3.12.2020

Plán: 1. Kap. 20 ⟶ Věta 20.1. ($\exists$ c.n.)

2. úlohy na regulaci

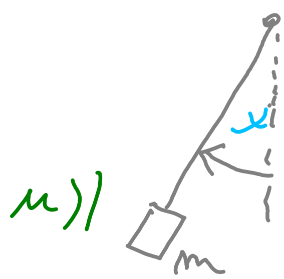
(3. aproximace c.n., aplikace na stabilitu)

ú 21)

$$x'' + \frac{g}{\ell}\sin x = u$$

$$x(0) = x_0$$

$$x'(0) = x_1$$



zjednodušení:

$$x'' + x = u$$

$$x(0) = x_0$$

$$x'(0) = x_1$$

přípustné regulace

$$u(t) : [0,T) \longrightarrow [-1,1]$$

cíl: najít $u(\cdot)$ t.ž. $x(t^*) = 0$, $x'(t^*) = 0$,

$t^* \geq 0$ minimální

(viz 18.III)

$$x' = y \quad \Rightarrow \quad \begin{aligned} x' &= y \\ y' &= -x + u \end{aligned} \;=\; \underbrace{\begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u$$

$n = 2$, $m = 1$

$\mathcal{K}(A|B) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ … rank $= 2$, $\sigma(A) = \{\pm i\}$

tj. Re ≤ 0

$\Rightarrow$ globální regul. V. 18.7

$\exists$ časově optim. reg. V. 18.8

navíc $u^*(t) = \pm 1$ s.v. V. 18.9.

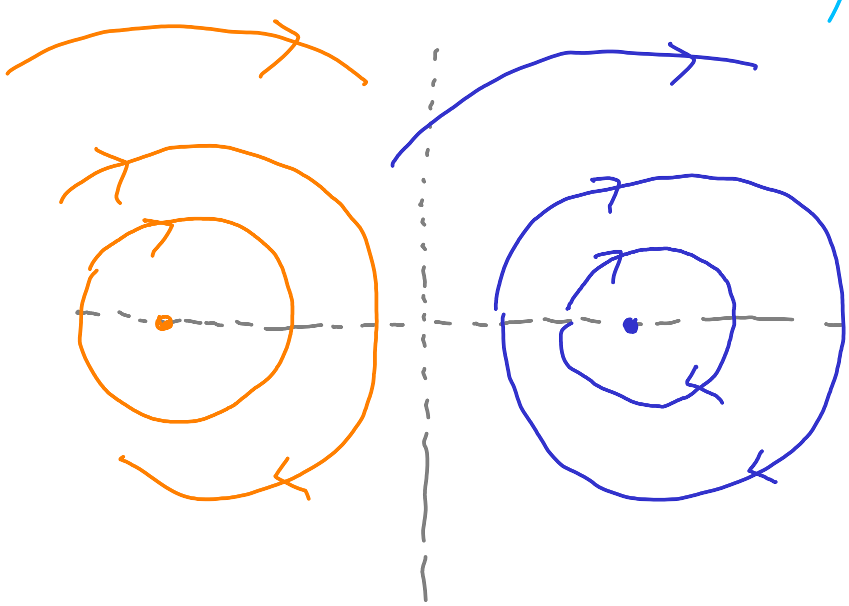
(bang-bang)

elementární úvahy: $u \equiv 1$ $\Leftrightarrow$ $x'' + x = 1$

$$x'' + (x-1) = 0 \quad \Big/ \, 2x'$$

$$\frac{d}{dt}\left[(x')^2 + (x-1)^2\right] = 0$$

$$\Rightarrow \quad y^2 + (x-1)^2 \equiv c$$



$u \equiv -1$:

Pozn: perioda oběhu $= 2\pi$

užití (PM)

(Věta 18.10)

$u^*(t)$ je optimální $\Longrightarrow$ $\exists\, h \in \mathbb{R}^2 \setminus \{0\}$ t.ž.

