Věta 14.1 [Tot. dif a spojitost, derivace.]

nechť $\exists\, df(a)$. Potom:

1. f je spojitá v bodě a

2. pro $\forall v$ je $\frac{\partial f}{\partial v}(a) = df(a)v$.

Dk. víme: $f(a+h) = f(a) + Lh + r(h)$,

kde $L = df(a)$ je lineární, a tedy

$$\|Lh\| \leq C\|h\| \to 0, \quad h \to 0$$

a dále $r(h) = o(\|h\|)$, $h \to 0$,

$$\Rightarrow r(h) = \underbrace{\frac{r(h)}{\|h\|}}_{\to 0} \cdot \underbrace{\|h\|}_{\to 0} \to 0, \quad h \to 0$$

CELKEM: $$f(a+h) = \underbrace{f(a) + \underbrace{Lh + r(h)}_{\to 0}}_{\to f(a),\ h \to 0}$$

neboli $f(x) \to f(a)$, $x \to a$

$\Leftrightarrow$ spojitost v bodě a

(„Věta 2.5")

$\underline{2.}$ $f(a+tv) = f(a) + \underbrace{L(tv)}_{tLv \text{ (linearita } L)} + r(tv)$

$\Rightarrow \frac{1}{t}\left[f(a+tv) - f(a)\right] = \underbrace{Lv + \frac{r(tv)}{t}}_{\to Lv}$

nebot: $\frac{r(tv)}{t} = \underbrace{\frac{r(tv)}{\|tv\|}}_{\to 0} \cdot \underbrace{\frac{\|tv\|}{t}}_{\frac{|t|\|v\|}{t} = (\operatorname{sgn} t)\|v\|} \to 0, \ t \to 0$

... omezené

Věta 14.2. [Derivace a spojitost, t. dif.]

$\underline{1.}$ nechť $\frac{\partial f}{\partial t_i}$ jsou omezené na $U(a)$, pro $\forall i$. Pak f je spojitá v bodě a.

$\underline{2.}$ nechť $\frac{\partial f}{\partial t_i}$ jsou spojité v bodě a, pro $\forall i$. Pak $\exists\, df(a)$.

Dk. BÚNO $a = 0$; nechť $h \in U(0)$ dáno.

pomocné body: $\underset{\sim}{h}^0 = (0, 0, \dots, 0) = 0$

$$\underset{\sim}{h}^1 = (h_1, 0, \dots, 0)$$

$$\underset{\sim}{h}^2 = (h_1, h_2, 0, \dots)$$

$$\underset{\sim}{h}^N = (h_1, h_2, \dots, h_N)$$

$$
\begin{aligned}
f(\underset{\sim}{h}) - f(0) &= f(\underset{\sim}{h}^N) - f(\underset{\sim}{h}^{N-1}) \\
&+ f(\underset{\sim}{h}^{N-1}) - f(\underset{\sim}{h}^{N-2}) \\
&\vdots \\
&+ f(\underset{\sim}{h}^1) - f(\underset{\sim}{h}^0) \\
&= \sum_{i=1}^{N} \underbrace{f(\underset{\sim}{h}^i) - f(\underset{\sim}{h}^{i-1})}_{P_i}.
\end{aligned}
$$

klíčové pozorování: označme

$$\varphi^i(t) = f\left(t h_i \underset{\sim}{e}^i + \underset{\sim}{h}^{i-1}\right), \quad i = 1, \dots N$$

$$P_i = \varphi^i(1) - \varphi^i(0) = (\varphi^i)'(\tau_i), \quad \tau_i \in (0,1)$$

Lagrange!!

$$= \frac{\partial f}{\partial x_i}(\underset{\sim}{\theta}^i) h_i, \quad \text{kde } \underset{\sim}{\theta}^i = \tau_i h_i \underset{\sim}{e}^i + \underset{\sim}{h}^{i-1}$$

CELKEM: $|f(\underline{h})-f(0)| \le \sum_{i=1}^{N} |P_i|$

$$\le \sum_{i=1}^{N} \underbrace{\left|\frac{\partial f}{\partial x_i}(\underline{\theta}^i)\right|}_{\le C} \underbrace{|h_i|}_{\|h\|} \le NC\cdot\|h\|$$

tedy: $f(\underline{h}) \to f(\underline{0})$, $\underline{h} \to 0$.

