants}, \mathbf{D}) = 2\nu(\text{invariants})(1 + |\mathbf{D}|^2)^{\frac{r-2}{2}} \mathbf{D},$$

- φ - electrostatic potential
- z_i - specific charge of c_i; Q - total charge; $\mathfrak{z} := (z_1, \dots, z_L)$

where

$$\mathbf{D} := \frac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T)$$

Balance of global energy

- We define
$$E := \frac{|\boldsymbol{v}|^2}{2} + e + \frac{|\nabla\varphi|^2}{2} \qquad \text{total energy},$$
where $e : Q \to \mathbb{R}_+$ denotes the internal energy.
- The balance of the total energy of the problem can be written in the following form (we assume that there are no sources)
$$\boxed{\partial_t E + \operatorname{div}((|\boldsymbol{v}|^2/2 + p + e + Q\varphi)\boldsymbol{v} - \varphi\nabla\partial_t\varphi) - \operatorname{div}(\mathbf{S}\boldsymbol{v}) + \operatorname{div}\mathbf{q}_E = \boldsymbol{f}\cdot\boldsymbol{v}}$$
where $\mathbf{q}_E : Q \to \mathbb{R}^d$ denotes the flux of the global energy not coming from the Cauchy stress.
- Balance of kinetic energy (only for “regular” solutions)
$$\boxed{\partial_t\left(\frac{|\boldsymbol{v}|^2}{2}\right) + \operatorname{div}(|\boldsymbol{v}|^2/2 + p)\boldsymbol{v} - \operatorname{div}(\mathbf{S}\boldsymbol{v}) + \mathbf{S}\cdot\nabla\boldsymbol{v} = \boldsymbol{f}\cdot\boldsymbol{v}}$$

Balance of electrostatic energy

- Multiply balance of total charge by φ

$$\varphi\left(\partial_t Q + \mathrm{div}(Q\boldsymbol{v}) + \mathrm{div}(\mathbf{q}_{\mathfrak{c}}\mathfrak{z})\right) = 0$$

leads to ($-\Delta\varphi = Q$)

$$\boxed{\partial_t\left(\frac{|\nabla\varphi|^2}{2}\right) + \mathrm{div}\left(-\varphi\nabla\partial_t\varphi + \varphi Q\boldsymbol{v} + \varphi\mathbf{q}_{\mathfrak{c}}\mathfrak{z}\right) - Q\boldsymbol{v}\cdot\nabla\varphi - \mathbf{q}_{\mathfrak{c}}\cdot(\nabla\varphi\otimes\mathfrak{z}) = 0}$$

- Write the equation for internal energy $e = E - |\boldsymbol{v}|^2/2 - |\nabla\varphi|^2/2$

$$\boxed{\partial_t e + \mathrm{div}(e\boldsymbol{v}) + \mathrm{div}(\mathbf{q}_E - \varphi\mathbf{q}_{\mathfrak{c}}\mathfrak{z}) + \mathbf{q}_{\mathfrak{c}}\cdot(\nabla\varphi\otimes\mathfrak{z}) = \mathbf{S}\cdot\nabla\boldsymbol{v}}$$

Constitutive equations

- We need to specify the structure of fluxes in terms of unknowns
- Set $(e, \mathfrak{c})$ the primary state variable and express all fluxes w.r.t them, i.e.,

$$\boxed{\begin{aligned} \mathbf{q}_{\mathfrak{c}} &:= \mathbf{q}^*_{\mathfrak{c}}(e, \mathfrak{c}, \varphi, \nabla e, \nabla \mathfrak{c}, \nabla \varphi \ldots), \\ \mathbf{q}_E &:= \mathbf{q}^*_E(e, \mathfrak{c}, \varphi, \nabla e, \nabla \mathfrak{c}, \nabla \varphi \ldots), \end{aligned}}$$

  where $\mathbf{q}^*_{\mathfrak{c}}$ and $\mathbf{q}^*_e$ are "proper" functions
- What is "proper" will be identified with the help of the entropy inequality

Entropy inequality

- We assume that there exists an entropy s, which is given by

$$\boxed{s := s^*(e, \mathfrak{c})},$$

where $s^* : \mathbb{R}_+ \times (0,1)^L \to \mathbb{R}$ is a concave smooth function, that fulfils

$$\boxed{\partial_t s + \operatorname{div}(s\boldsymbol{v}) + \operatorname{div} \mathbf{q}_s \geq 0}$$

