(P1)

$f(x)=\sqrt[3]{x^2(x-6)}$

spojitá v $D_f=\mathbb{R}$ (složení $\sqrt[3]{y}$ s polynomem)

$f(x)=\sqrt[3]{x^3-6x^2}=x\cdot\sqrt[3]{1-6/x}\to\pm\infty$, pro $x\to\pm\infty$

derivace: $x\neq 0,6$: $f'(x)=\dfrac{3x^2-12x}{3\sqrt[3]{x^4(x-6)^2}}=\boxed{\dfrac{x-4}{\sqrt[3]{x(x-6)^2}}}$

$x=0$ $\dfrac{f(x)-f(0)}{x-0}=\dfrac{\sqrt[3]{x^2(x-6)}}{\sqrt[3]{x^3}}=\dfrac{\sqrt[3]{x-6}}{\sqrt[3]{x}}\to\mp\infty,\ x\to 0\pm$

$x=6$ $f'(6)=\lim\limits_{x\to 6}f'(x)=\lim\limits_{x\to 6}\underbrace{(x-4)}_{\to +2}\cdot\underbrace{\dfrac{1}{\sqrt[3]{x(x-6)^2}}}_{\to +\infty}=+\infty$

monotonie: (+ extrémy) $f'(x)=0\iff x=4$ $f(x)$ roste v $(-\infty,0]$ a $[2,+\infty)$

klesá v $[0,2]$

$f(0)=0$ ... ostré lok. max

$f(2)=-2^{4/3}$ ... ostré lok. max

$\nexists$ globální extrém; $H_f=\mathbb{R}$

Konvexita: $(x\neq 0,6)$ $f''(x)=\left(\dfrac{x-4}{\sqrt[3]{x(x-6)^2}}\right)'=\dfrac{1}{\left(\sqrt[3]{x^2(x-6)^4}\right)}\cdot[*]$

kde $[*]=\sqrt[3]{x(x-6)^2}-(x-4)\cdot\underbrace{\left(\sqrt[3]{x(x-6)^2}\right)'}_{[**]}$

$$[**] = \frac{\left(x(x-6)^2\right)'}{3\sqrt[3]{x^2(x-6)^4}} = \frac{3(x-2)(x-6)}{3\sqrt[3]{x^2(x-6)^4}} = \frac{(x-2)}{\sqrt[3]{x^2(x-6)}}$$

$$\rightarrow [*] = \frac{1}{\sqrt[3]{x^2(x-6)}} \cdot \Big(\underbrace{x(x-6)-(x-4)(x-2)}_{-8}\Big)$$

CELKEM: $f''(x) = \dfrac{-8}{\sqrt[3]{x^4(x-6)^5}}$ > 0 pro $x<0$ nebo $x \in (0,6)$

