Věta 8.3 (Operace s malým o.)

(1) $f(x) = o(x^n)$, $g(x) = x^m$, $x \to 0$; kde $m \geq n$.

$\Rightarrow a f(x) + b g(x) = o(x^n)$, $x \to 0$

(2) $f(x) = o(x^n)$, $x \to x_0$

$\Rightarrow a x^m f(x) = o(x^{n+m})$, $x \to 0$

(3) $f(x) = o(x^n)$, $g(x) = o(x^m)$, $x \to 0$

$\Rightarrow f(x) g(x) = o(x^{n+m})$, $x \to 0$

(4) $f(x) = o(x^n)$, $g(x) \sim x^m$, $x \to 0$, $m \geq 1$

$\Rightarrow f(g(x)) = o(x^{mn})$, $x \to 0$

Dk. (1) cíl: $\dfrac{a f(x) + b g(x)}{x^n} \to 0$, $x \to 0$

... upravme: $$\underbrace{a \frac{f(x)}{x^n}}_{\to 0} + \underbrace{b x^{m-n}}_{(*)} \cdot \underbrace{\frac{g(x)}{x^m}}_{\to 0}$$

(*): omezené pro $x \to 0$, užiji VoAL
(neboť $m - n \geq 0$) resp. Věta 2.4

(2) cíl: $\underbrace{\frac{a x^m f(x)}{x^{m+n}}}_{=} \to 0, \; x \to 0$

... upravme: $a \frac{f(x)}{x^n} \to a \cdot 0$

(3) $\frac{f(x) \cdot g(x)}{x^{m+n}} = \frac{f(x)}{x^n} \cdot \frac{g(x)}{x^m} \to 0 \cdot 0$

(4) víme: $\frac{f(y)}{y^n} \to 0, \; y \to 0$

$\frac{g(x)}{x^m} \to C, \; x \to 0$ kde $C \in \mathbb{R}$, $\neq 0$

cíl: $\frac{f(g(x))}{x^{mn}} \to 0, \; x \to 0$

upravme: $\frac{f(g(x))}{(g(x))^n} \cdot \left(\frac{g(x)}{x^m} \right)^n = P_1 \cdot P_2$

$P_2 \to C^n$ ... dle VoLSF;

vnější: $h(r) = r^n$ spojité v $C \in \mathbb{R}$

vnitřní: $g(t)/t^m \to C$

$P_1 \to 0$ ... opět dle VoLSF

vnitřní fce: $g(t) = \frac{g(t)}{x^m} \cdot t^m \to 0 \cdot 0$ (VoAL)

<u>navíc</u>: $\frac{g(t)}{x^m} \neq 0$ (a tedy $g(t) \neq 0$)

na jistém $P(0,\delta) \Leftarrow$ Lemma 2.1

vnější fce: $\frac{f(y)}{y^n} \to 0, \; y \to 0$.

<u>Př.</u> 1 $\frac{x^3}{\sin x \cdot \ln(1+x^2)} \to 1, \; x \to 0$

... víme: $\sin x = x + o(x)$

$\ln(1+y) = y + o(y)$

$\Rightarrow \ln(1+x^2) = x^2 + o(x^2), \; x \to 0$

$$\sin x \cdot \ln(1+x^2) = (x+o(x))(x^2+o(x^2))$$
$$= x^3 + \underbrace{x\,o(x^2) + x^2 o(x) + o(x)o(x^2)}_{o(x^3)}, \quad x \to 0$$

$$f(x) = \frac{x^3}{x^3+o(x^3)} = \frac{1}{1+\underbrace{\frac{o(x^3)}{x^3}}_{\to 0}} \to \frac{1}{1+0} \quad (\text{VoAL})$$

② $\lim\limits_{x\to+\infty} x\left(\sqrt{1+x^2} - 3\sqrt[3]{1+x^3} + 2\sqrt[4]{1+x^4}\right) = ?$

$$= \lim_{y\to 0+} \frac{1}{y^2}\left(\sqrt{1+y^2} - 3\sqrt[3]{1+y^3} + 2\sqrt[4]{1+y^4}\right)$$

m'me: $(1+x)^a = 1+ax+o(x), \quad x \to 0(\pm)$

$$\Rightarrow \sqrt{1+y^2} = 1+\frac{1}{2}y^2+o(y^2)$$
$$\sqrt[3]{1+y^3} = 1+\frac{1}{3}y^3+o(y^3)$$
$$\sqrt[4]{1+y^4} = 1+\frac{1}{4}y^4+o(y^4)$$

$$\Rightarrow g(y) = \frac{1}{y^2}\Big(1+\frac{1}{2}y^2+o(y^2) - 3\left(1+\frac{1}{3}y^3+o(y^3)\right)$$
$$+2(1+o(y^2))\Big) = \frac{y^2/2+o(y^2)}{y^2} \to \frac{1}{2}, \quad y \to 0.$$