$$h \cdot \underbrace{e^{-tA}}_{*)} B u^*(t) = \max_{\eta \in [-1,1]} h \cdot e^{-tA} B \eta$$

pro s.v. t

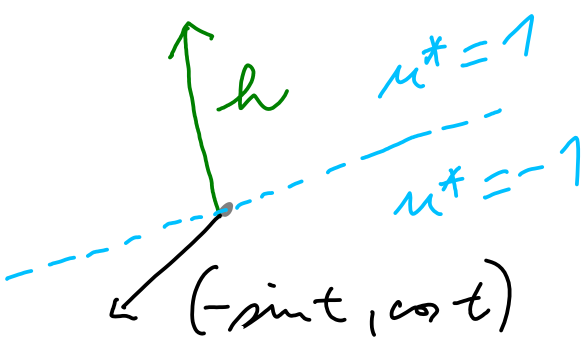
*) $-A = \begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}$ $\quad e^{-tA} = \begin{pmatrix} \cos t, & -\sin t \\ \sin t, & \cos t \end{pmatrix}$

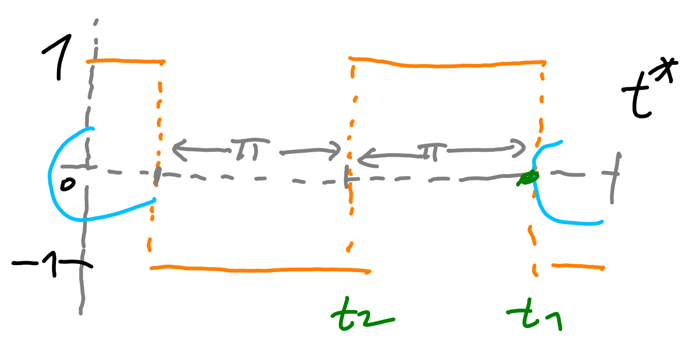
$$e^{-tA} B = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

(PM): $h \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} u^*(t) = \max_{|\eta| \le 1} h \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \eta$

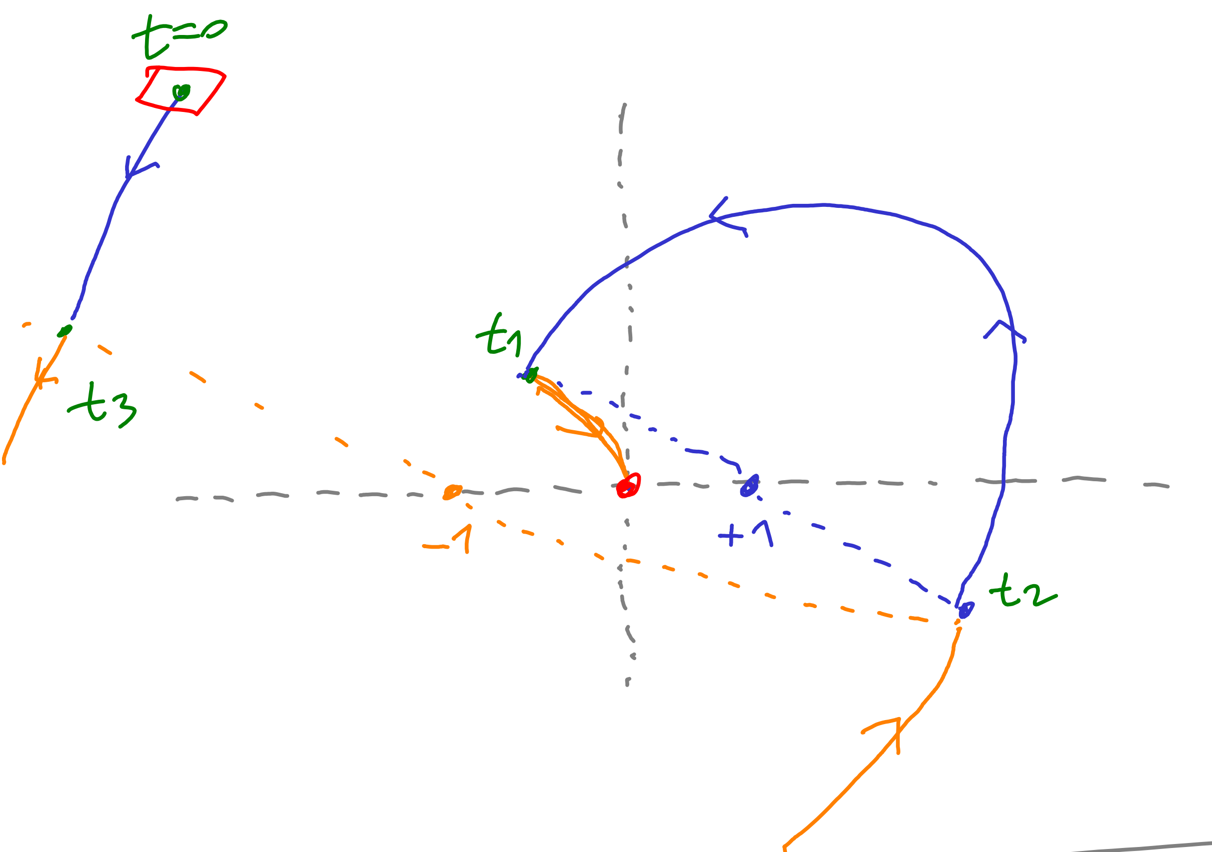
$\Longleftrightarrow$ $u^*(t) = \operatorname{sgn}\left(h \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \right)$

