Group representations 1

Characters, introduction

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Multiplicity a reminder

Definition

Let R be a semisimple artinian ring and let M be a finitely generated R-module. Then there are simple modules $S_1, S_2, \ldots, S_m$ such that $M \simeq \oplus_{i=1}^m S_i$. If S is a simple left R-module, the *multiplicity of S in M* is defined as

$$|\{i \in \{1, \ldots, k\} \mid S_i \simeq S\}|$$

Remark

The multiplicity is correctly defined. That is, if $S_1, \ldots, S_n$, $T_1, \ldots, T_m$ are simple left R-modules and $\oplus_{i=1}^n S_i \simeq \oplus_{i=1}^m T_i$ then for every simple left R-module S

$$|\{i \in \{1, \ldots, n\} \mid S_i \simeq S\}| = |\{i \in \{1, \ldots, m\} \mid T_i \simeq S\}|$$

multiplicity = dimension, a reminder

Proposition

Assume that R is a semisimple artininian finite dimensional $\mathbb{F}$-algebra, $\mathbb{F}$ algebraically closed. For any simple left R-module the multiplicity of S in ${}_R R$ coincides with $\dim_{\mathbb{F}}(S)$.

Proof.

Use the Wedderburn-Artin theorem, there are $k \in \mathbb{N}_0$ and $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that

$$R \simeq \mathrm{M}_{n_1}(\mathbb{F}) \times \cdots \times \mathrm{M}_{n_k}(\mathbb{F})$$

as $\mathbb{F}$-algebras. So we may assume

$$R = \mathrm{M}_{n_1}(\mathbb{F}) \times \cdots \times \mathrm{M}_{n_k}(\mathbb{F})$$

and in this case we know that there are simple left modules $S_1, S_2, \ldots, S_k$ pair-wise non-isomorphic such that ${}_R R \simeq S_1^{n_1} \oplus S_2^{n_2} \oplus \cdots \oplus S_k^{n_k}$. So the multiplicity of S_i in R is n_i which coincides with $\dim_{\mathbb{F}}(S_i)$.

The regular representation

Let G be a finite group, $\mathbb{F}$ an algebraically closed field with $\mathrm{char}(\mathbb{F}) \nmid |G|$. Let $S_1, S_2, \ldots, S_k$ be simple left $\mathbb{F}G$ modules such that every simple left $\mathbb{F}G$ module is isomorphic to exactly 1 of these modules. We already know that k is the number of conjugacy classes in G. For $1 \leq i \leq k$ let $d_i := \dim_{\mathbb{F}}(S_i)$.
By the previous proposition the multiplicity of S_i in ${}_{\mathbb{F}G}\mathbb{F}G$ is exactly d_i, so $\mathbb{F}G \simeq \oplus_{i=1}^k S_i^{d_i}$ as left $\mathbb{F}G$-modules.
Now look at dimensions of these modules: $\dim_{\mathbb{F}}(\mathbb{F}G) = |G|$, $\dim_{\mathbb{F}}(\oplus_{i=1}^k S_i^{d_i}) = \sum_{i=1}^k d_i^2$.
Thus we conclude

$$|G| = \sum_{i=1}^{k} d_i^2 \,.$$

Summary in the language of $\mathrm{Rep}_{\mathbb{F}}(G)$

Theorem

Let G be a finite group, $\mathbb{F}$ an algebraically closed field, $\mathrm{char}(\mathbb{F}) \nmid |G|$.

1. *Every representation of G over $\mathbb{F}$ is equivalent to a direct sum of irreducible representations.*
2. *If k is the number of conjugacy classes of G, then there are $\varphi_1, \varphi_2, \cdots, \varphi_k \in \mathrm{Rep}_{\mathbb{F}}(G)$ irreducible such that every irreducible representation of G over $\mathbb{F}$ is equivalent to exactly one of these representations.*
3. *If d_i is the degree of φ_i then the multiplicity of φ_i in $\mathrm{reg}_{\mathbb{F}}(G)$ is d_i*

$$\mathrm{reg}_{\mathbb{F}}(G) \simeq \overbrace{\varphi_1 \oplus \cdots \oplus \varphi_1}^{d_1} \oplus \cdots \oplus \overbrace{\varphi_k \oplus \cdots \oplus \varphi_k}^{d_k}$$

4. $|G| = \sum_{i=1}^{k} d_i^2$

Representations of finite abelian groups

Proposition

Let G be a finite group and $\mathbb{F}$ algebraically closed field such that $\mathrm{char}(\mathbb{F}) \nmid |G|$. Then G is abelian if and only if every irreducible representation of G over $\mathbb{F}$ has degree 1.

