Věta 23.6. [Cauchyho věta.] TRIK

Důkaz č. 1: $f(z) = F_1 + iF_2 =: v_1 - iv_2$

V. 23.2: (C.R.) $\frac{\partial F_1}{\partial x} = \frac{\partial F_2}{\partial y}, \frac{\partial F_1}{\partial y} = -\frac{\partial F_2}{\partial x}$

$\Leftrightarrow$ $\operatorname{div} \underset{\sim}{v} = 0, \operatorname{rot} \underset{\sim}{v} = 0$

$$\int_\varphi g(z)\,dz = \int_a^b g(\varphi(t))\varphi'(t)\,dt$$

$$= \int_a^b (v_1\circ\varphi - i v_2\circ\varphi)(\varphi_1' + i\varphi_2')$$

$$= \int_a^b (v_1\circ\varphi)\varphi_1' + (v_2\circ\varphi)\varphi_2'$$

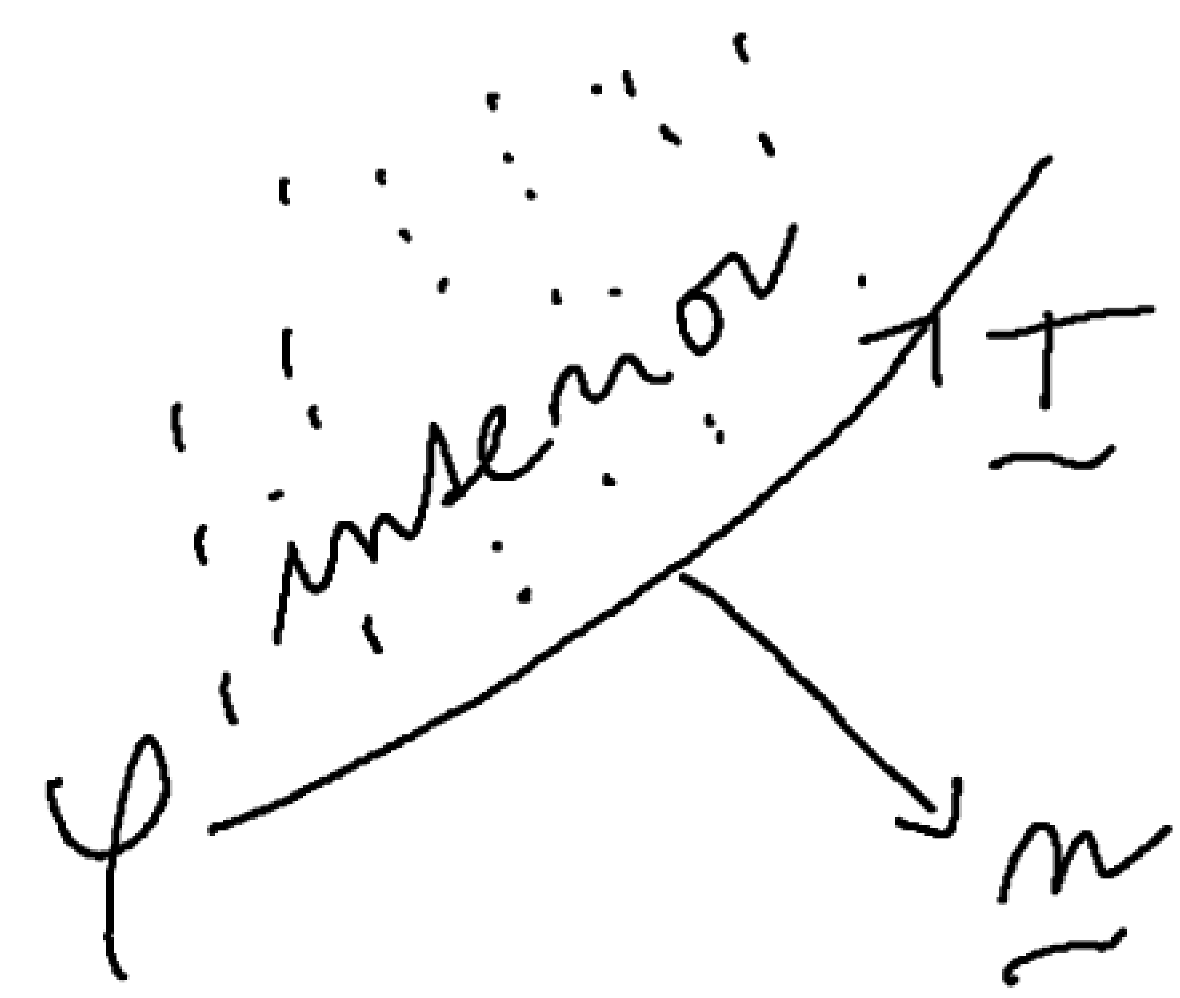
$$+ i\int_a^b (v_1\circ\varphi)\varphi_2' - (v_2\circ\varphi)\varphi_1'$$