Def. Pro $f \in C^n(\mathcal{U}(x_0), \delta)$ definuji:

$$R^f_{n+1, x_0}(x) = f(x) - T^f_{n, x_0}(x) \quad , x \in \mathcal{U}(x_0)$$

tzv. Taylorův zbytek po n-tém členu.

Věta 8.4. Nechť $f \in C^{n+1}(\mathcal{U}(x_0, \delta))$, nechť $x \in P(x_0, \delta)$ je pevné. Pak $\exists \theta$ mezi x, x_0 t. ž.

$$R_{n+1}(x) = \frac{f^{(n+1)}(\theta)}{(n+1)!}(x - x_0)^{n+1}$$

resp. $\exists \theta$ mezi x, x_0 t. ž.

$$R_{n+1}(x) = \frac{f^{(n+1)}(\theta)}{n!}(x - \theta)^n (x - x_0).$$

Dk. TRIK: pomocná fce $\varphi(t)$, $t \in [x_0, x]$

kde $\varphi(t) = f(t) - T^f_{n, t}(x)$

$$= f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x - t)^k$$

vidíme: $\varphi(x_0) = f(x_0) - T^f_{n, x_0}(x) = R_{n+1}(x)$

$\varphi(x) = f(x) - T^f_{n, x}(x) = 0$

pomocný výpočet: $\varphi'(t) = \frac{d}{dt}\varphi(t)$

$$= -\left\{ f(t) + \sum_{r=1}^{n} \frac{1}{r!} f^{(r)}(t)\cdot(x-t)^r \right\}'$$

$$= -f(t) - \sum_{r=1}^{n} \frac{1}{r!}\left(f^{(r+1)}(t)\cdot(x-t)^r - f^{(r)}(t)\, r\,(x-t)^{r-1}\right)$$

$$= \underbrace{-f'(t) - \sum_{r=1}^{n} \frac{f^{(r+1)}(t)}{r!}(x-t)^r}_{-\sum_{l=1}^{n+1} \frac{f^{(l)}}{(l-1)!}(x-t)^{l-1}} + \sum_{r=1}^{n} \frac{f^{(r)}}{(r-1)!}(x-t)^{r-1}$$

$$= \frac{f^{(n+1)}(t)}{(n+1)!}(x-t)^n$$

. . . .

Věta 6.8 (Cauchy) $\Rightarrow$ $\boxed{\dfrac{\varphi(x_0)-\varphi(t)}{\psi(x_0)-\psi(t)} = \dfrac{\varphi'(\theta)}{\psi'(\theta)}}$

kde volím: (L): $\psi(t) = (x-t)^{n+1}$

resp. (C) $\psi(t) = t, \quad t \in [x_0, x]$.

ad(L): $LS = \dfrac{R_{n+1}(x)}{(x-x_0)^{n+1}}$, $\varphi'(t) = -(n+1)(x-t)^n$

$$PS = \frac{-\frac{f^{(n+1)}(\theta)}{n!}\cdot(x-\theta)^n}{-(n+1)\cdot(x-\theta)^n} = \frac{f^{(n+1)}(\theta)}{(n+1)!}$$

ad(C): $LS = \dfrac{R_{n+1}(x)}{x_0-x}$, $\varphi'(t) = 1$

$$PS = -\frac{f^{(n+1)}(\theta)}{n!}(x-\theta)^n$$

Příklad $f(x) = e^x$, $x_0 = 0$, $x \in \mathbb{R}$ pevné:

$$R_{n+1}(x) = e^x - \sum_{k=0}^{n} \frac{x^n}{n!} = \frac{e^\theta}{(n+1)!} x^{n+1},\ \text{kde } \theta \in [0, x]$$

$$\Rightarrow |R_{n+1}(x)| \leq c \cdot \frac{M^{n+1}}{(n+1)!}$$

$$\Rightarrow R_{n+1}(x) \to 0,\ n \to \infty \ (x \text{ pevné})$$