Def. $Q_{k,x_0}(x) = \frac{1}{k!}(x-x_0)^k$, $x_0 \in \mathbb{R}$, $k \geq 0$ celé

např.: $Q_{0,x_0}(x) = 1$ (neboť $0! = a^0 = 1$)

$Q_{1,x_0}(x) = x - x_0$

$Q_{2,x_0}(x) = \frac{1}{2}(x-x_0)^2 \ \dots$

Lemma 8.1. Platí:

(1) $Q_{k,x_0}(x)$ je polynom stupně k

(2) $Q'_{0,x_0}(x) = 0$, $Q'_{k,x_0}(x) = Q_{k-1,x_0}(x)$, $k \geq 1$

(3) $Q^{(l)}_{k,x_0}(x_0) = \begin{cases} 1, & l = k \\ 0, & l \neq k \end{cases}$

Dk. (1) neboť: $(x-x_0)^k = x^k + \dots$

(2) $1' = 0$, $\left(\frac{1}{k!}(x-x_0)^k\right)' = \underbrace{\frac{k}{k!}}_{=\frac{1}{(k-1)!}}(x-x_0)^{k-1}$

(3) plyne z (2), neboť:

- $Q^{(k)}_{k,x_0} = Q_{0,x_0} = 1$
- $l > k \Rightarrow Q^{(l)}_{k,x_0} \equiv 0$

$l < k \Rightarrow$ piš $k = l + s$, $s \in \mathbb{N}$

$$Q^{(l)}_{k,x_0}(x_0) = Q_{s,x_0}(x_0) = 0$$

Věta 8.1 Buď $f \in C^n(\mathcal{U}(x_0))$.

Pak $f(x) = T^f_{n,x_0}(x) + o\left((x-x_0)^n\right)$,

pro $x \to x_0$.

Navíc: jednoznačnost.

Dk. BÚNO: $x_0 = 0$, ... označme:

$$p(x) = T^f_{n,0}(x) = \sum_{k=0}^{n} \underbrace{\frac{f^{(k)}(0)}{k!} x^k}_{= f^{(k)}(0) \cdot Q_k(x)}$$

dle L. 8.1. $\Rightarrow$ $Q_k^{(l)}(0) = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases}$

a tedy:

$$p^{(l)}(0) = \sum_{k=0}^{n} f^{(k)}(0)\, Q_k^{(l)}(0) = f^{(l)}(0),$$

pro $s = 0, 1, \dots n$.

$\underline{\underline{1.}}$ $\dfrac{f(x)-h(x)}{x^n} \to 0\ ,\ x \to 0.$

.... l'Hospital „$\frac{0}{0}$", neboť $f(x) \to f(0)$
 $h(x) \to h(0)$

$\Rightarrow \dfrac{f'(x)-h'(x)}{n x^{n-1}}$ ... opakuji tento postup ...

$\Rightarrow \dfrac{f^{(n)}(x)-h^{(n)}(x)}{n!} \to \underline{\underline{0}}$

$\underline{\underline{2.}}$ buď $g(x)$ polynom st $\leq n$ t.ž.

$\dfrac{f(x)-g(x)}{x^n} \to 0\ ,\ x \to 0.$

<u>cíl</u>: $h(x) \equiv g(x)$

pomocná úvaha: $\dfrac{h(x)-g(x)}{x^n} =$

$$= \underbrace{\frac{f(x)-g(x)}{x^n}}_{\to 0} - \underbrace{\frac{f(x)-h(x)}{x^n}}_{\to 0} \to 0 \qquad x \to 0$$

pišme $p(t) = \sum_{k=0}^{n} a_k x^k$

$$q(t) = \sum_{k=0}^{n} b_k x^k$$

$\boxed{??}$ $\exists\, s \in \{0, \dots, n\}$ t.ž. $a_s \neq b_s$;

BÚNO: buď s nejmenší takové

$$\Rightarrow p(t) - q(t) = \sum_{k=s}^{n} \underbrace{(a_k - b_k)}_{=\, c_k} x^k$$

$$= \underbrace{c_s}_{\neq 0} x^s + \sum_{k=s+1}^{n} c_k x^k$$

$$\frac{p(t)-q(t)}{x^s} = \overbrace{\frac{p(t)-q(t)}{x^n}}^{\to 0} \cdot x^{\overset{\geq 0}{n-s}} \to 0$$

leč zároveň:

$$\frac{p(t)-q(t)}{x^s} = c_s + \underbrace{\sum_{k=s+1}^{\infty} c_k x^{\overset{>0}{k-s}}}_{\to 0}$$

$$\to c_s \neq 0.$$

SPOR

Příkl. ① $f(x) = e^x$, $x_0 = 0$.