$$= \int_{\langle\varphi\rangle} \underset{\sim}{v}\cdot d\underset{\sim}{s} + i\int_{\langle\varphi\rangle} (\underset{\sim}{v}\cdot\underset{\sim}{n})\,ds = K_1 + iK_2$$

neboť: $\underset{\sim}{\varphi}' = (T_1 \circ \varphi, T_2 \circ \varphi) \cdot \|\underset{\sim}{\varphi}'\| = \underset{\sim}{T}\, ds$

$$\underset{\sim}{n} = (n_1, n_2) = (T_2, -T_1)$$

$$\Rightarrow \boxed{\underset{\sim}{n}\, ds = (\varphi_2', -\varphi_1')}$$



Věty 19.6, 19.7 (Gauss & Green):

$$K_2 = \int_{\text{int}\,\varphi} \underbrace{\operatorname{div} \underset{\sim}{v}}_{0}\, dx\,dy = 0$$

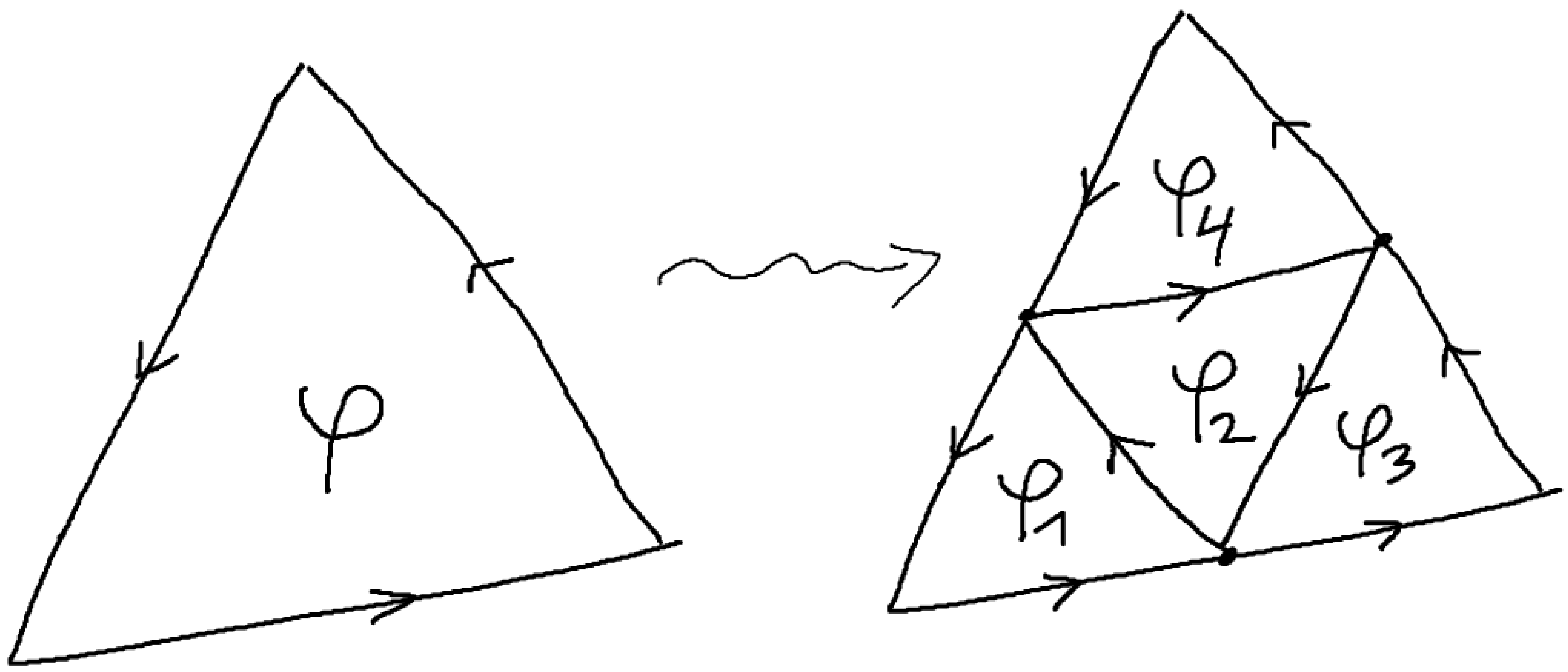
$$K_1 = \int_{\text{int}\,\varphi} \underbrace{\operatorname{rot} \underset{\sim}{v}}_{0}\, dx\,dy = 0.$$