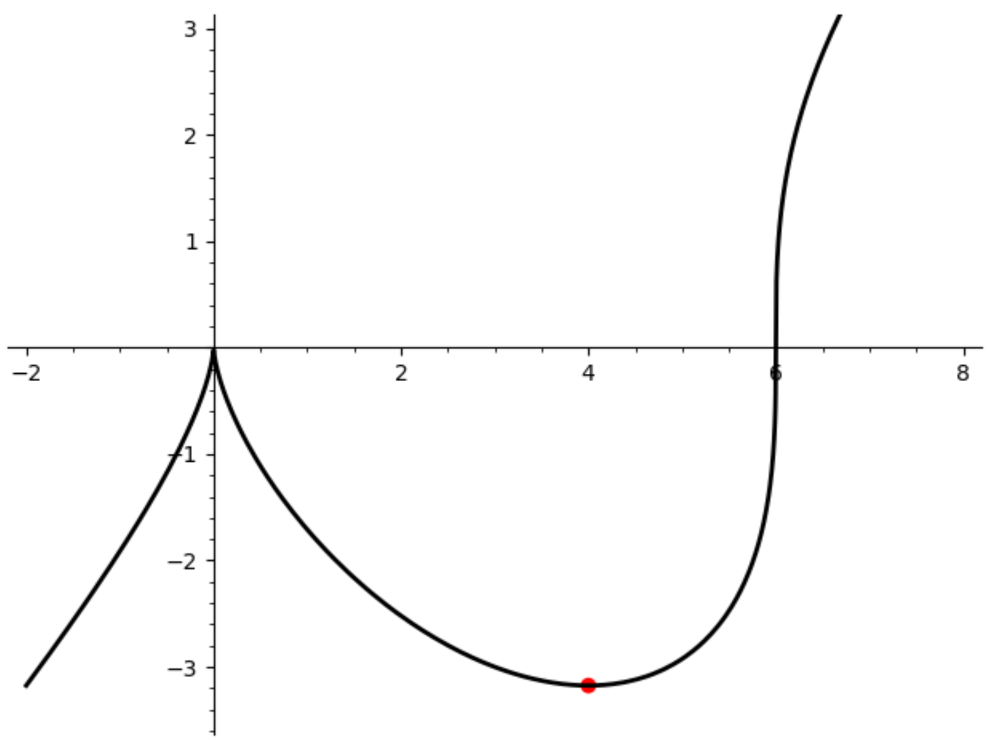
< 0 pro $x > 6$

$\Rightarrow$ $f(x)$ ryze konvexní v $(-\infty, 0]$ a v $[0,6]$,

leč NE v $(-\infty, 6]$ $\left(f'_-(0) > f'_+(0)\right)$

$f(x)$ ryze konkávní v $[6, +\infty)$ inflexní bod

$\left(\exists f'(6)\right)$



P2) $f(x) = 5\sin x \cdot \cos^3 x$ $\qquad \mathcal{D}_f = \mathbb{R}$, lichá, π-periodická

(neb: $\sin(x+\pi) = -\sin x$
$\cos(x+\pi) = -\cos x$)

derivace

$\Rightarrow$ stačí pro $x \in [0, \frac{\pi}{2}]$

$$f'(x) = 5\cos^4 x - 15\sin^2 x \cos^2 x$$
$$= 5\cos^2 x\left(\cos^2 x - 3\sin^2 x\right), \quad \forall x \in \mathbb{R}$$

$f'(x) = 0 \Leftrightarrow \cos x = 0$, tj. $x = \frac{\pi}{2}$ $\vee$ $\operatorname{tg} x = \frac{1}{\sqrt{3}}$, tj. $x = \pm\frac{\pi}{6}$

(to vše mod π)

$\Rightarrow$ $f(x)$ roste v $[-\frac{\pi}{6}, \frac{\pi}{6}]$, klesá v doplňcích, tj. $[\frac{\pi}{6}, \pi - \frac{\pi}{6}]$

$f(\pm\frac{\pi}{6}) = \frac{15}{16}\sqrt{3} = \pm 1{,}62 \ldots$ globální max/min.