Proof.

Let k be the number of conjugacy classes in G. Note that G is abelian if and only if $k = |G|$.
Let $\varphi_1, \varphi_2, \ldots, \varphi_k \in \mathrm{Rep}_{\mathbb{F}}(G)$ be the list of all irreducible representations of G over $\mathbb{F}$ up to equivalence (that is, every irreducible representation of G over $\mathbb{F}$ is equivalent to exactly one representation on the list). Let d_i be the degree of φ_i. Then $|G| = \sum_{i=1}^{k} d_i^2$.
If $|G| = k$, then every d_i is one which implies that every irreducible representation of G over $\mathbb{F}$ has degree 1.
Conversely, if $d_i = 1$ for every $i \in \{1, \ldots, k\}$, we get $|G| = k$, so G is an abelian group. □

Representations of finite abelian groups

Remark

Recall we saw an example of an irreducible representation $\varphi \in \mathrm{Rep}_{\mathbb{Q}}(\mathbb{Z}_3)$ *of degree 2.*

Example

Find all irreducible complex representations of a finite abelian group.

Idea: We have to find $\mathrm{Hom}(G, \mathbb{C}^*)$ for a finite abelian group. We know the size of this set. We use the theorem on the structure of finite abelian groups and the universal property of coproducts.

Characters, informal intro

Characters can be seen as numerical invariants associated to group representations of finite degree.
Sometimes one can use characters to decide whether a given representation is irreducible or not or whether two given representations are equivalent or not.

Characters, the definition

Definition

Let G be a group and let $\psi\colon G \to \mathrm{GL}(n, \mathbb{F})$ be a matrix representation of G over a field $\mathbb{F}$. *Character* of ψ is a function $\chi_\psi\colon G \to \mathbb{F}$ defined by the rule

$$\chi_\psi\colon g \mapsto \mathrm{Tr}(\psi(g)), g \in G$$

Remark

If $n = 1$ then ψ and χ_ψ are essentially the same.

Definition

Let G be a group and let $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ be a representation of G over $\mathbb{F}$ of finite degree (that is $\dim_{\mathbb{F}}(V) < \infty$). *Character* of φ is a function $\chi_\varphi\colon G \to \mathbb{F}$ defined by the rule

$$\chi_\varphi\colon g \mapsto \mathrm{Tr}([\varphi(g)]_B), g \in G$$

where B is a basis of V.

Basic properties of characters

Proposition

Let G be a group and let $\mathbb{F}$ be a field.

a) *Equivalent representations of G over $\mathbb{F}$ have equal characters.*
b) *Character of any representation of G over $\mathbb{F}$ is constant on conjugacy classes of G.*
c) *Character of a direct sum of representations is the sum of characters of the summands*
d) *If $g \in G$ is an element of order $n \in \mathbb{N}$, $\mathbb{E}$ an extension of $\mathbb{F}$ containing all roots of $x^n - 1$ and $\psi\colon G \to \mathrm{GL}(d, \mathbb{F})$ a matrix representation of degree d. Then $\chi_\psi(g) = \sum_{i=1}^{d} \lambda_i$, where $\lambda_1, \ldots, \lambda_d \in \mathbb{E}$ are roots of $x^n - 1$.*
e) *If $\mathbb{F} = \mathbb{C}$ and $g \in G$ is an element of finite order, the $\chi_\psi(g^{-1}) = \overline{\chi_\psi(g)}$ for every matrix representation $\psi\colon G \to \mathrm{GL}(d, \mathbb{C})$ of G over $\mathbb{C}$.*

Proofs of a) and b)

We consider matrix representations only (proofs for linear representations are essentially the same).

The crucial property is that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for every pair of matrices $A \in \mathrm{M}_{k,\ell}(\mathbb{F})$ and $B \in \mathrm{M}_{\ell,k}(\mathbb{F})$.

a) Assume $\psi_1, \psi_2 \colon G \to \mathrm{GL}(d, \mathbb{F})$ are equivalent representations of G over $\mathbb{F}$.