náhled:





rychlé ne optimální regulování: (obrázek !!)



Ú23 (kvadratický regulátor)

$x' = x + u$; $u(t) : [0,T] \to \mathbb{R}$ měřitelné

$x(0) = x_0$

cíl: $\max P[u(\cdot)] = -\int_0^T x^2(t) + \alpha u^2(t)\,dt$

$\left(\Leftrightarrow \min \tilde{P}[u(\cdot)] = \int_0^T x^2(t) + \alpha u^2(t)\,dt \right)$

Kz 18.IV: $H = \lambda(x+u) - x^2 - \alpha u^2$

(PM): $\max\limits_{\eta \in \mathbb{R}} H(x, \lambda, \eta) \ldots$

$$\mathcal{L}\quad H = \lambda x - x^2 + \lambda u - \alpha u^2$$

$$\frac{\partial H}{\partial u} = \lambda - 2\alpha u \implies u(t) = \frac{\lambda(t)}{2\alpha}$$

(ADJ) $$\lambda' = -\frac{\partial H}{\partial x} = -\lambda + 2x, \quad \lambda(T) = 0$$

$(g \equiv 0)$

CELKEM: $u(t) = \dfrac{\lambda(t)}{2\alpha}$, kde

$$\lambda' = -\lambda + 2x$$
$$x' = x + \frac{\lambda}{2\alpha}$$

poč. podm.: $x(0) = x_0$

$\lambda(T) = 0$

? nabývání maxima

Poznámky ke *-slabé konv.

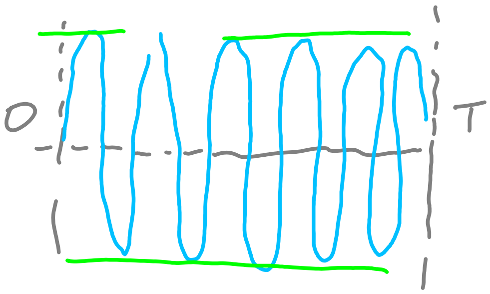
$$u_n(t) \xrightarrow{*} u(t) \text{ v } L^\infty(0,T)$$

$$\iff \int_0^T \varphi(t) u_n(t)\,dt \to \int_0^T \varphi(t) u(t)\,dt$$

pro $\forall\, \varphi(t) \in L^1(0,T)$

zeme

1) $u_n(t) = \sin nt \longrightarrow 0$, leč $\|u_n\|_{L^\infty} = 1$



(Riemann–Lebesgue)

pozor!: $$\int_0^T (u_n(t))^2\,dt \to \frac{T}{2} \neq 0$$

tj. $u(\cdot) \longmapsto \int_0^T u^2(t)\,dt$ není spojitý vůči *-slabé konv.

Tvrzení: nechť $r : \mathbb{R} \to \mathbb{R}$ je C^1, konvexní.
nechť $u_n(t) \xrightarrow{*} u(t)$ v $L^\infty(0,T)$.