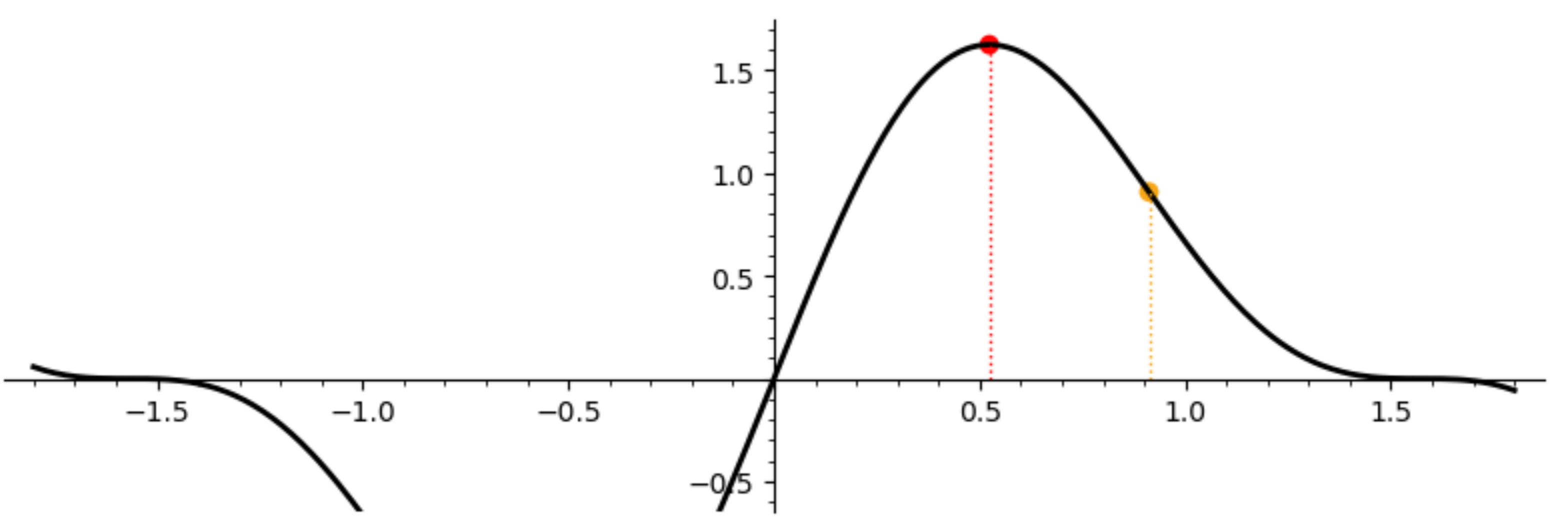
($f(\frac{\pi}{2}) = 0$ není (lok.) extrém)

$f''(x) = $ (po úpravách) $= 10\sin x \cdot \cos x\left(3\sin^2 x - 5\cos^2 x\right)$, $x \in \mathbb{R}$

$\Rightarrow$ $f''(x) = 0 \Leftrightarrow x = 0$ $\vee$ $x = \frac{\pi}{2}$ $\vee$ $\operatorname{tg} x = \pm\sqrt{\frac{5}{3}}$

(mod π)

tj. $x = \pm 0{,}91$



P3) $f(x)=(x-1)e^{-|x-1|}$ $\qquad D_f=\mathbb{R}$, spojitá, vůči $x=0$ „lichá"

$x\to+\infty$: $f(x)=(x-1)\cdot e^{-x+1}=\dfrac{(x-1)\cdot e}{e^x}\to 0$

(analogicky $x\to-\infty$)

derivace, $x\neq 1$

$$f'(x)=1\cdot e^{-|x-1|}+(x-1)\underbrace{\left(e^{-|x-1|}\right)'}_{e^{-|x-1|}\cdot \operatorname{sgn}(x-1)}=e^{-|x-1|}\left(1-\underbrace{(x-1)\cdot\operatorname{sgn}(x-1)}_{|x-1|}\right)$$

$$f'(1)=\lim_{x\to 1}f'(x)=e^0(1-0)=e$$

$$\Rightarrow\ f'(x)=e^{-|x-1|}\left(1-|x-1|\right),\ \forall x\in\mathbb{R}$$

monotonie (text)