$\pm \frac{\partial f}{\partial x_i}(0)$

ad 2. $$f(\underline{h})-f(\underline{0}) = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\underline{\theta}^i)\, h_i$$

$$= \underbrace{\sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(0)\, h_i}_{Df(\underline{0})h} + \underbrace{\sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}(\underline{\theta}^i) - \frac{\partial f}{\partial x_i}(\underline{0})\right) h_i}_{=:\, r(h)}$$

... očekávaný tvar tot. dif ...

... zbývá ukázat: $\frac{r(h)}{\|h\|} \to 0,\ h \to 0$

zřejmě platí: $\underline{h} \to 0 \Rightarrow \underline{\theta}^i \to 0$

a tedy $$\frac{|R(h)|}{\|h\|} \le \sum_{j=1}^{N} \underbrace{\left| \frac{\partial f}{\partial x_i}(\underset{\sim}{\theta}^i) - \frac{\partial f}{\partial x_i}(\underset{\sim}{0}) \right|}_{\to 0} \underbrace{\frac{|h_i|}{\|h\|}}_{\le 1}$$

... spojitost $\frac{\partial f}{\partial x_i}$ v bodě $a=0$ ....

Věta 14.3 [Tot. dif součtu a superpozice]

1. nechť $\exists\, df(a), dg(a)$. Potom $\exists\, d(f+g)(a)$ a rovná se $df(a) + dg(a)$.
2. nechť $\exists\, df(a)$, nechť $\exists\, dg(b)$, kde $b = f(a)$. Potom $\exists\, d(g \circ f)(a)$ a rovná se
$$dg(f(a))\, df(a).$$

Dk. 1. víme, že platí

$$f(a+h) = f(a) + Lh + R(h)$$
$$g(a+h) = g(a) + Kh + w(h),$$

kde $L = df(a)$, $K = dg(a)$

$$R(h), w(h) = o(\|h\|),\ h \to 0$$

$$\Rightarrow (f+g)(a+h) = f(a+h) + g(a+h)$$

$$= \underbrace{f(a) + g(a)}_{(f+g)(a)} + \underbrace{Lh + Kh}_{(L+K)h} + \underbrace{r(h) + w(h)}_{\substack{= \\ o(\|h\|),\\ h\to 0}}$$

a tedy: $L + K = d(f+g)(a)$

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2. dle předpokladu:

$$f(a+h) = f(a) + Lh + r(h)$$

$$g(b+k) = g(b) + Kk + w(k)$$

kde $L = df(a)$, $K = dg(b)$, $b = g(a)$

$$\underset{h\to 0}{r(h) = o(\|h\|)}, \quad \underset{k\to 0}{w(k) = o(\|k\|)}$$

$$\Rightarrow (g\circ f)(a+h) = g(f(a+h))$$

$$= g(\underbrace{f(a)}_{b} + \underbrace{Lh + r(h)}_{k})$$

$$= g(b) + K(Lh + r(h)) + w(Lh + r(h))$$

$= (g \circ f)(a) + KLh + \eta^1(h) + \eta^2(h)$

kde $KL = dg(b)\,df(a)$

$\eta^1(h) = KR(h)$

$\eta^2(h) = \omega(Lh + R(h))$

... stačí ukázat:

$\eta^{1,2}(h) = o(\|h\|), \ h \to 0$

$$\frac{\eta^1(h)}{\|h\|} = \frac{KR(h)}{\|h\|} = K\left(\frac{R(h)}{\|h\|}\right) \to K0 = 0$$