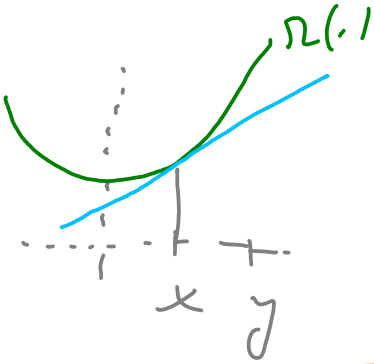
Pak:
$$\liminf_{n\to\infty} \int_0^T r(u_n(t))\,dt \geq \int_0^T r(u(t))\,dt$$

Neboli: konvexita $\Rightarrow$ *-slabá zdola polospojitost.

Důsledek: $\exists$ minima (přechod v minimalizující posl.)

Dz.: convexitate: $\Omega(y) \geq \Omega(x) + \Omega'(x)(y-x)$

$\forall y, x \in \mathbb{R}$



$$\Omega(u_n(t)) \geq \Omega(u(t)) + \Omega'(u(t))\,(u_n(t) - u(t))$$

$$\int_0^T \Omega(u_n(t))\,dt \geq \int_0^T \Omega(u(t))\,dt + \int_0^T \underbrace{\Omega'(u(t))}_{=\,\phi(t) \in L^1} \cdot \underbrace{(u_n(t) - u(t))}_{\overset{*}{\rightharpoonup} 0}\,dt$$

liminf $n \to \infty$ (applied to the first two terms)

last integral $\to 0$ $\square$

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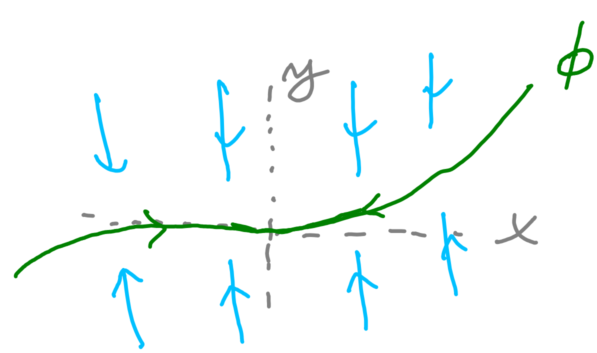
(1) $x' = Ax + f(t,y)$ ... centrali

$\quad y' = By + g(t,y)$ ... ~~stabili~~

(2) $p' = Ap + f(p, \phi(p))$

$\uparrow$ C.V.



Princip redukované stability:

$(0,0)$ je stabilní (resp. asymp. stab.) pro (1)

$\Longleftrightarrow$ 0 má analogickou vlastnost pro (2).

Princip aproximace c.v.:

víme: ϕ je c.v. $\Longleftrightarrow$ platí (RED), tj.:

(navíc $\phi \in C^1$)

$p(t)$ řeší (2) $\Rightarrow$ $(p(t), \phi(p(t)))$ řeší (1), kde $y(t) = \phi(p(t))$

rovnice $(1)_2$: $\left(\phi(p(t))\right)' = B\phi(p(t)) + g(p(t), \phi(p(t)))$

$$\nabla\phi(p(t))\, p'(t) = \nabla\phi(p(t))\left[A p(t) + f(p(t), \phi(p(t)))\right]$$

Pozorování: $\phi \in C^1$, pak ϕ splní (INV) $\Longleftrightarrow$ (RED)

jinými slovy: $M\phi \equiv 0$, kde (DR)

$$[M\phi](x) = \nabla\phi(x)\left[Ax + f(x, \phi(x))\right] - B\phi(x) - g(x, \phi(x)).$$

Pozn.: $\boxed{M\phi \equiv 0}$ $\Leftrightarrow$ odpovídající derivace $\boxed{\phi(x) - y}$ podél (1) je nula.

Princip aproximace c.v.

Nechť $\psi(x): \mathbb{R}^n \to \mathbb{R}^m$ je C^1,

nechť $\psi(0) = 0$, $\nabla \psi(0) = 0$.

Nechť $M\psi(x) = \mathcal{O}(|x|^q)$, $x \to 0$

pro jisté $q > 1$.

Pak c.v. ϕ splňuje $\phi(x) = \psi(x) + \mathcal{O}(|x|^q)$

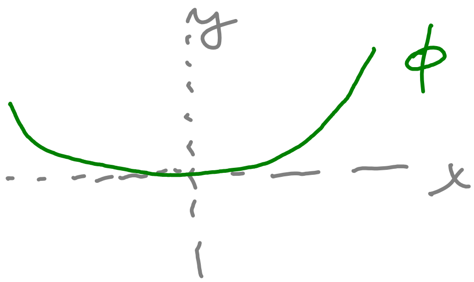
$x \to 0$.

