28.5.07 : 1. ter.

(1) $$\int_0^{2\pi} \frac{\cos^2 x}{2b\sin x + i(1-b^2)}\,dx\,; \qquad b \in (1, \infty)$$

(2) $$\left[\frac{1}{(x+i\pi)(x^2+\lambda^2)}\right]^{\wedge}(\xi)\,; \qquad \lambda>0;\ \lambda \neq \pi.$$

(3) $$f(z) = \frac{1}{z\sinh z}$$ $\forall$ residua & typy singularit

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pozn. (1) & (2): schéma
singularity
výpočet reziduí (metoda !!)

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(1)

① $g(z) = \dfrac{(z^2+1)^2}{4z^2(z+b)(bz-1)}$;

$I = 2\pi i \cdot \left(\mathrm{res}_0 + \mathrm{res}_{1/b} \right)$

$= \dfrac{i\pi}{b^2}$,

② $g(z) = \dfrac{e^{-2\pi i \xi z}}{(z+i\lambda)(z+i\pi)\cdot(z-i\lambda)}$

$$\hat{f}(\xi) = \begin{cases} \dfrac{-i\pi}{\lambda(\lambda+\pi)} e^{2\pi\xi\lambda}, & \xi < 0 \\ \dfrac{-i\pi}{\lambda(\lambda+\pi)} & \xi = 0 \\ \dfrac{-2\pi i}{\lambda-\pi} \left(\dfrac{e^{-2\pi^2\xi}}{\lambda+\pi} - \dfrac{e^{-2\pi\lambda\xi}}{2\lambda} \right) & \xi > 0 \end{cases}$$

③ $z_k = k\pi i$; $k \in \mathbb{Z}$

$\mathrm{res}_0 = 0$ 0 2-méreszű; $\mathrm{res}_0 = 0$

$k\pi i$; $k \neq 0$ ... : $\mathrm{res} = \dfrac{(-1)^k}{k\pi i}$, mat 8b

① $$\int_0^{2\pi} \frac{\cos^2 x}{2b\sin x + i(1-b^2)}\,dx; \quad b \in (1,\infty).$$

$$I = \int_\varphi g(z)\,dz; \qquad \varphi = e^{it};\ t\in[0,2\pi]$$

$$g(z) = \frac{\left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^2}{\frac{b}{i}\left(z-\frac{1}{z}\right)+i(1-b^2)}\cdot\frac{1}{iz} = \frac{(z^2+1)^2}{4z^2\left(bz^2+(b^2-1)z-b\right)}$$

[2]

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singularity: $\quad z_0 = 0 \quad$ (2-nás. pól.)

$$D = (b^2-1)^2 - 4b^2 = (b^2+1)^2;$$

$$z_{1,2} = \frac{1-b^2 \pm (b^2+1)}{2b} = \begin{cases} \frac{1}{b} = z_1 \\ -b = z_2. \end{cases}$$

rozklad: $$g(z) = \frac{(z^2+1)^2}{4z^2(z+b)(bz-1)};$$

[2]

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uvnitř φ leží $z_0, z_1 \Rightarrow I = 2\pi i\left(\operatorname{res}_0 + \operatorname{res}_{1/b}\right).$ [1]

$$\operatorname{res}_0 g = \left[\frac{(z^2+1)^2}{4(bz^2+(b^2-1)z-b)}\right]'\Bigg|_{z=0}$$

$$= \frac{1}{4\cdot(\ldots)^2}\Big[2(z^2+1)\cdot 2z\cdot 4(\ldots) - (z^2+1)^2\cdot 4\cdot(2bz+b^2-1)\Big]\Bigg|_{z=0}$$

$$= \frac{1}{4\cdot(-b)^2}\cdot\left(-4(b^2-1)\right) = \frac{1-b^2}{4b^2}$$

[2]

$$\operatorname{res}_{\frac{1}{b}} g = \frac{(z^2+1)^2}{4z^2(z+b)\cdot b}\Bigg|_{z=\frac{1}{b}} = \frac{\left(\frac{1}{b^2}+1\right)^2}{\frac{4}{b^2}\left(\frac{1}{b}+b\right)\cdot b} = \frac{b^2+1}{4b^2};$$

[1]

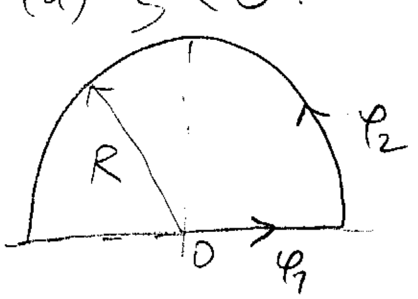
$$I = 2\pi i\left(\frac{1-b^2}{4b^2}+\frac{1+b^2}{4b^2}\right) = \boxed{\frac{i\pi}{b^2}}$$

② $f(x)=\frac{}{(x+i\pi)(x^2+\lambda^2)}$; $\lambda>0$. // $\lambda\neq\pi$.

$$g(z)=\frac{e^{-2\pi i\xi z}}{(z+i\pi)(z^2+\lambda^2)}=\frac{e^{-2\pi i\xi z}}{(z+i\pi)(z+i\lambda)(z-i\lambda)}$$

[1]

(a) $\xi<0$:



$\int_{\varphi_1} g\to\hat f(\xi)$; $\boxed{z\to+\infty}$

$\int_{\varphi_2} g\to 0$; neboť: $|e^{-2\pi i\xi z}|\le 1$ $\left(\begin{matrix}\xi<0\\ \operatorname{Im} z>0\end{matrix}\right)$

$|f(z)|\le\frac{c}{|z|^3}$;

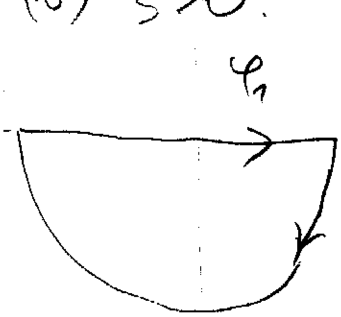
Lemma o půlkružnici [2]

$\Rightarrow$ $\hat f(\xi)=2\pi i\cdot \operatorname{res}_{i\lambda} g(z)=$

$$=2\pi i\cdot\left.\frac{e^{-2\pi i\xi z}}{(z+i\pi)(z+i\lambda)}\right|_{z=i\lambda}=\frac{-i\pi}{\lambda(\lambda+\pi)}\cdot e^{2\pi\xi\lambda}$$

[1]

(b) $\xi>0$:



$\int_{\varphi_1} g\to\hat f(\xi)$, $\int_{\varphi_2}\to 0$ ($\xi>0$!!) ($\operatorname{Im} z<0$!!)

$\hat f(\xi)=-2\pi i\cdot\left(\operatorname{res}_{-i\pi}+\operatorname{res}_{-i\lambda}\right)$ [3]

$$=\frac{-2\pi i}{\lambda-\pi}\cdot\left(\frac{e^{-2\pi^2\xi}}{\lambda+\pi}-\frac{e^{-2\pi\lambda\xi}}{2\lambda}\right)$$

$\hat f(0)=\frac{-i\pi}{\lambda(\lambda+\pi)}$ · [1]

$$2\pi i \cdot \operatorname{res}_{i\lambda} g = 2\pi i \cdot \frac{e^{-2\pi i \xi z}}{(z+i\pi)(z+i\lambda)}\bigg|_{z=i\lambda}$$

$$= \frac{\cancel{2}\pi \cancel{i}}{\cancel{i}(\lambda+\pi)\cdot(\cancel{2}i\lambda)} \cdot e^{-2\pi i \xi (i\lambda)}$$

$$= \frac{-i\pi}{\lambda(\lambda+\pi)} \cdot e^{2\pi\lambda\xi}, \quad \xi < 0.$$

$$2\pi i \left[\operatorname{res}_{-i\lambda} g + \operatorname{res}_{-i\pi} g\right] = -\hat{f}(\xi) =$$

$$2\pi i \left[\frac{e^{-2\pi i \xi z}}{(z+i\pi)(z-i\lambda)}\bigg|_{z=-i\lambda} + \frac{e^{-2\pi i \xi z}}{z^2+\lambda^2}\bigg|_{z=-i\pi}\right]$$

$$= 2\pi i \left[\frac{e^{-2\pi\lambda\xi}}{i(\pi-\lambda)\cdot(-2i\lambda)} + \frac{e^{-2\pi^2\xi}}{\lambda^2-\pi^2}\right]$$

$$= \frac{2\pi i}{\lambda-\pi} \cdot \left(\frac{e^{-2\pi^2\xi}}{\lambda+\pi} - \frac{e^{-2\pi\lambda\xi}}{2\lambda\xi}\right)$$

позн. $\xi \to 0+$: $\dfrac{2\pi i}{\lambda-\pi}\left(\dfrac{1}{\lambda+\pi} - \dfrac{1}{2\lambda}\right)$

$$= \frac{2\pi i}{\lambda-\pi} \cdot \frac{\lambda-\pi}{2\lambda(\lambda+\pi)} = \frac{\pi i}{\lambda(\lambda+\pi)}!$$

max 5b

(3) $f(z) = \dfrac{1}{z \sinh z}$;

$$\sinh z = \frac{1}{2}\left(e^{z} - e^{-z}\right) = 0$$

$$e^{2z} - 1 = 0$$

$$z = k\pi i;\ k \in \mathbb{Z}.$$

[1]

(a) $z = 0$: 2-rendű pól:

$$\operatorname{res}_0 f(z) = \lim_{z\to 0} \left[z^2 f(z)\right]';$$

[1]

$$\left[z^2 f(z)\right]' = \left(\frac{z}{\sinh z}\right)' = \frac{\sinh z - z\cosh z}{(\sinh z)^2}$$

$\sinh z = z + o(z^2)$

$\cosh z = 1 + o(z)$

$$= \frac{z - z(1 + o(z)) + o(z^2)}{(z + o(z))^2} = \frac{o(z^2)}{z^2 + o(z^2)} \to \underline{\underline{0}}.$$

[1]

(b) $z_k = k\pi i;\ k \in \mathbb{Z} \setminus \{0\}$.

$$\operatorname{res}_{z_k} f = \frac{1}{z} \cdot \frac{1}{(\sinh z)'}\bigg|_{z = z_k} = \frac{1}{z \cosh z}\bigg|_{z = z_k}$$

$$= \frac{1}{k\pi i\, \frac{1}{2}\left(e^{z} + e^{-z}\right)} = \frac{(-1)^k}{k\pi i}$$

$$e^{\pm k\pi i} = (-1)^k$$

[2]