- we introduce a temperature $\theta : Q \to \mathbb{R}_+$ defined as

$$\boxed{\theta := \theta^*(e, \mathfrak{c}), \qquad \text{where} \qquad \theta^* := \frac{1}{\partial_e s^*}}$$

- we introduce the vector of re-scaled chemical potentials $\zeta : Q \to \mathbb{R}^L$ as

$$\boxed{\zeta := \zeta^*(e, \mathfrak{c}), \qquad \text{where} \qquad \zeta^* := -\partial_{\mathfrak{c}} s^*}$$

Entropy inequality & constraints

We "deduce" the entropy inequality from the internal energy balance and from the equations for $\mathfrak{c}$

- multiplying the internal energy balance by $\frac{1}{\theta} = \partial_e s^*(e, \mathfrak{c})$

$$\partial_e s^*(e,\mathfrak{c})\partial_t e + \partial_e s^*(e,\mathfrak{c})\nabla e \cdot \boldsymbol{v} + \operatorname{div} \frac{\mathbf{q}_E - \varphi \mathbf{q}_\mathfrak{c} \mathfrak{z}}{\theta} = \frac{\mathbf{S}\cdot\mathbf{D} - \mathbf{q}_\mathfrak{c}\cdot(\nabla\varphi\otimes\mathfrak{z})}{\theta} + (\mathbf{q}_E - \varphi\mathbf{q}_\mathfrak{c}\mathfrak{z})\cdot\nabla\frac{1}{\theta}$$

- multiplying the equation for c_i by $-\zeta_i = \partial_{c_i} s^*(e, \mathfrak{c})$ and summing over $i = 1, \dots, L$ we get

$$\partial_\mathfrak{c} s^*(e,\mathfrak{c})\cdot\partial_t\mathfrak{c} + \nabla\mathfrak{c}\cdot(\boldsymbol{v}\otimes\partial_\mathfrak{c} s^*(e,\mathfrak{c})) - \operatorname{div}(\mathbf{q}_\mathfrak{c}\zeta) = -\mathfrak{r}\cdot\zeta - \mathbf{q}_\mathfrak{c}\cdot\nabla\zeta$$

- summing the result

$$\partial_t s + \operatorname{div}(s\boldsymbol{v}) + \operatorname{div}\Big(\underbrace{\frac{\mathbf{q}_E - \varphi\mathbf{q}_\mathfrak{c}\mathfrak{z}}{\theta} - \mathbf{q}_\mathfrak{c}\zeta}_{\mathbf{q}_s}\Big)$$

$$= \underbrace{\frac{\mathbf{S}\cdot\mathbf{D}}{\theta} + \left((\mathbf{q}_E - \varphi\mathbf{q}_\mathfrak{c}\mathfrak{z})\cdot\nabla\frac{1}{\theta} - \mathbf{q}_\mathfrak{c}\cdot\nabla\zeta - \frac{\mathbf{q}_\mathfrak{c}\cdot(\nabla\varphi\otimes\mathfrak{z})}{\theta}\right) - \mathfrak{r}\cdot\zeta}_{\geq 0}$$

Constitutive laws - only linear case

For simplicity, we consider that the diffusion flux and the heat flux are linear functions of the chemical potential gradients, the temperature gradients and the electrostatic potential gradients, then necessarily

$$\mathbf{q}_{\mathfrak{c}}^i := -\sum_{j=1}^{L} M^{ij}(\mathfrak{c},\theta,\varphi)\left(\nabla\zeta^j + \frac{z_j}{\theta}\nabla\varphi\right) - \mathfrak{m}^i(\mathfrak{c},\theta,\varphi)\nabla\frac{1}{\theta}$$

$$\mathbf{q}_E := \varphi\mathbf{q}_{\mathfrak{c}\mathfrak{z}} - \kappa(\mathfrak{c},\theta)\nabla\theta - \sum_{i=1}^{L}\mathfrak{m}^i(\mathfrak{c},\theta)\left(\nabla\zeta^i + \frac{z_i}{\theta}\nabla\varphi\right)$$

with $M : [0,1]^L \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^{L\times L}_{sym}$, $\mathfrak{m} : [0,1]^L \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^L$ and $\kappa : [0,1]^L \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ being continuous mappings.

Constitutive laws - constraints & assumptions

Due to the requirements on $\mathfrak{c}$ we specify certain algebraic notation

- We define the so-called Gibs simplex as

$$G := \{x \in \mathbb{R}^L;\ x_i \geq 0 \text{ for } i = 1, \dots L,\ \sum_{i=1}^{L} x_i = 1\}$$

  and our goal is to look for $\mathfrak{c} \in G$ a.e. in Q.