Problém: V. 19.6, 19.7 chtějí $\underset{\sim}{v} \in C^1$, tj. $D\underset{\sim}{v}$ spojitý; my víme (Věta 23.2) pouze $\exists D\underset{\sim}{v}$ (a nic více)

Důkaz č. 2

KROK 1: φ .... trojúhelník

polož $I = \int_\varphi f(z)\,dz$; **cíl: $I = 0$**



$$\int_\varphi f(z)\,dz = \sum_{j=1}^{4} \int_{\varphi_j} f(z)\,dz$$ (princip skládání)

tvrdíme: $\exists j$ t.ž. $\left| \int_{\varphi_j} f(z)\,dz \right| \geq \frac{|I|}{4}$

dk: ?? $|I_j| < \frac{|I|}{4}$, $j = 1, \ldots 4$.

$$\Rightarrow |I| \leq \sum_{j=1}^{4} |I_j| < 4 \cdot \frac{|I|}{4} = |I|$$

↑ SPOR

BÚNO $j=1$:

$$\Rightarrow \quad l(\varphi_1) \leq \frac{1}{2} L(\varphi)$$

$$d(\varphi_1) \leq \frac{1}{2} d(\varphi)$$

leč: $$\left| \int_{\varphi_1} f(z)\,dz \right| \geq \frac{1}{4} |I|$$

kde: $L(\varphi)$ .... délka křivky

$d(\varphi)$ ... diametr (= délka max. strany)

Indukcí: posloupnost $\{\varphi_n\}_{n=1}^{\infty}$

$$L(\varphi_n) \leq \frac{1}{2^n} L(\varphi)$$

$$d(\varphi_n) \leq \frac{1}{2^n} d(\varphi)$$

leč: $$\left| \int_{\varphi_n} f(z)\,dz \right| \geq \frac{1}{4^n} |I|$$

Označme dále: $T_n = \langle \varphi_n \rangle \cup \operatorname{int} \varphi_n$

<u>Limitní bod</u>: volme $z_n \in \langle \varphi_n \rangle$

(libovolné)