... spojitost K (lineární)

$$\frac{\eta^2(h)}{\|h\|} = \begin{cases} 0, & Lh + R(h) = 0 \\ \underbrace{\dfrac{\omega(Lh+R(h))}{\|Lh+R(h)\|}}_{\to 0 \text{ dle „VoLSF"}} \cdot \underbrace{\dfrac{\|Lh+R(h)\|}{\|h\|}}_{\text{omezené, neboť...}} \end{cases}$$

$$\frac{\|Lh+r(h)\|}{\|h\|} \le \underbrace{\frac{\|Lh\|}{\|h\|}}_{\le C} + \underbrace{\frac{\|r(h)\|}{\|h\|}}_{\to 0}$$

neboť: $\|Lh\| \le C\|h\|$

<u>Pozn.</u> sol. dif. součinu?

nechť: $f(t), g(t) : \mathcal{U}(a) \to \mathbb{R}$

<u>TRIK:</u> $f(t)\cdot g(t) = (\Phi \circ F)(t)$

kde $\Phi : (u,v) \longmapsto uv$

$F : t \longmapsto (f(t), g(t))$

zřejmě $\Phi \in C^1$, $\nabla\Phi = \left(\frac{\partial\Phi}{\partial u}, \frac{\partial\Phi}{\partial v}\right) = (v,u)$

... Věta 14.2 $\Rightarrow$ $d\Phi(u,v) : \mathbb{R}^2 \to \mathbb{R}$

$(x_1, x_2) \to vx_1 + ux_2$

... dále: $dF(a) = (df(a), dg(a))$

... V. 14.3 $\Rightarrow$ $d(f\cdot g)(a) = d(\Phi\circ F)(a)$

$= d\Phi(F(a))\,dF(a) = g(a)\,df(a) + f(a)\,dg(a)$

Pozn. ... jiný konec důkazu Věty 14.3, 2.

cíl: $\eta^2(h) = \omega(Lh + r(h)) = o(\|h\|)$

tj: $\forall \varepsilon > 0 \ \exists \delta > 0$ t.ž. $0 < \|h\| < \delta$

$$\Rightarrow \frac{\|\omega(Lh + r(h))\|}{\|h\|} < \varepsilon$$

neboli $\|\omega(Lh + r(h))\| < \varepsilon \|h\|$

... pomocné úvahy:

$$\|Lh + r(h)\| \leq \|Lh\| + \|r(h)\|$$

$$\leq C\|h\| + \underbrace{\frac{\|r(h)\|}{\|h\|}}_{<1,\ \text{je-li } h \text{ blízko } 0} \cdot \|h\|$$

$\Rightarrow \exists \delta_1 > 0$ t.ž. $0 < \|h\| < \delta_1$

$$\Rightarrow \|Lh + r(h)\| < (C+1)\|h\|$$

dále: $\omega(x) = o(x)$, $x \to 0$

a tedy: $\exists \Delta > 0$ t.ž. $0 < \|x\| < \Delta$

$$\Rightarrow \|\omega(x)\| < \frac{\varepsilon}{C+1}\|x\|$$

... dále, polož $\delta_2 = \dfrac{\Delta}{C+1}$

a konečně $\boxed{\delta = \min\{\delta_1, \delta_2\}}$.

Nechť nyní $0 < \|h\| < \delta$ .... chceme položit $x = Lh + r(h)$, ale je třeba rozlišit dva případy:

(i) $Lh + r(h) = 0$ ... pak ale zřejmě

$$w(Lh + r(h)) = 0 < \varepsilon \|h\|,$$

(neboť $w(0) = 0$)

(ii) $Lh + r(h) = x \neq 0$, pak máme

$$0 < \|Lh + r(h)\| < (C+1)\underbrace{\|h\|}_{< \delta_2 = \frac{\Delta}{C+1}} < \Delta$$

a tedy máme:

$$\|w(Lh + r(h))\| < \frac{\varepsilon}{C+1} \underbrace{\|Lh + r(h)\|}_{< (C+1)\|h\|}$$

$$< \varepsilon \|h\|$$