A1. [5b] Vypočítejte integrál (substituce $y = e^x$, per-partes)

$$\int e^x \ln(1 + e^x)\, dx$$

A2. [5b] Vypočítejte integrál (substituce $y = \sqrt[6]{x}$)

$$\int \frac{1 - \sqrt[3]{x}}{1 + \sqrt[2]{x}} \cdot \frac{dx}{\sqrt[6]{x^5}}$$

.... Kde výsledky platí? Uveďte vždy maximální otevřené intervaly.

B1. [5b] Vypočítejte integrál (per-partes)

$$\int \ln(1 - x^{-1})\, dx$$

B2. [5b] Vypočítejte integrál (substituce $y = \sqrt{x}$)

$$\int \frac{1}{1 - x^2} \cdot \frac{dx}{\sqrt{x}}$$

.... Kde výsledky platí? Uveďte vždy maximální otevřené intervaly.

A1. $\int e^x \ln(1+e^x)\,dx = \left| \begin{array}{l} 1.\ \text{VoS} \\ y = e^x \\ dy = e^x dx \end{array} \right| = \int \underbrace{1}_{u'} \cdot \underbrace{\ln(1+y)}_{v}\,dy = \left| \begin{array}{l} \text{per-} \\ \text{partes} \end{array} \right.$

$u = y \quad v' = \frac{1}{1+y}$

$= y \ln(1+y) - \underbrace{\int \frac{y}{1+y}\,dy}_{I}$

$I = \int \frac{1+y-1}{1+y}\,dy = \int 1 - \frac{1}{1+y}\,dy = y - \ln(1+y)$ v $(-\infty,-1)$
$(-1,+\infty)$

protože $e^x: \mathbb{R} \to (0,\infty)$, je: $\int e^x \ln(1+e^x) = e^x \ln(1+e^x)$
$- e^x + \ln(1+e^x)$ v $\mathbb{R}$.

A2. $\int \frac{1-\sqrt[3]{x}}{1+\sqrt[2]{x}} \frac{dx}{\sqrt[6]{x^5}} = \left| \begin{array}{ll} y = \sqrt[6]{x} & \sqrt[3]{x} = y^2 \\ dy = \frac{dx}{6\sqrt[6]{x^5}} & \sqrt[2]{x} = y^3 \end{array} \right| \overset{c}{=}$

$6\int \frac{1-y^2}{1+y^3}\,dy$; $\frac{1-y^2}{1+y^3} = \frac{(1+y)(1-y)}{(1+y)(1-y+y^2)} = \frac{1-y}{1-y+y^2}$;

$\int \frac{1-y}{1-y+y^2} = -\frac{1}{2}\int \frac{(2y-1)\,dy}{1-y+y^2} + \frac{1}{2}\int \frac{dy}{1-y+y^2} = I_1 + I_2$

$I_1 = -\frac{1}{2}\int \frac{dz}{z} \left| \begin{array}{l} z = 1-y+y^2 \\ dz = (2y-1)\,dy \end{array} \right| = -\frac{1}{2}\ln|z|$
$= -\frac{1}{2}\ln(1-y+y^2)$.

I_2: $1-y+y^2 = \left(y-\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left(\left(\frac{2y-1}{\sqrt{3}}\right)^2 + 1\right)$

tedy: $I_2 = \frac{1}{2}\cdot\frac{4}{3}\int \frac{dy}{1+\left(\frac{2y-1}{\sqrt{3}}\right)^2} = \underbrace{\frac{1}{2}\cdot\frac{4}{3}\cdot\frac{\sqrt{3}}{2}}_{\frac{1}{\sqrt{3}}} \operatorname{arctg}\left(\frac{2y-1}{\sqrt{3}}\right)$

celkem: $-3\ln\left(1-\sqrt[6]{x}+\sqrt[3]{x}\right) + 2\sqrt{3}\operatorname{arctg}\left(\frac{2\sqrt[6]{x}-1}{\sqrt{3}}\right)$

$x \in (0,\infty)$ nebo $(-\infty,0)$

B1. $\int \underbrace{1}_{u'} \cdot \underbrace{\ln\left(1-\frac{1}{x}\right)}_{v} dx = x \ln\left(1-\frac{1}{x}\right) - \int \frac{dx}{x-1} = x\ln\left(1-\frac{1}{x}\right) - \ln|x-1|$

$u = x \; ; \; v' = \frac{1}{1-x^{-1}} \cdot \frac{+1}{x^2}$

? Def: $1-\frac{1}{x} > 0 \iff x \in (-\infty, 0)$ nebo $(1, +\infty)$

B2. $\int \frac{dx}{1-x^2} \cdot \frac{1}{\sqrt{x}} \quad \left| \begin{array}{l} y = \sqrt{x} \\ dy = \frac{dx}{2\sqrt{x}} \end{array} \right| = \int \frac{2\,dy}{(1-y^4)}$

$\frac{1}{1-y^4} = \frac{1}{(1-y)(1+y)(1+y^2)} = \frac{A}{y+1} + \frac{B}{y-1} + \frac{Cy+D}{y^2+1}$

$\Rightarrow A = \frac{1}{4}, \; B = -\frac{1}{4}, \; C = 0, \; D = \frac{1}{2}$

$2\int \frac{dy}{1-y^4} = \frac{1}{2}\int \frac{dy}{y+1} - \frac{1}{2}\int \frac{dy}{y-1} + \int \frac{dy}{1+y^2}$

$= \underbrace{\frac{1}{2}\ln\left|\frac{y+1}{y-1}\right| + \operatorname{arctg} y}_{F(y)} \; ; \quad y \; (-\infty, -1)$
$(-1, 1)$
$(1, +\infty)$

Tj. $\int \frac{dx}{(1-x^2)\sqrt{x}} = F(\sqrt{x}) \; ; \quad x \in (0, 1)$ nebo $(1, +\infty)$