10.5.2007

1. [8b] Pomocí reziduové věty spočítejte

$$\int_0^{2\pi} \frac{\sin x \, dx}{\sin x + ai},$$

kde $a \in (1, \infty)$.

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2. [8b] Najděte Fourierovu transformaci funkce

$$f(x) = \frac{1}{(x^2 - 2i)^2}.$$

Alespoň v jednom případě zdůvodněte podrobně, proč křivkový integrál konverguje k nule.

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3. [5b] Nalezněte reziduum ve všech singularitách funkce:

$$f(z) = \frac{\sin(z + z^2)}{z^3(1 - z)}.$$

3 : min 11 b

2 : 14 b

1 : 17 b

1) $I = \int_0^{2\pi} \frac{\sin x}{\sin x + ai}$; $a > 1$.

$I = \int_\varphi g(z)\,dz$; φ.. jedn. kružnice;

$$g(z) = \frac{\frac{1}{2i}\left(z - \frac{1}{z}\right)}{\frac{1}{2i}\left(z - \frac{1}{z}\right) + ai} \cdot \frac{1}{iz}$$

$$= \frac{-i\,(z^2-1)}{z\,(z^2-2az-1)}.$$

[2]

singularity: $z_0 = 0$;

$z^2 - 2az - 1 = (z-a)^2 - (1+a^2)$;

$z_{1,2} = a \pm \sqrt{1+a^2}$.

[1]

$z_1 > 1 \Rightarrow z_1 \notin \operatorname{int} \varphi$

$z_2 = a - \sqrt{1+a^2}$; zjevně $z_2 < 0$

$z_2 > -1$?

$a - \sqrt{1+a^2} > -1$

$a + 1 > \sqrt{1+a^2}$

$a^2 + 2a + 1 > 1 + a^2$

$2a > 0$ .. platí.

$I = 2\pi i \cdot \left\{ \operatorname{res}_0 g + \operatorname{res}_{z_2} g \right\}$.

[1]

$$g(z) = \frac{-i\,(z^2-1)}{z\,(z^2-2az-1)}\,;$$

$$\mathrm{res}_0\, g = \left.\frac{-i\,(z^2-1)}{(z^2-2az-1)}\right|_{z=0} = \frac{-i\,(-1)}{-1} = -i$$

[1]

$$\mathrm{res}_{z_2}\, g = \left.\frac{-i\,(z^2-1)}{z}\cdot\frac{1}{(z^2-2az-1)'}\right|_{z=z_2}$$

$$= \frac{-i\,(z_2^2-1)}{z_2}\cdot\frac{1}{(2z_2-2a)}\,;$$

aside: $z_2^2-1 = (a-\sqrt{1+a^2})^2-1 = a^2-2a\sqrt{1+a^2}+\cancel{1}+a^2-\cancel{1}$
$= 2a\,(a-\sqrt{1+a^2})$

$$2z_2-2a = -2\sqrt{1+a^2}\,;$$

$$\mathrm{res}_{z_2}\, g = \frac{-i\,\cancel{2}a\cdot\cancel{(a-\sqrt{1+a^2})}}{\cancel{a-\sqrt{1+a^2}}}\cdot\frac{1}{-\cancel{2}\sqrt{1+a^2}} = \frac{+ia}{\sqrt{1+a^2}}.$$

[2]

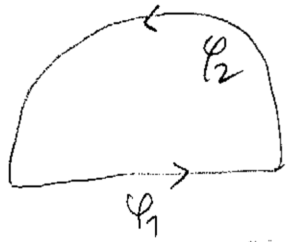
$$I = 2\pi i\left(-i+\frac{ia}{\sqrt{1+a^2}}\right) = 2\pi\left(1-\frac{a}{\sqrt{1+a^2}}\right).$$

[1]

2) $f(x) = \dfrac{1}{(x^2-2i)^2}$;

$$g(z) = \frac{e^{-2\pi i \xi z}}{(z^2-2i)^2} \; ; \qquad [1]$$

$\xi < 0$:



$$\int_{\varphi_1} g(z)\,dz \longrightarrow \hat{f}(\xi).$$

$$\left.\begin{array}{l} z \in \langle \varphi_2 \rangle : \; |g(z)| \leq \dfrac{1}{(R^2-2)^2} \; ; \\ L(\varphi) = \pi R \end{array}\right\} \Rightarrow \int_{\varphi_2} \longrightarrow 0. \qquad [2]$$

singularity: $2i = (1+i)^2$;

$$g(z) = \frac{e^{-2\pi i \xi z}}{\big(z-(1+i)\big)^2 \big(z+(1+i)\big)^2} \; ; \qquad [2]$$

$$\hat{f}(\xi) = 2\pi i \cdot \operatorname{res}_{1+i} g(z) = \left[\frac{e^{-2\pi i \xi z}}{(z+1+i)^2}\right]'_{z=1+i} \cdot 2\pi i \qquad (1)$$

$$[\;\;]' = e^{-2\pi i \xi z} \cdot \left(\frac{-2\pi i \xi}{(z+\lambda)^2} - \frac{2}{(z+\lambda)^3} \right) \Bigg|_{z=\lambda}$$

$$= \frac{e^{-2\pi i \xi (1+i)}}{\underbrace{[2(1+i)]^2}_{4i}} \cdot \left(-2\pi i \xi - \frac{2}{1+i} \right)$$

$\boxed{\text{need: } f(\xi) = f_0(-|\xi|)}$

$$= \frac{\pi}{2} \cdot e^{2\pi \xi} \cdot e^{-2\pi i \xi} \cdot \left(-2\pi i \xi - \frac{2}{1+i} \right) ; \qquad [2]$$

3) $\displaystyle f(z) = \frac{\sin(z+z^2)}{z^3(1-z)}$ allem [5b]

(a): $z = 1$: $\displaystyle f(z) = \frac{g(z)}{z-1}$, kee $\displaystyle g(z) = -\frac{\sin(z+z^2)}{z^3}$

$$\operatorname{res}_1 f(z) = g(1) = -\sin 2 \qquad [2]$$

(b) $z=0$: $\displaystyle \frac{1}{1-z} = 1 + z + z^2 + \cdots \quad \forall |z|<1$

$$\frac{1}{z^3(1-z)} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + \cdots \quad 0<|z|<1$$

$$\sin(z+z^2) = z + z^2 - \underbrace{\frac{(z+z^2)^3}{6} + \cdots}_{\frac{z^3}{6} + \cdots}$$

$$g(z) = \left(\frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + \cdots\right)\left(z + z^2 + \frac{z^3}{6} + \cdots\right)$$

$$= \frac{1}{z} + \frac{1}{z}$$

$$\Rightarrow \operatorname{res}_0 g = 2 \qquad [3]$$

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variante $\displaystyle \operatorname{res}_0 g = \frac{1}{2}\left(\frac{\sin(z+z^2)}{1-z}\right)''\Bigg|_{z=0} = \cdots = 2$