- We denote

$$\ell := (1, \dots, 1) \in \mathbb{R}^L$$

  and define the orthogonal projection $\mathcal{P}_\ell$ as

$$\mathcal{P}_\ell x := x - \frac{x \cdot \ell}{|\ell|^2}\ell.$$

Constitutive laws - constraints & assumptions

- diffusion flux

$$\sum_{i=1}^{L} M^{ij} = \sum_{i=1}^{L} \mathfrak{m}^i = 0 \implies \ell \text{ is eigenvector of } M \text{ with eigenvalue } 0$$

$$\alpha(\theta)|\mathcal{P}_\ell x|^2 \le \sum_{i,j=1}^{L} M^{ij}(\mathcal{P}_\ell x)_i(\mathcal{P}_\ell x)_j = \sum_{i,j=1}^{L} M^{ij} x_i x_j \le \alpha^{-1}|x|^2, \qquad |\mathfrak{m}| \le \alpha^{-1}\min(1,\theta)$$

with $\alpha(\theta) \to 0$ as $\theta \to 0$.

- heat flux - for some $\beta \in (0,1]$

$$c_1 \le \frac{\kappa(\theta)}{1+\theta^{-\beta}} \le c_2$$

- the production rate $\mathfrak{r}$

$$\sum_{i=1}^{L} r_i = 0 \implies \mathcal{P}_\ell \mathfrak{r}^*(\mathfrak{c},\theta,\zeta) = \mathfrak{r}^*(\mathfrak{c},\theta,\zeta)$$

- entropy

$$\boxed{s^*(e,\mathfrak{c}) = s^*_e(e) + s^*_{\mathfrak{c}}(\mathfrak{c})} \qquad \text{the most restrictive one} \qquad \implies \boxed{\psi(\theta,\mathfrak{c}) = \psi_1(\theta) + \theta\psi_2(\mathfrak{c})}$$

Constitutive laws - entropy

- the assumptions on s_e^*: it is strictly concave, strictly increasing $\mathcal{C}^2$ function such that $s_e^*(0) = 0$ and

$$\boxed{(s_e^*)'(e) \to \infty \text{ as } e \to 0_+} \quad \Longrightarrow \quad \boxed{e = c_v(\theta)\theta}$$

  with bounded c_v but $c_v(\theta) \to 0$ as $\theta \to 0_+$

- the assumption on $s_\mathfrak{c}^*$: it is strictly concave $\mathcal{C}^2$ function such that

$$\boxed{C_1|x|^2 \leq -\sum_{i,j=1}^{L} \partial^2_{c_i c_j} s_\mathfrak{c}^*(\mathfrak{c}) x_i x_j}$$

  In addition, we assume that for all $K > 0$ there exists $\varepsilon > 0$ such that

$$\boxed{|\partial_\mathfrak{c} s_\mathfrak{c}(\mathfrak{c})| \leq K \implies c_i \geq \varepsilon \text{ for all } i = 1, \ldots, L} \qquad \text{entropy does not like } 0$$

A priori estimates

We formally derive a priori estimates - all boundary integral vanish - no-slip or slip bc for $\boldsymbol{v}$, Neumann or Newton for $\mathfrak{c}$ and θ

- kinetic energy

$$\boxed{\sup_{t\in(0,T)} \|\boldsymbol{v}(t)\|_2^2 + \int_0^T \|\nabla \boldsymbol{v}\|_r^r \le \|\boldsymbol{v}_0\|_2^2 + C(\boldsymbol{f},\dots)}$$

- total energy

$$\boxed{\sup_{t\in(0,T)} \|e(t)\|_1 \le \|e_0\|_1 + C(\boldsymbol{f},\dots)}$$

- entropy inequality (Dafour and Sorret effects are not “visible”)

$$\frac{d}{dt}\int_\Omega s \ge \int_\Omega \frac{\kappa|\nabla\theta|^2}{\theta^2} + \sum_{i,j=1}^{L} M^{ij}\nabla\zeta^i\cdot\nabla\zeta^j - \mathfrak{r}\cdot\zeta$$

which leads to (under proper boundary conditions)

$$\boxed{\sup_{t\in(0,T)} (\|\theta(t)\|_1 + \|\boldsymbol{v}(t)\|_2) + \int_0^T \|\ln\theta\|_{1,2}^2 + \|\nabla(\theta^{-\beta/2})\|_2^2 + \|\mathcal{P}_\ell\zeta\|_{1,2}^2 + \le C(\boldsymbol{f},\dots)}$$

