3. termín.

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① $$\int_0^{2\pi} \frac{\sin x}{1 + a\sin x}\,dx; \quad a \in (0,1)$$

② $$\left[\frac{1}{e^{x} + e^{-x+2} + e^{2} + 1}\right]^{\wedge}(\xi) =$$

③ $$f(z) = \frac{1}{z\,\mathrm{tg}\,z}$$

Typ singularity & min. dva členy Laurentova rozvoje v bodě

(a) $z = 0$

(b) $z = \frac{\pi}{2}$

① $I = \int_0^{2\pi} \frac{\sin x}{1+a\cdot\sin x}\,dx; \quad a\in(0,1)$

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typový příklad: $\int_0^{2\pi} R(\cos x,\sin x)\,dx = \int_{\varphi} g(z)\,dz$ ← jednotková kružnice

$$g(z)=\frac{1}{iz}R\left(\frac12\left(z+\frac1z\right),\frac{1}{2i}\left(z-\frac1z\right)\right).$$

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zde $g(z)= \dfrac{\frac{1}{2i}\left(z-\frac1z\right)}{1+\frac{a}{2i}\left(z-\frac1z\right)}\cdot\dfrac{1}{iz}$ ; po úpravě $\cdot\,\dfrac{2iz}{2iz}$

$$= \frac{-i(z^2-1)}{z(az^2+2iz-a)} \;;= \frac{-i(z^2-1)}{az\left(z^2+\frac{2i}{a}z-1\right)}$$

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Singularity: (= nulové body jmenovatele) $(a\neq 0)$

$z_0=0$;

$z_{1,2}$ = kořeny polynomu $z^2+\frac{2i}{a}z-1$

$$D=\left(\frac{2i}{a}\right)^2-4\cdot1(-1) = 4-\frac{4}{a^2} = 4\underbrace{\left(1-\frac{1}{a^2}\right)}_{<0};$$

neboť $a^2<1$.

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tj. $D=\left(\pm2i\sqrt{\frac1{a^2}-1}\right)^2$

$\cancel{z_{1,2}=}$

$$z_{1,2}=\frac12\left(-\frac{2i}{a}\pm2i\sqrt{\frac1{a^2}-1}\right)$$

$$= i\left(\pm\sqrt{\frac{1}{a^2}-1}-\frac1a\right).$$

? leží uvnitř $\varphi$ ?

$$z_1 = -i\left(\underbrace{\frac{1}{a}}_{>1} + \underbrace{\sqrt{\frac{1}{a^2}-1}}_{>0}\right) \quad \text{NE;}$$

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$$z_2 = -i\left(\underbrace{\sqrt{\frac{1}{a^2}-1} - \frac{1}{a}}_{\in(\cancel{0}-1,0)}\right) \quad \text{ANO}$$

ověření:

$$\sqrt{\frac{1}{a^2}-1} - \frac{1}{a} < 0$$

$$\sqrt{\frac{1}{a^2}-1} < \frac{1}{a} \quad \Big/^2$$

$$\frac{1}{a^2} - 1 < \frac{1}{a^2} \checkmark$$

$$\sqrt{\frac{1}{a^2}-1} - \frac{1}{a} > -1$$

$$\sqrt{\frac{1}{a^2}-1} > \frac{1}{a} - 1 \quad \Big/^2$$

$$\frac{1}{a^2} - 1 > \frac{1}{a^2} - \frac{2}{a} + 1$$

$$\frac{2}{a} > 2$$

$$1 > a \checkmark$$

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celkově tedy: $I = 2\pi i\left(\mathrm{res}_0\, g + \mathrm{res}_{z_2}\, g\right)$.