<u>Tvrdím</u>: $\{z_n\} \subset \mathbb{C}$ je cauchyovská

<u>dk</u>. $\forall \varepsilon > 0 \; \exists n_0 \; \forall m, n \geq n_0 : |z_m - z_n| < \varepsilon$

$\varepsilon > 0$ dáno: volme $n_0 \in \mathbb{N}$

t.ž. $\frac{1}{2^{n_0}} < \varepsilon / d(\varphi)$

$m, n \geq n_0 \Rightarrow z_m, z_n \in T_{n_0}$,

a tedy $|z_m - z_n| \leq d(\varphi_{n_0})$

$$\leq \frac{1}{2^{n_0}} d(\varphi) < \varepsilon .$$

tedy: (úplnost $\mathbb{C} \simeq \mathbb{R}^2$)

$\exists z_0 \in \mathbb{C}$ t.ž. $z_n \to z_0, n \to \infty$.

a navíc $z_0 \in T_1 \subset \Omega$.

Závěr KROKU 1:

$f(z) \in \mathcal{H}(\Omega)$, $z_0 \in \Omega \Rightarrow \exists f'(z_0) \in \mathbb{C}$

tj. $f(z) = f(z_0) + f'(z_0)(z - z_0) + g(z)$

kde $g(z) = o(z - z_0)$, $z \to z_0$

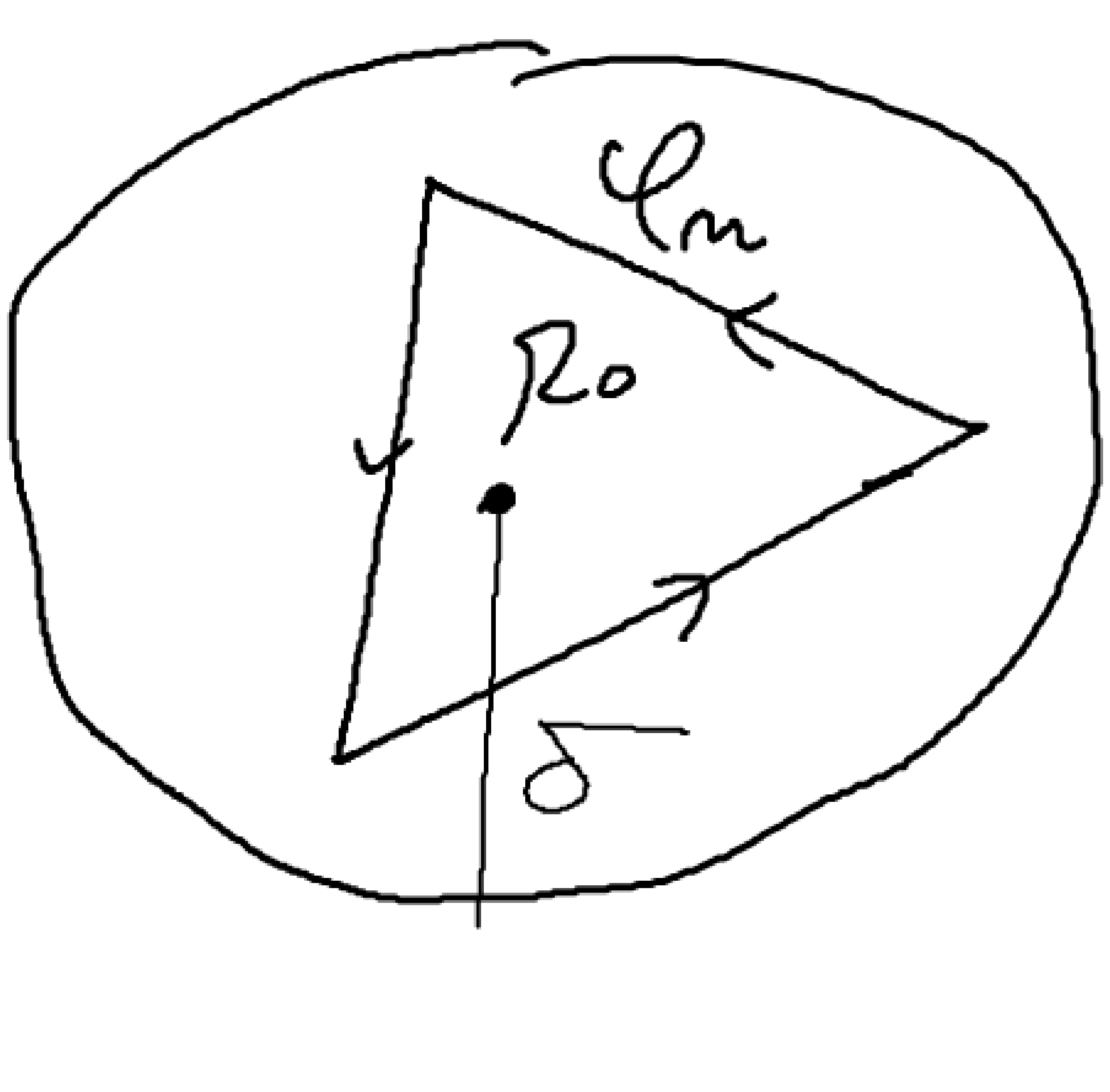
tj. $\dfrac{g(z)}{z - z_0} \to 0$, $z \to z_0$