Příklad: $x' = -x^3 + y^2$ $\qquad n = m = 1$

$y' = -2y + x^2$ $\qquad A = (0),\ B = (-2)$

Pozn.: linearizace $\sim (0,0)$: $M = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$, $\sigma = \{-2, 0\}$

? stabilita



? aproximace c.v.

$\left[\Pi\phi = ?\right]$

formálně: $\phi(x) - y \;\Big|\; \frac{d}{dt}$

$$\phi'(x)x' - y' = \phi'(x)\left[-x^3 + (\phi(x))^2\right] - \left(-2\phi(x) + x^2\right)$$

$$\Rightarrow \quad \Pi\phi(x) = \phi'(x)\left(\phi^2(x) - x^3\right) + \underline{2\phi(x) - x^2}$$

zkusme: $\psi(x) = \frac{1}{2}x^2 \Rightarrow \Pi\psi(x) = O(|x|^4)$, $x \to 0$

Princip aproximace: $\phi(x) = \frac{1}{2}x^2 + O(|x|^4)$

redukovaná rce: $p' = -p^3 + (\phi(p))^2$ (*)

$$\Rightarrow \quad p' = -p^3 + \left(\frac{1}{2}\cancel{x}^{\,p\,2} + O(|p|^4)\right)^2$$

$$p' = -p^3 + O(|p|^4) = p^3\underbrace{\left(-1 + O(p)\right)}_{<0 \text{ blízko } 0}$$

$\Rightarrow$ 0 asymp. stab. pro (*)

$\Rightarrow$ (0,0) —"— , původní úlohy

10.12.

ú4) Uvažuji $x' = F(x)$ v okolí $0 \in \mathbb{R}$.

Nechť $F(x) = ax^k + O(x^{k+1})$, $x \to 0$,

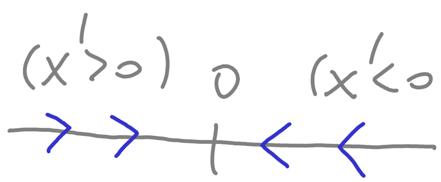
kde $a \neq 0$, $k \in \mathbb{N}$.

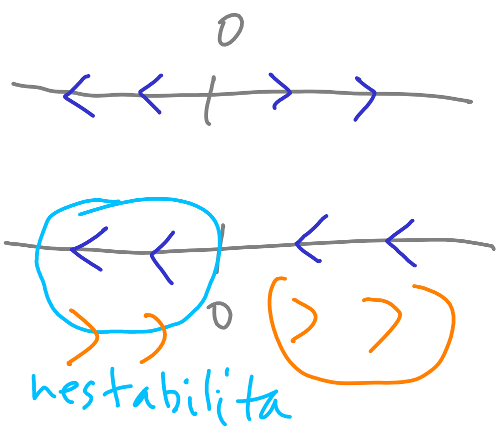
Pozn: (i) $a < 0$, k liché $\Rightarrow$ 0 asympt. stabilní

(ii) ostatní případy $\Rightarrow$ 0 nestabilní

Dk: $x' = ax^k + O(x^{k+1}) = ax^k(1 + O(x))$

ax^k: má znaménko $x'(t)$; $(1 + O(x))$: > 0 blízko 0

(i) 

(ii) 

($a > 0$, k liché)

($a < 0$, k sudé)

Př. 1

$x' = \underbrace{ax^3}_{f} + x^2y$

$y' = -y + \underbrace{y^2 + xy - x^3}_{g}$

$n = m = 1$

$A = 0$, $B = -1$

Věta 20.1. (+ lokal.) $\Rightarrow \exists$ lok. c. v. $y = \phi(x)$ $(\phi: \mathbb{R} \to \mathbb{R})$

aproximace: $(\phi(x) - y)' = \phi'(x)x' - y'$

$= \phi'(x)\left[ax^3 + x^2 y\right] - \left[-y + y^2 + xy - x^3\right]$

$\Rightarrow \quad \Pi\phi(x) = \phi'(x)\left(ax^3 + x^2\phi(x)\right) + \phi(x) - \phi^2(x) - x\phi(x) - x^3$

(1) zkusme: $\psi(x) \equiv 0 \Rightarrow \Pi\psi(x) = -x^3 = O(x^3)$, $x \to 0$

$\Rightarrow \quad \phi(x) = 0 + O(x^3)$

redukovaná rce: $p' = a p^3 + p^2 \phi(p)$

$p' = a p^3 + p^2 O(p^3)$, kde $p^2 O(p^3) = O(p^5)$

$a \neq 0$: jsem hotov!