That means, there exists a matrix $X \in \mathrm{GL}(d, \mathbb{F})$ such that for every $g \in G$

$$\psi_1(g) = X\psi_2(g)X^{-1}\,.$$

Apply trace:

$$\chi_{\psi_1}(g) = \mathrm{Tr}(X\psi_2(g)X^{-1}) = \mathrm{Tr}(\psi_2(g)X^{-1}X) = \chi_{\psi_2}(g)$$

b) Let $\psi \colon G \to \mathrm{GL}(d, \mathbb{F})$ be a matrix representation of G over $\mathbb{F}$ and let $g, h \in G$ in the same conjugacy class. That is, there exists $x \in G$ such that $h = xgx^{-1}$. Then

$$\chi_\psi(h) = \mathrm{Tr}(\psi(x)\psi(g)\psi(x)^{-1}) = \mathrm{Tr}(\psi(g)\psi(x)^{-1}\psi(x)) = \chi_\psi(g)$$

Proof of c)

Again we prove this part for matrix representations only. For $i = 1, 2, \ldots, k$ let $\psi_i \colon G \to \mathrm{GL}(d_i, \mathbb{F})$ be a matrix representation of G over $\mathbb{F}$. We want to prove

$$\chi_{\psi_1 \oplus \psi_2 \oplus \cdots \oplus \psi_k} = \sum_{i=1}^{k} \chi_{\psi_i}$$

But this is obvious from the definition of direct sum of representations: Recall $\psi := \psi_1 \oplus \psi_2 \oplus \cdots \oplus \psi_k \colon G \to \mathrm{GL}(\sum_{i=1}^{k} d_i, \mathbb{F})$ maps an element $g \in G$ to a block-diagonal matrix $\psi_1(g) \oplus \psi_2(g) \oplus \cdots \oplus \psi_k(g)$. Then the sum of diagonal entries of $\psi(g)$ is the sum of all diagonal entries of matrices $\psi_1(g), \psi_2(g), \ldots, \psi_k(g)$. In other words

$$\chi_\psi(g) = \sum_{i=1}^{k} \chi_{\psi_i}(g)$$

Proof of d)

Since $g^n = 1$ and ψ is a group homomorphism, we obtain $\psi(g)^n = E$. If $\lambda \in \overline{\mathbb{E}}$ is an eigenvalue of $\psi(g)$, then $\lambda^n = 1$. Indeed, let $v \in \overline{\mathbb{E}}^d$ be a nonzero vector satisfying $\psi(g)v = \lambda v$. Then

$$v = (\psi(g))^n v = \lambda^n v$$

which implies λ is a root of $x^n - 1 \in \mathbb{F}[x]$, so by our assumption $\lambda \in \mathbb{E}$.

From linear algebra we know that $\mathrm{Tr}(\psi(g))$ is the sum of all roots of characteristic polynomial of $\psi(g)$ regarding their multiplicity. Since each of these roots is an eigenvalue of $\psi(g)$, we obtain $\lambda_1, \lambda_2, \ldots, \lambda_d$ such that $\lambda_i^n = 1$ for every $1 \leq i \leq d$ and

$$\chi_\psi(g) = \sum_{i=1}^{d} \lambda_i$$

Proof of e)

Again consider $g \in G$ such that $g^n = 1$ and let $\psi \colon G \to \mathrm{GL}(d, \mathbb{C})$ be a homomorphism of groups. Then $\psi(g)^n = E$. We know that every $d \times d$ matrix over $\mathbb{C}$ is similar to an upper triangular matrix (for example to its Jordan canonical form).
Let $X \in \mathrm{GL}(d, \mathbb{C})$ be such that $Y := X\psi(g)X^{-1}$ is upper triangular.
Let $\lambda_1, \ldots, \lambda_d$ be elements on the diagonal of Y. Then

$$\chi_\psi(g) = \mathrm{Tr}(\psi(g)) = \mathrm{Tr}(Y) = \sum_{i=1}^{d} \lambda_i$$

Note also that

$$\chi_\psi(g^{-1}) = \mathrm{Tr}(\psi(g)^{-1}) = \mathrm{Tr}(X\psi(g)^{-1}X^{-1}) = \mathrm{Tr}(Y^{-1})$$

Of course, Y^{-1} is also upper triangular with entries $\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_d^{-1}$ on the diagonal.
Recall that in part d) we proved $\lambda_i^n = 1$, in particular $|\lambda_i| = 1$ for every $1 \leq i \leq d$.