$\varepsilon > 0$ dáno: $\exists\, \delta > 0$ t.ž.

$$\left| \frac{g(z)}{z - z_0} \right| < \varepsilon \text{ pro } \forall z \in P(z_0, \delta)$$

$$|g(z)| < \varepsilon |z - z_0| ;\ \forall z \in U(z_0, \delta)$$

fixuji $n \in \mathbb{N}$ t.ž. $T_n \subset U(z_0, \delta)$



… lze, neboť:

$z_n \to z_0$;

$z_0, z_n \in T_n$, $d(T_n) \to 0$

víme: $\left| \int_{\varphi_n} f(z)\,dz \right| \geq \frac{1}{4^n} |I|$

leč na druhou stranu:

$f(z) = p(z) + q(z)$, tj.

$$\int_{\varphi_n} f(z)\,dz = \int_{\varphi_n} p(z)\,dz + \int_{\varphi_n} q(z)\,dz = K_1 + K_2$$

ad K_1: $p(z) = P'(z)$, kde

$$P(z) = f(z_0)\cdot z + f'(z_0)\frac{(z-z_0)^2}{2}$$

Věta 23.5, bod 6 $\Longrightarrow$

$$K_1 = \int_{\varphi_n} P'(z)\,dz = P(k.\,b.\,\varphi_n) - P(z.\,b.\,\varphi_n)$$
$$= 0 \quad (\varphi_n \text{ uzavřené})$$

ad K_2: provedeme odhad:

$$|q(z)| < \varepsilon |z - z_0| \leq \varepsilon \cdot d(\varphi_n)$$

...

a sedy (dle Věty 23.5, bod 5)

$$|K_2| = \left| \int_{\varphi_n} g(z)\,dz \right| < \varepsilon\, d(\varphi_n) \cdot L(\varphi_n)$$

$$\leq \varepsilon \frac{1}{2^n} d(\varphi) \cdot \frac{1}{2^n} L(\varphi).$$

<u>CELKEM sedy</u>:

$$\frac{1}{4^n} |I| \leq \left| \int_{\varphi_n} f(z)\,dz \right| \leq \frac{\varepsilon}{4^n} d(\varphi) L(\varphi)$$

$$|I| \leq \varepsilon\, d(\varphi) L(\varphi), \text{ leč}$$

$\varepsilon > 0$ libovolné

$\Rightarrow$ $\boxed{I = 0}$

KROK 2: φ mnohoúhelník

KROK 3: φ obecné (po částech C^1)