- $a > 0$ ... nestabilní
- $a < 0$ ... asymp. stab.

$a = 0$ ... ??? $p' = p^2 O(p^3)$ $\gtrless 0$?

(ii) lepší aproximace: $\psi(x) = x^3$

$$\Rightarrow M\psi(x) = 3x^2\left(ax^3 + x^2\cdot x^3\right) + x^3 - (x^3)^2 - \underbrace{x\cdot x^3}_{x^4} - x^3$$

$$= x^4 + O(x^5)$$

$$M\psi(x) = O(x^4) \Rightarrow \phi(x) = x^3 + O(x^4)$$

red. rce: $p' = \underbrace{ap^3}_{=0} + p^2\left(p^3 + O(p^4)\right)$

$$p' = p^5 + O(p^6) \overset{\text{lich}}{\Longrightarrow} \underline{\text{nestabilní}}$$

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p. 3

$x' = -r^l \quad (l \geq 2)$ } centr. $m = 1$, $A = 0$

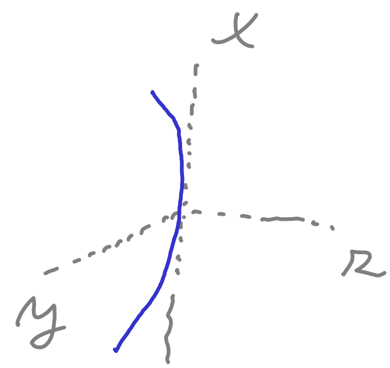
$y' = -y + x^2$

$r' = -2r - x^2$ } stabilní $m = 2$, $B = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$

V. 20.7. (+lok) $\Rightarrow \exists$ c.v. $(\mathbb{R}^n \to \mathbb{R}^m)$

$y = \phi_1(x)$

$r = \phi_2(x)$



aproximare: $\phi_1(t) - y$, $\phi_2(t) - z$ $\Big/ \ \dfrac{d}{dt}$

$$\phi_1' \cdot x' - y' = \phi_1' \cdot (-\phi_2^{\ell}) - (-\phi_1 + x^2)$$
$$\phi_2' \cdot x' - z' = \phi_2' (-\phi_2^{\ell}) - (-2\phi_2 - x^2)$$

$$M\phi(t) = 0 \iff \begin{cases} M_1(\phi)(x) = -\phi_1'\phi_2^{\ell} + \phi_1 - x^2 = 0 \\ M_2(\phi)(x) = -\phi_2'\phi_2^{\ell} + 2\phi_2 + x^2 = 0 \end{cases}$$

[illegible]: $\psi_1(x) = x^2$, $\psi_2(x) = -\frac{1}{2}x^2$

$$\Rightarrow M_1(\psi)(x) = -2x \cdot \left(-\tfrac{1}{2}x^2\right)^{\ell}$$
$$M_2(\psi)(x) = -x\left(-\tfrac{1}{2}x^2\right)^{\ell}$$

[illegible]: $M\psi(x) = O(x^{2\ell+1})$, $x \to 0$

[illegible] $2\ell + 1 \geq 5$ $(\ell \geq 2)$

[illegible]: $p' = -\left(\underbrace{\phi_2(p)}_{= -\frac{1}{2}p^2 + O(p^5)}\right)^{\ell}$

$$p' = -\left(-\frac{1}{2}p^2 + O(p^5)\right)^{\ell}$$

$$p' = -\left(-\frac{1}{2}\right)^{\ell} p^{2\ell} + O(p^7)$$

liché: $\Rightarrow$ nestabilní dle iii2.

Pozn.: máme aproximace

$$\phi_1(x) = x^2 + O(x^5)$$

$$\phi_2(x) = -\frac{1}{2}x^2 + O(x^5)$$