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Výpočet reziduí: $$g(z) = \frac{-i\,(z^2-1)}{a z\,(z-z_1)(z-z_2)}$$

$\Rightarrow$ jednoduché póly; 1. pravidlo (V. 22.13)

$$\operatorname{res}_0 g = \operatorname{res}_0 \frac{h(z)}{z} = h(0); \quad \text{kde } h(z) = \frac{-i(z^2-1)}{a\left(z^2+\frac{2i}{a}z-1\right)}$$

$$= \frac{-i(-1)}{a\,(-1)} = -\frac{i}{a}.$$

$$\operatorname{res}_{z_2} g = \operatorname{res}_{z_2} \frac{1}{z-z_2}\cdot\left(\frac{-i(z^2-1)}{az(z-z_1)}\right)$$

$$= \left(\frac{-i(z^2-1)}{az(z-z_1)}\right)\Bigg|_{z=z_2}.$$

trik: (výpočet jmenovatele) $z_2$ je kořen $z^2+\frac{2i}{a}z-1$;

tj.: $z_2^2-1 = -\frac{2i}{a}z_2$

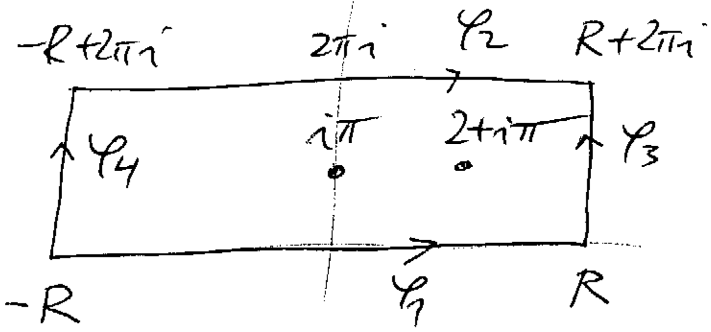
$$= \frac{-i\left(-\frac{2i}{a}\cancel{z_2}\right)}{a\cdot\cancel{z_2}(z_2-z_1)} = \frac{-2}{a^2\left(2i\sqrt{\frac{1}{a^2}-1}\right)} = \frac{i}{a\sqrt{1-a^2}}.$$

celkem:
$$I = 2\pi\left(\frac{1}{a}-\frac{1}{a\sqrt{1-a^2}}\right) = \frac{-2\pi a}{\sqrt{1-a^2}\left(1+\sqrt{1-a^2}\right)}.$$

(2) $\left[\frac{1}{e^x+e^{2-x}+e^2+1}\right]^\wedge(\xi) = ?$

$f(x) = \frac{1}{e^x+e^{2-x}+e^2+1}$

$$g(z) = \frac{e^{-2\pi i \xi z}}{e^z+e^{2-z}+e^2+1};$$




$$\varphi := \varphi_1 \dotplus \varphi_3 \dot- \varphi_2 \dot- \varphi_4;$$

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$$\int_{\varphi_1} g(z)\,dz = \int_{-R}^{R} f(t)\, e^{-2\pi i \xi t}\,dt \to \hat{f}(\xi); \quad R\to\infty$$

$$\int_{\varphi_2} g(z)\,dz = \int_{-R}^{R} f(t+2\pi i)\, e^{-2\pi i \xi (t+2\pi i)}\,dt$$

$$= \int_{-R}^{R} f(t)\cdot e^{-2\pi i \xi t}\cdot e^{4\pi^2 \xi}\,dt \to e^{4\pi^2\xi}\cdot \hat{f}(\xi);$$

$R\to\infty$.

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$\int_{\varphi_3} g(z)\,dz \to 0$; nebot' $\varphi_3 = R + t\cdot 2\pi i;\ t\in[0,1]$

$$\int_{\varphi_3} g = \int_0^1 \underbrace{e^{-2\pi i \xi (R+t\cdot 2\pi i)}}_{\text{omezené}} \frac{2\pi i\cdot dt}{\underbrace{e^{R+t\cdot 2\pi i}}_{\to\infty} + \underbrace{e^{2-R\cdots}}_{\to 0} + e^2+1}$$

podobně $\int_{\varphi_4} g \to 0$.

celkem: $\int_\varphi g(z)\,dz \to \hat{f}(\xi)\left(1-e^{4\pi^2\xi}\right)$.