Further estimates

- from entropy inequality, we see that (we control $\ln \theta$)

$$\theta > 0 \text{ a.e.}$$

- According to the assumption on M and $\mathfrak{m}$ the quantity $\mathfrak{c} \cdot \ell$ satisfies the transport equation with initial data identically equal to one:

$$\boxed{\mathfrak{c} \cdot \ell = 1 \text{ a.e}}$$

- Due to the entropy estimates we can now look onto the equation for internal energy as on the heat equation with right hand side being in $L^1 \cap (W^{1,2})^*$. Hence the standard procedure leads to

$$\boxed{\int_Q \frac{|\nabla \theta|^2}{(1+\theta)^{1+\varepsilon}} \le C(\varepsilon^{-1})}$$

- Using the assumption on s^* one can deduce that $|\zeta| \le C(1 + |\mathcal{P}_\ell \zeta|)$:

$$\boxed{\int_Q |\zeta|^q + |\mathfrak{r}|^{q'} \le C} \overset{s \text{ does not like } 0}{\Longrightarrow} c_i > 0 \text{ a.e. for } i = 1, \dots, L \Longrightarrow \boxed{\|\mathfrak{c}\|_\infty \le C}$$

Existence of a weak solution

Theorem

For any "reasonable" data there exists a weak solution.

Balance of mass - modelling - compressible

- Compressible mixtures

$$\varrho := \sum_{i=1}^{L} \rho_i .$$

- We consider that the only macroscopic velocity is the barycentric one and we intend to describe all quantities and laws in terms of $\boldsymbol{v}$ instead of $\boldsymbol{v}_i$. We introduce the diffusion flux $\mathbf{q}_{\mathfrak{c}} := (\mathbf{q}_{\mathfrak{c}}^1, \dots, \mathbf{q}_{\mathfrak{c}}^L) : Q \to \mathbb{R}^{d\times L}$ which models

$$\mathbf{q}_{\mathfrak{c}}^i :\stackrel{\sim}{=} \rho_i \boldsymbol{v}_i - \rho_i \boldsymbol{v}, \qquad i = 1, \dots, L$$

and the balance of mass for the i-th constituent takes the form

$$\partial_t \boldsymbol{\rho} + \operatorname{div}(\boldsymbol{\rho v}) + \operatorname{div} \mathbf{q}_{\mathfrak{c}} = \mathfrak{r}$$

- The necessary compatibility conditions

$$\sum_{i=1}^{L} r_i = 0, \qquad \sum_{i=1}^{L} \mathbf{q}_{\mathfrak{c}}^i = \mathbf{0}$$

Linear momentum

We consider the **Newtonian** fluid described by

$$\boxed{\partial_t(\varrho \boldsymbol{v}) + \operatorname{div}(\varrho \boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div}(2\mu(\theta)(\mathbf{D} - \mathbf{I} \operatorname{div} \boldsymbol{v}/3) + \lambda(\theta)\mathbf{I} \operatorname{div} \boldsymbol{v}) = -\nabla p(\theta, \boldsymbol{\rho}) + \varrho \boldsymbol{f},}$$

where

- p - is the pressure
- $\boldsymbol{f}$ - the external body forces;

where

$$\mathbf{D} := \frac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T)$$

We require the growth conditions for viscosities

$$\mu(\theta) \sim (1+\theta)$$
$$0 \leq \lambda(\theta) \lesssim (1+\theta)$$

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Constitutive equations

- Free energy

$$\rho\psi := \theta \sum_{i=1}^{L} \frac{\rho_i}{m_i} \log \frac{\rho_i}{m_i} + F(\boldsymbol{\rho}) - c_W \varrho\theta \log\theta - G(\theta)$$

here m_i are molar masses, the first term corresponds to the ideal fluid, the function F is assumed to be convex and giving the "sufficient" growth for ϱ, e.g.

$$F(\boldsymbol{\rho}) := \left(\sum_{i=1}^{L} \frac{\rho_i}{m_i}\right)^{\gamma}, \quad \text{or easier case } F(\boldsymbol{\rho}) := \sum_{i=1}^{L} \left(\frac{\rho_i}{m_i}\right)^{\gamma}$$

- further quantities

$$\mu_i = \frac{\partial(\varrho\psi)}{\partial\rho_i}, \quad \varrho e = \varrho\psi - \theta\frac{\partial(\varrho\psi)}{\partial\theta}, \quad -\varrho s = \frac{\partial(\varrho\psi)}{\partial\theta}, \quad p = -\varrho\psi + \sum_{i=1}^{L} \rho_i \frac{\partial(\varrho\psi)}{\partial\rho_i}.$$