$f'(x)=0 \Leftrightarrow 1=|x-1|$, tj. $x=0 \vee 2$

$f'(x)>0$ ($<$) $\qquad 1>|x-1|$ ($<$), tj. $x\in(0,2)$

$x\in(-\infty,0)\vee(2,+\infty)$

$\Rightarrow$ $f(x)$ roste v $[0,2]$, klesá v $(-\infty,0]$ a v $[2,+\infty)$

globální extrémy

$(f=\pm e^{-1}=\pm 0.37)$

Konvexita

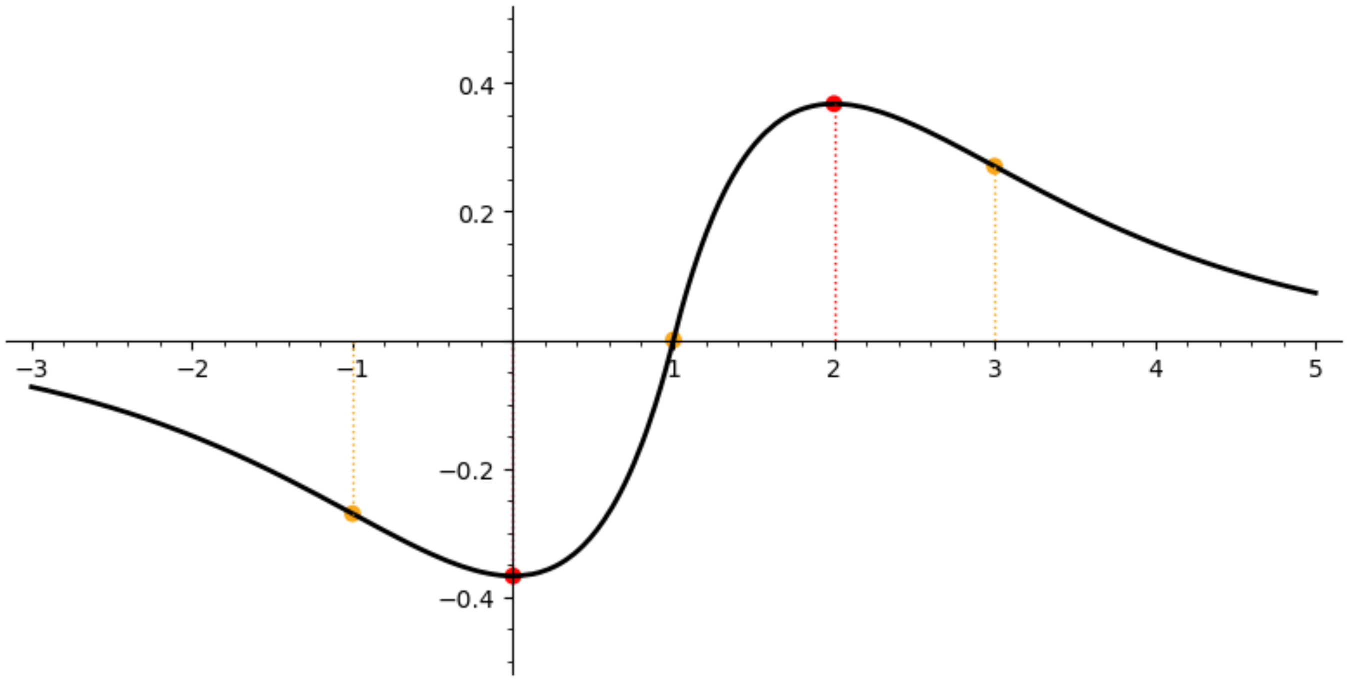
$x>1$: $f''(x)=\left(e^{1-x}(2-x)\right)'=e^{1-x}(x-3)\gtrless 0 \Leftrightarrow x>3$

resp. $x\in(1,3)$

$x<1$: $f''(x)=\left(e^{x-1}\cdot x\right)'=e^{x-1}(x+1)\gtrless 0\Leftrightarrow x\in(-1,1)$

resp. $x<-1$

$\Rightarrow$ inflexní body: $x=-1,1,3$



P4 $f(x) = x+2+\dfrac{3}{|x+1|}$

$\mathcal{D}_f = \mathbb{R}\setminus\{-1\}$, spojitá v $\mathcal{D}_f$

(bez symetrií)

$f(x)\to\pm\infty$, $x\to\pm\infty$

(neboť $\frac{1}{|x+1|}\to 0$)

$f(x)\to+\infty$, $x\to -1\pm$ $\left(\frac{1}{0+}=+\infty\right)$

derivace $(x\neq -1)$

$$f'(x) = 1 - \frac{3}{(|x+1|)^2}\cdot\left(|x+1|'\right) = 1 - \frac{3\cdot \operatorname{sgn}(x+1)}{(x+1)^2}$$

monotonie (+extr.)