$f^{(k)}(x) = e^x$, tj. $f^{(k)}(0) = e^0 = 1$

$k = 0, 1, 2, \dots$

$$\Rightarrow T^{e^x}_{n,0}(x) = \sum_{k=0}^{n} \frac{x^k}{k!}, \text{ nebolí}$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots \frac{x^n}{n!} + o(x^n), \ x \to 0$$

② $f(x) = \sin x$, $x_0 = 0$.

$f^{(k)}(x) = \cos x, -\sin x, -\cos x, \sin x, \dots$

$f^{(k)}(0) = 0, 1, 0, -1$, atd.

$$\Rightarrow T^{\sin}_{2n+1,0}(x) = \sum_{l=0}^{n} (-1)^l \frac{x^{2l+1}}{(2l+1)!}, \text{ tj.}$$

$$\sin x = x - \frac{x^3}{6} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$$

$x \to 0$

③ $f(x) = (1+x)^a$, $x_0 = 0$, $a \in \mathbb{R}$ pevné

$f'(x) = a(1+x)^{a-1}$

$\vdots$

$f^{(k)}(x) = a \cdot (a-1) \cdots (a-k+1) \cdot (1+x)^{a-k}$

$$\Rightarrow f^{(k)}(0) = a(a-1)\cdots(a-k+1), \text{ a tedy}$$

$$T_{n,0}^{(1+x)^a}(x) = \sum_{k=0}^{n} \binom{a}{k} x^k$$

$$= 1 + ax + \frac{a(a-1)}{2}x^2 + \frac{a(a-1)(a-2)}{6}x^3 + \cdots$$

Věta 8.2. Nechť $F \in C^{n+1}(\mathcal{U}(x_0))$, nechť $F'(x) = f(x)$. Potom:

(1) $\left(T_{n+1,x_0}^{F}(x)\right)' = T_{n,x_0}^{f}(x)$

(2) $\int T_{n,x_0}^{f}(x)\,dx = T_{n+1,x_0}^{F}(x) + C$,

při vhodné volbě C.

Dk. (1) $T_{n+1,x_0}^{F}(x) = \sum_{k=0}^{n+1} F^{(k)}(x_0)\, Q_{k,x_0}(x)$

$$\frac{d}{dx}: \sum_{k=0}^{n+1} F^{(k)}(x_0)\, \underbrace{Q'_{k,x_0}(x)}$$

$0, \; k=0$

$Q_{k-1,x_0}(x), \; k \geq 1$

$$= \sum_{k=1}^{n+1} F^{(k)}(x_0)\, Q_{k-1,x_0}(x) = \underbrace{\sum_{l=0}^{n} F^{(l+1)}(x_0)\, Q_{l,x_0}(x)}_{=\, T^f_{n,x_0}(x)}$$

leč: $F^{(l+1)} = (F')^{(l)} = f^{(l)} \Rightarrow T^f_{n,x_0}(x)$

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(2) $$\int T^f_{n,x_0}(x)\,dx = \sum_{k=0}^{n} f^{(k)}(x_0) \underbrace{\int Q_{k,x_0}(x)\,dx}_{Q_{k+1,x_0}(x)}$$

$$= \sum_{k=0}^{n} \underbrace{f^{(k)}(x_0)}_{F^{(k+1)}(x_0)} Q_{k+1,x_0}(x) + C$$

$$= \sum_{l=1}^{n+1} F^{(l)}(x_0)\, Q_{l,x_0}(x) + C = T^F_{n+1,x_0}(x)$$

zvolíme-li $C = F(x_0)$

$$\left(= F(x_0) \cdot \underbrace{Q_{0,x_0}(x)}_{1} \right)$$

Příkl. (1) $\cos x = (\sin x)'$, a tedy

$$T^{\cos}_{2n,0}(x) = \left(T^{\sin}_{2n+1,0}(x)\right)' = \sum_{k=0}^{n} (-1)^k \frac{x^{2n}}{(2n)!},$$

$$\Rightarrow \cos x = 1 - \frac{x^2}{2} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$$

$$x \to 0$$

(2) $\ln x = \int \frac{dx}{1+x}$, $x \in \mathcal{U}(0,1)$

$$T^{(1+x)^{-1}}_{n,0}(x) = \sum_{k=0}^{n} (-1)^k x^k, \text{ neboť}$$

$$\binom{-1}{k} = \frac{(-1)(-2)\cdots(-1-k+1)}{k!} = \frac{(-1)\cdot(-2)\cdots(-k)}{k!}$$

$$\Rightarrow T^{\ln(1+x)}_{n+1,0}(x) = \int \sum_{k=0}^{n} (-1)^k x^k + C$$

$$= \sum_{k=0}^{n} (-1)^k \frac{x^{k+1}}{k+1} = \sum_{l=1}^{n+1} (-1)^{l+1} \frac{x^l}{l}, \text{ tj.}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

$$(C = \ln 1 = 0)$$