- Constitutive equations: for $i = 1, \dots, N$,

$$\mathbf{q}_i = -\sum_{j=1}^{L} M_{ij} \nabla \frac{\mu_j}{\theta} - M_i \nabla \frac{1}{\theta}, \qquad \mathbf{q}_\theta := -\kappa(\theta)\nabla\theta - \sum_{i=1}^{L} M_i \nabla \frac{\mu_i}{\theta}.$$

Necessary conditions - theorem

- The quantities $(M_{ij})_{i,j=1,\dots,L}$, $(M_i)_{i=1,\dots,L}$ satisfy

$$\sum_{i=1}^{N} M_{ij} = \sum_{i=1}^{L} M_i = 0 \qquad j = 1,\dots,L.$$

This is required by mass conservation.

- Moreover we assume that (maximal coercivity)

$$\sum_{i,j=1}^{L} M_{ij} z_i z_j \geq c|P_\ell \boldsymbol{z}|^2 \qquad \boldsymbol{z} \in \mathbb{R}^L$$

- Further, we assume that the reaction term $\mathfrak{r} = \mathfrak{r}(P_\ell(\boldsymbol{\mu}/\theta), \theta)$ and again has maximal coercivity

$$\mathfrak{r}(P_\ell(\boldsymbol{z}), \theta) \cdot \boldsymbol{z} \geq C|P_\ell \boldsymbol{z}|^2$$

Theorem

For steady case and reasonable data, there is always a weak solution.

Necessary conditions - theorem

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$$\mathfrak{r}(P_\ell(\boldsymbol{z}), \theta) \cdot \boldsymbol{z} \geq C|P_\ell \boldsymbol{z}|^2$$

Theorem

For steady case and reasonable data, there is always a weak solution.

Notes to proof

- sequence of regularized problems for $(\boldsymbol{\rho}^n, \theta^n, \boldsymbol{v}^n)$ with a priori bounds ($\varrho^n := \sum_{i=1}^L \rho_i^n$)

$$\|\boldsymbol{v}^n\|_{1,2} + \|\theta^n\|_{1,q} + \|P_\ell(\boldsymbol{\mu}^n/\theta^n)\|_{1,2} + \|\varrho^n\|_{\gamma+\varepsilon} \le C$$

- standard identification of limiting equation, it remains to identify the nonlinearities

$$\begin{aligned}
\boldsymbol{v}^n &\to \boldsymbol{v} && \text{a.e. in } Q\\
\theta^n &\to \theta && \text{a.e. in } Q\\
P_\ell(\boldsymbol{\mu}^n/\theta^n) &\to \overline{P_\ell(\boldsymbol{\mu}/\theta)} && \text{a.e. in } Q\\
P_\ell(\boldsymbol{\mu}^n) &\to \overline{P_\ell(\boldsymbol{\mu})} && \text{a.e. in } Q\\
\boldsymbol{\rho}^n &\rightharpoonup \boldsymbol{\rho} && \text{weakly}\\
p^n &\rightharpoonup \overline{p(\boldsymbol{\rho}, \theta)} && \text{weakly}
\end{aligned}$$

- how to get the compactness of $\boldsymbol{\rho}^n$: assume that $\mu_i \sim \rho_i$ then

$$(\rho_1^n, \dots, \rho_L^n) - \varrho^n(1, \dots, 1)/L \to \overline{(\rho_1, \dots, \rho_L) - \varrho(1, \dots, 1)/L} \text{ a.e. in } Q$$

  it is enough to show

$$\varrho^n \to \varrho \text{ a.e. in Q}$$

Main key sub-results

- The most problematic is the compactness of densities ρ_i. But $P_\ell(\mu_j)$ is compact so what is enough is

$$\lim_{k\to\infty} \lim_{n,m\to\infty} \| T_k(\varrho^n) - T_k(\varrho^m)\|_2 = 0$$

- It is true as far as we can renormalize the equation for ϱ - ok for large γ
- The standard trick which works for lower gamma is based on the control of

$$\sup_k \lim_{n\to\infty} \| T_k(\varrho^n) - T_k(\varrho)\|_{2+\varepsilon} \leq C$$

but we only have (for the difficult choice of F)

$$\boxed{\sup_k \lim_{n,m\to\infty} \|\sqrt{\theta}(T_k(\varrho^n) - T_k(\varrho^m))\|_2 \leq C}$$

But using once again the effective viscous flux equation, we get that the above estimate is sufficient for the renormalization.