Proof of e)

Hence $\lambda_i^{-1} = \overline{\lambda_i}$ and therefore

$$\chi_\psi(g^{-1}) = \sum_{i=1}^{d} \overline{\lambda_i} = \overline{\sum_{i=1}^{d} \lambda_i} = \overline{\chi_\psi(g)}\,.$$

Remark

Recall that the Jordan canonical form of $\psi(g)$ is diagonal, since if B is a Jordan block with nonzero eigenvalue and size bigger than 1, then $B^n \neq E$ for every $n \in \mathbb{N}$.

Remark

From the proof it follows that if $\psi\colon G \to \mathrm{GL}(d,\mathbb{C})$ is a complex representation of G and $g \in G$ has finite order, then $|\chi_\psi(g)| \leq d$. We will recall this fact later.

Schur's lemma in $\mathrm{Rep}_{\mathbb{F}}(G)$

Lemma

(Schur) Let G be a finite group and let $\mathbb{F}$ be a field.

a) *Assume that $\varphi, \psi \in \mathrm{Rep}_{\mathbb{F}}(G)$ are irreducible. Then every $\theta \in \mathrm{Rep}_{\mathbb{F}}(G)(\varphi, \psi)$ is either 0 or an isomorphism.*

b) *Assume that $\mathbb{F}$ is algebraically closed and $\varphi \in \mathrm{Rep}_{\mathbb{F}}(G)$ is irreducible. Then every $\theta \in \mathrm{Rep}_{\mathbb{F}}(G)(\varphi, \varphi)$ is a scalar multiple of identity.*

Remark

Recall that every irreducible representation of a finite group G over $\mathbb{F}$ corresponds to a simple left $\mathbb{F}G$-module. Every such a module is obtained as a factor of $\mathbb{F}G$ modulo a maximal left ideal, so every irreducible representation of a finite group has finite dimension (you can prove this directly showing that every nonzero representation of a finite group has a nonzero invariant subspace of finite dimension).

Ideas of the proof of Schur's lemma

a) Let $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ and $\psi\colon G \to \mathrm{Aut}_{\mathbb{F}}(U)$ be representations of G over $\mathbb{F}$.
Recall $\theta \in \mathrm{Rep}_{\mathbb{F}}(G)(\varphi, \psi)$ means that $\theta \in \mathrm{Hom}_{\mathbb{F}}(V, U)$ satisfies that for every $g \in G$

$$\theta \circ [\varphi(g)] = [\psi(g)] \circ \theta$$

For this it is easy to see that $\mathrm{Ker}\ \theta$ is φ-invariant subspace of V and $\mathrm{Im}\ \theta$ is ψ-invariant subspace of U.
If φ is irreducible then θ is either a monomorphism or zero.
If ψ is irreducible then θ is either onto or zero.
If both representations are irreducible then θ is either an isomorphism or zero.
b) Assume $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ irreducible and $\mathbb{F}$ algebraically closed. Since $\dim_{\mathbb{F}}(V)$ is finite, $\theta \in \mathrm{End}_{\mathbb{F}}(V)$ has an eigenvalue $\lambda \in \mathbb{F}$. This means that $\theta(v) = \lambda v$ for some $v \neq 0$.
Consider $\theta' := \theta - \lambda 1_V \in \mathrm{Rep}_{\mathbb{F}}(G)(\varphi, \varphi)$. By a) this is is either an isomorphism or zero. Now $v \in \mathrm{Ker}\ \theta$ and hence $\theta' = 0$.

Schur's lemma in matrix form

Lemma

(Schur) Let G be a group, $\mathbb{F}$ be a field.

a) *Assume $\varphi\colon G \to \mathrm{GL}(n, \mathbb{F})$, $\psi\colon G \to \mathrm{GL}(m, \mathbb{F})$ are irreducible matrix representations of G over $\mathbb{F}$. Let $C \in \mathrm{M}_{m,n}(\mathbb{F})$ satisfies*

$$C\varphi(g) = \psi(g)C$$

for every $g \in G$. Then either $C = 0$ or $m = n$ and C is regular.

b) *Assume that $\mathbb{F}$ is algebraically closed, $\varphi\colon G \to \mathrm{GL}(n, \mathbb{F})$ irreducible and $C \in \mathrm{M}_n(\mathbb{F})$ satisfies*

$$C\varphi(g) = \varphi(g)C$$

for every $g \in G$. Then $C = \lambda E$ for some $\lambda \in \mathbb{F}$.