$x<-1$ $\quad f'(x) = 1+\dfrac{3}{(x+1)^2} > 0$ vždy

$x>-1$ $\quad f(x) = 1-\dfrac{3}{(x+1)^2} = 0 \Leftrightarrow (x+1)^2 = 3$, tj.

$x = -1\pm\sqrt{3}$

$\Rightarrow$ $f(x)$ roste v $(-\infty,-1)$ a v $[-1+\sqrt{3},+\infty)$

klesá v $(-1,-1+\sqrt{3})$

ostré lok. min.

$\mathcal{H}_f = \mathbb{R}$

$\nexists$ glob. extrémy

konvexita $f''(x) = -3\left(\frac{\operatorname{sgn}(x+1)}{(x+1)^2}\right)' \overset{*)}{=} -3\operatorname{sgn}(x+1)\cdot\left(\frac{1}{(x+1)^2}\right)'$

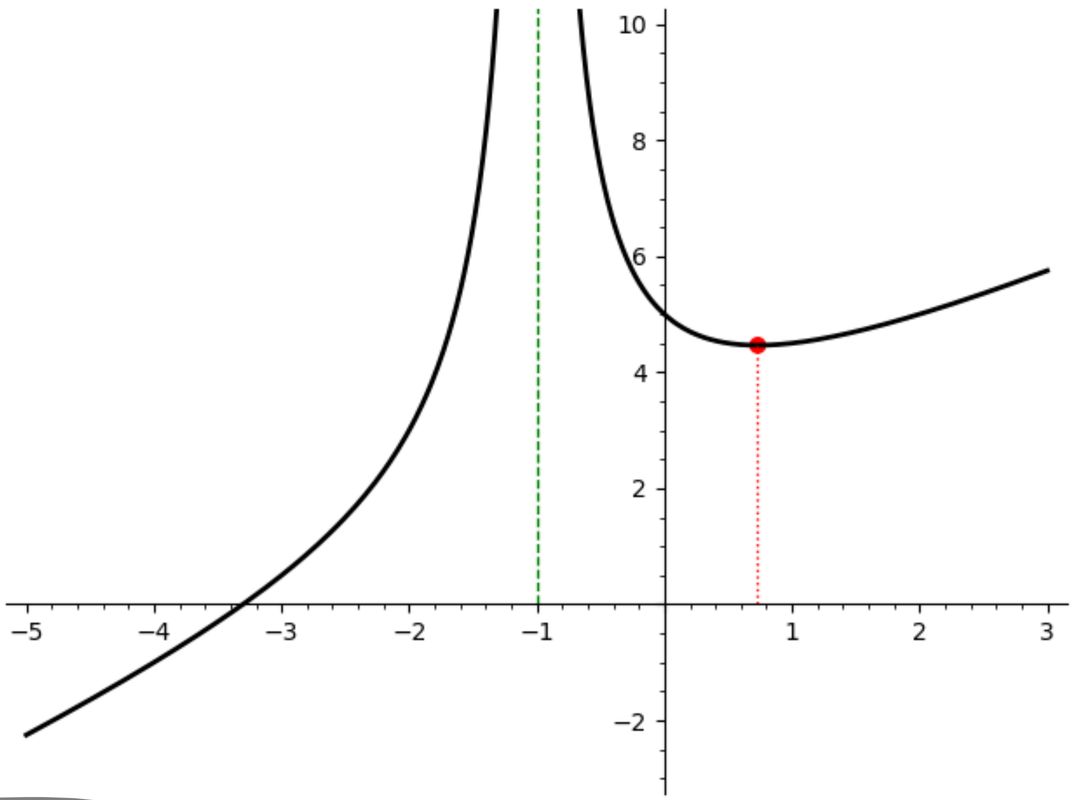
$(x \neq -1)$

$$= 6\,\frac{\operatorname{sgn}(x+1)}{(x+1)^3} = \frac{6}{|x+1|^3} > 0$$

*) $\operatorname{sgn}(x+1)$ je konst. v $(-\infty,-1)$ a v $(-1,+\infty)$

$\Rightarrow$ $f(x)$ ryze konvexní v $(-\infty,-1)$ a v $(-1,+\infty)$

(NE: $f(x)$ konvexní v D_f)