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? singularity $g$: $\quad e^z + e^{2-z} + e^2 + 1 = 0 \;\big/\cdot e^z$

polož $y = e^z$ $\qquad e^{2z} + (e^2+1)e^z + e^2 = 0$;

$$y^2 + (e^2+1)y + e^2 = 0$$

$$(y+1)(y+e^2) = 0$$

(a) $y = e^z = -1$;

$$e^z = e^{i\pi} \Leftrightarrow z = i\pi + 2k\pi i;$$

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(b) $y = \cancel{e}\, e^z = -e^2 = e^{2+i\pi}$

$$\Leftrightarrow z = 2 + i\pi + 2k\pi i;$$

$k \in \mathbb{Z}$

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uvnitř $\varphi$ leží pouze $i\pi$; $2+i\pi$.

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celkem tedy $\quad \hat{f}(\xi) = \dfrac{2\pi i}{1-e^{4\pi^2\xi}}\left(\mathrm{res}_{i\pi}\, g + \mathrm{res}_{2+i\pi}\, g\right)$

neboť $\int_\varphi g = 2\pi i\left(\mathrm{res}_{i\pi}\, g + \mathrm{res}_{2+i\pi}\, g\right)$; $\quad R$ dost velké.

výpočet reziduí: [V. 22.13; bod 2].

$$\operatorname{res}_{i\pi} g = \frac{e^{-2\pi i \xi z}}{(\qquad)'}\Bigg|_{z=i\pi} = \frac{e^{-2\pi i \xi z}}{e^{z} \cancel{+e} - e^{2-z}}\Bigg|_{z=i\pi}$$

$$= \frac{e^{2\pi^2 \xi}}{-1+e^{2}};$$

$$\operatorname{res}_{2+i\pi} g = \frac{e^{-2\pi i \xi z}}{e^{z}-e^{2-z}}\Bigg|_{z=2+i\pi} = \frac{e^{2\pi^2\xi}\cdot e^{-4\pi i \xi}}{-e^{2}+1};$$

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$$\text{celkem: } \hat{f}(\xi) = \frac{2\pi i \cdot e^{2\pi^2\xi}}{e^{2}-1}\,\frac{\left(-e^{-4\pi i \xi}+1\right)}{1-e^{4\pi^2\xi}};$$

pro $\xi \neq 0$.

spojitost:

$$f(0) = \lim_{\xi \to 0} \hat{f}(\xi) \underset{\text{l'Hospital}}{=} \frac{\cancel{2\pi} i}{e^{2}-1}\,\frac{+4\cancel{\pi} i}{-\cancel{4\pi^2}} = \frac{+2}{e^{2}-1}$$

(3) $f(z) = \dfrac{1}{z \cdot \operatorname{tg} z}$ ;

$\cos 0 \neq 0$

(a) $z_0 = 0$ : $f = \dfrac{\cos z}{z \cdot \sin z}$ ; $\sin z \approx z$ ; $z \to 0$ ;

tj: $\underline{2}$-násobný pól.

$$f(z) = \frac{a}{z^2} + \frac{b}{z} + c + \cdots$$

tedy: $\cos z = f(z) \cdot z \sin z$

$$\left(1 - \frac{z^2}{2} + \cdots\right) = \left(\frac{a}{z^2} + \frac{b}{z} + c + \cdots\right) \cdot \left(z^2 - \frac{z^4}{6} + \cdots\right)$$

porovnání mocnin: $z^0$ : $1 = a \cdot 1$ ; $\to$ $a = 1$

$z^1$ : $0 = b$

$z^2$ : $-\frac{1}{2} = -\frac{1}{6} + c$ $\longrightarrow$ $c = -\frac{1}{3}$

celkem: $f = \dfrac{1}{z^2} - \dfrac{1}{3} + \cdots$

(b) $z_0 = \frac{\pi}{2}$ : odstranitelná singularita!!

Laurent = Taylor ; $f(z) = a_0 + a_1\left(z - \frac{\pi}{2}\right) + a_2\left(z - \frac{\pi}{2}\right)^2 + \cdots$

$a_0 = f\left(\frac{\pi}{2}\right) = 0$

$a_1 = f'\left(\frac{\pi}{2}\right) = \cdots = \frac{\pi}{2}$ atd.