Proof of the matrix form of Schur's lemma, part a)

It is possible to use linear representations associated to matrix reps and apply the previous version of Schur's lemma (at least for finite groups). Let us write here a direct argument.

a) Consider $\mathrm{Ker}\ C := \{v \in \mathbb{F}^n \mid Cv = 0\} \leq \mathbb{F}^n$,
$\mathrm{Im}\ C := \{Cv \mid v \in \mathbb{F}^n\} \leq \mathbb{F}^m$.
An easy computation shows that $\mathrm{Ker}\ C$ is a φ-invariant subspace, i.e.,

$$v \in \mathrm{Ker}\ C \Rightarrow \forall g \in G\ [\varphi(g)]v \in \mathrm{Ker}\ C\,.$$

If $Cv = 0$ then $C[\varphi(g)]v = [\psi(g)]Cv = 0$.
Similarly $\mathrm{Im}\ C$ is ψ-invariant subspace: If $w = Cv$ for some $v \in \mathbb{F}^n$, then $[\psi(g)]w = [\psi(g)]Cv = C([\varphi(g)]v)$. That is,

$$w \in \mathrm{Im}\ C \Rightarrow \forall g \in G\ [\varphi(g)]w \in \mathrm{Im}\ C\,.$$

If φ is irreducible, then either $\mathrm{Ker}\ C = 0$ or $\mathrm{Ker}\ C = \mathbb{F}^n$. If ψ is irreducible, then either $\mathrm{Im}\ C = 0$ or $\mathrm{Im}\ C = \mathbb{F}^m$.
If both representations are irreducible, then either $C = 0$ or $n = m$ and C is regular

Proof of Schur's lemma, part b)

In this case C is a square matrix. Since $\mathbb{F}$ is algebraically closed, there exists $\lambda \in \mathbb{F}$ an eigenvalue of C.
Note that for every $g \in G$ we have
$[\varphi(g)](C - \lambda E) = (C - \lambda E)[\varphi(g)]$. So we can apply part a) for the matrix $C - \lambda E$. This matrix is not regular, so it has to be zero. That is $C = \lambda E$.

Some applications

Corollary

Let G be a finite group. If $\varphi\colon G \to \mathrm{GL}(n, \mathbb{F})$ and $\psi\colon G \to \mathrm{GL}(m, \mathbb{F})$ are irreducible matrix representations of G over $\mathbb{F}$ which are not equivalent and $X \in \mathrm{M}_{m,n}(\mathbb{F})$ is an arbitrary matrix then

$$\sum_{g \in G} \psi(g) X \varphi(g^{-1}) = 0$$

Proof.

Let $Y := \sum_{g \in G} \psi(g) X \varphi(g^{-1})$. Then for every $h \in G$ is

$$\psi(h) Y = \sum_{g \in G} \psi(hg) X \varphi(g^{-1} h^{-1} h) = [\sum_{g \in G} \psi(hg) X \varphi((hg)^{-1})] \varphi(h) \,.$$

In other words $\psi(h) Y = Y \varphi(h)$ for every $h \in G$. By Schur's lemma, $Y = 0$ or $m = n$ and Y is regular. If Y is regular, we obtain $\psi(h) = Y \varphi(h) Y^{-1}$ for every $h \in G$, that is, φ and ψ are equivalent.

Some applications, cont.

Corollary

Let G be a finite group. If $\varphi\colon G \to \mathrm{GL}(n, \mathbb{F})$ is an irreducible matrix representation and $\mathbb{F}$ is algebraically closed, then for every $X \in \mathrm{M}_n(\mathbb{F})$ there exists $\lambda \in \mathbb{F}$ such that

$$\sum_{g \in G} \varphi(g) X \varphi(g^{-1}) = \lambda E$$

(the proof is similar as the proof of the previous corollary)

The end

That's all for today.
Thnk you for your attention.