PS $f(x) = \arcsin\left(\frac{2x}{x^2+1}\right)$ .... lichá, spojitá v $D_f = \mathbb{R}$

$h(x) = \frac{2x}{x^2+1}$

($\arcsin y : [-1,1] \to \mathbb{R}$ spojitá, lichá

$h(x) : \mathbb{R} \to [-1,1]$ spojitá, lichá)

*)

*) $|h(x)| \leq 1 \quad \forall x \in \mathbb{R}$

$\Leftrightarrow \quad \pm 2x \leq x^2+1$

$0 \leq (x \pm 1)^2$ ✓ (navíc zřejmě $|h(x)| = 1 \Leftrightarrow x = \pm 1$)

derivace $(x \neq \pm 1)$: $f'(x) = \underbrace{\frac{1}{\sqrt{1-h^2(x)}}}_{= \sqrt{\frac{(x^2+1)^2}{(x^2-1)^2}}} \cdot \underbrace{h'(x)}_{= \frac{2(x^2+1) - 2x \cdot 2x}{(x^2+1)^2} = \frac{2(1-x^2)}{(x^2+1)^2}}$

$$\Rightarrow \quad f'(x) = \frac{|x^2+1|}{|x^2-1|} \cdot \frac{2(1-x^2)}{(x^2+1)^2} = \frac{-2\,\mathrm{sgn}(x^2-1)}{x^2+1}, \quad \forall x \neq \pm 1$$

$\left(\begin{array}{l} \mathrm{sgn}(x^2-1) = \pm 1 \\ \text{na } P_\pm(1) \end{array}\right)$ (neboť $|x^2+1| = (x^2+1)$, $|x^2-1| = (x^2-1) \cdot \mathrm{sgn}(x^2-1)$)

$$f'_\pm(1) = \lim_{x \to 1\pm} f'(x) = \lim_{x \to 1\pm} \frac{\mp 2}{x^2+1} = \mp \frac{1}{2}, \quad f'_\pm(-1) = \pm \frac{1}{2}$$

(analogicky nebo z lichosti $f(x)$)

monotonie $f'(x) < 0$ v $(-\infty, -1)$ a v $(1, +\infty)$

$f'(x) > 0$ v $(-1, 1)$

$\Rightarrow$ $f(x)$ klesající v $(-\infty, -1]$ a v $[1, +\infty)$

rostoucí v $[-1, 1]$

navíc: $f(x) \to 0$, $x \to \pm\infty$ $\left(h(x) = \frac{\frac{1}{x}}{1 + \frac{1}{x^2}} \to 0, x \to +\infty \right.$ $\left. \arcsin 0 = 0, \text{ spojitá} \right)$

$\Rightarrow$ $f(\pm 1) = \pm \frac{\pi}{2}$ globální extrémy

$$H_f = f([-1,1]) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

konvexita $(x \neq \pm 1)$ ... opět $\operatorname{sgn}(x^2-1) \equiv$ konst.

$$\Rightarrow f''(x) = -2\operatorname{sgn}(x^2-1)\cdot\left(\frac{1}{x^2+1}\right)' = \frac{2x\cdot\operatorname{sgn}(x^2-1)}{(x^2+1)^2}$$

$\Rightarrow$ $f''(x) > 0$ resp. < 0 na $(-1,0)$ a $(1,+\infty)$ resp.
na $(-\infty,-1)$ a $(0,1)$

$\Rightarrow$ $f(x)$ ryze konvexní v $[-1,0]$ a v $[1,+\infty)$
konkávní v $(-\infty,1]$ a v $[0,1]$

inflexní bod: $x=0$ (NE $x=\pm 1$, neboť $\nexists f'(\pm 1)$)